CS 184/284A Spring 2023 **Name:** _____

Exam 1 (Online) **SID Number:** _____

March 7th, 2023

Time Limit: 120 minutes

- Please carefully read, sign and submit the Honor Code described on the next page.
- You will need your own blank paper, a pen, a web browser, your class Gradescope account, and an internet-connected device capable of taking and uploading photos (a smartphone is ideal).
- You will have 2 hours to complete and submit your exam on Gradescope. You can choose to start and complete the exam at any time during the 24 hours of March 7th 2023, California time (i.e. 12:01am to 11:59pm).
- During the 24 hour period, you must not communicate with anyone about the exam, including posting to or reading from the internet any information about the exam questions. This is a strict requirement.
- This exam is open book, and open internet, subject to the limitations described above.
- This exam is 120 minutes, and has a total of 110 points. There are 19 pages (including this cover page) and 8 parts. Problem difficulty varies throughout the exam, so move on if you find yourself stuck.
- Show your work and what you know for partial credit.
- For some questions, you will have to write the solution by hand and upload a picture/file to Gradescope. For these, please write out each answer on a separate page and upload a separate file for each question. Uploading time is built into the 2 hours, but if you encounter uploading difficulties you may email your photos to ath@berkeley.edu with subject 'CS184 Exam'.
- For fairness to students taking the exam at different times, the teaching staff will not answer any questions or post any corrections about the exam during the 24 hour period.

Problem	Points	Score
1	0	
2	20	
3	17	
4	12	
5	18	
6	17	
7	14	
8	12	
Total:	110	

1. Honor Code and Academic Integrity Certification

The Honor Code is the commitment and work of students individually and as a community to uphold honesty and integrity. This commitment is comprehensive. Specifically on exams, it means not giving or receiving any help, or using any resources that are not permitted. The Honor Code requires that students take an active role in seeing to it that others as well as themselves uphold the letter and spirit of the Honor Code.

By uploading my signature I certify that:

- All the work submitted in my name for this exam is my work alone.
- I have not given or received any help during this exam.
- I have not used any un-permitted resources during this exam (see cover page for permitted resources).
- I have timed myself and used no more than the allotted time to write my answers for this exam.
- If I become aware of any Honor Code violations related to this exam, I will inform the course staff immediately.

I understand that a false statement on this declaration is a violation of the Honor Code, and would be subject to the highest level of disciplinary sanction under the University Code of Conduct.

1.i. Certification

UPLOAD PHOTO **On a sheet of your own paper, please sign to provide your certification of this statement, write your name, SID number and the date. Take a photo of your page and upload to Gradescope Exam 1.**

Signature: _____

Name: _____

SID Number: _____

Date: _____

2. (Total : 20 points) True / False

Mark each statement true or false. (1 point each)

(2a) (1 point) **T** In rasterization, we can obtain an antialiased image by rasterizing at a higher resolution and downsampling to the desired resolution.

(2b) (1 point) **F** Pasteurized coordinates are needed to allow perspective to be expressed as matrix multiplication.

(2c) (1 point) **T** Any sequence of scales, rotations, and shears can be represented as a single transformation matrix.

(2d) (1 point) **F** If A, B, and C are transformation matrices then $A \cdot B \cdot C = C \cdot B \cdot A$ unless one of the matrices is a rigid-body transformation.

(2e) (1 point) **T** A rotation in a 2D plane can be thought of as a rotation in 3D where the third dimension is perpendicular to the 2D plane.

(2f) (1 point) **F** Perspective projection causes objects to appear smaller so that the height of an object is proportionate to $1/\text{distance}^2$.

(2g) (1 point) **F** When using a Z-buffer for hidden surface removal, the polygons must be sorted front-to-back before the rendering pass starts.

(2h) (1 point) **F** Each iteration of Loop Subdivision triples the number of triangles.

(2i) (1 point) **T** Any cubic polynomial can be expressed using a weighted combination of the cubic Bézier basis functions.

(2j) (1 point) **T** Barycentric interpolation can be used to take values defined at a triangle's vertices and interpolate the values over the triangle.

(2k) (1 point) **F** Bi-linear interpolation is a fast method for finding the intersection of two rays in 3D space.

(2l) (1 point) **T** When evaluating a tensor-product Bézier patch the de Casteljau method and the matrix multiplication method produce the same results, aside from possible floating point error.

(2m) (1 point) **T** Computing the intersection of a ray and a sphere requires solving a quadratic equation.

(2n) (1 point) **T** K-D Trees enable computation of the intersection between a ray and a scene, with computational complexity that is logarithmic, rather than linear, in the number of scene elements.

(2o) (1 point) **F** In Monte Carlo integration, importance sampling guarantees lower noise.

(2p) (1 point) **F** A Monte Carlo estimator that relies on only one random sample will be noisy and biased.

(2q) (1 point) **F** One nit is one candela per square meter per steradian.

(2r) (1 point) **T** Irradiance decreases as the angle increases between the light source and the normal vector to the receiving surface.

(2s) (1 point) **T** A rendering that simulates only direct illumination can be darker than one that simulates global illumination.

(2t) (1 point) **T** Russian Roulette is a method that can compute an unbiased estimate of the infinite recursion in the rendering equation, in a finite amount of time.

3. (Total : 17 points) Transformation Matrices

You have a 3D scene with a camera in it and would like to compute a viewing matrix for the camera using the "look-at" method. The camera is located at $\mathbf{c}$. The camera is upright and pointed in direction $\mathbf{d}$. Your scene is set up with $+\hat{\mathbf{y}}$ being the vertical axis.

Note: A hat $\hat{}$ indicates a normalized vector. Do not assume a vector is normalized unless there is a hat over it or a formula explicitly includes normalization.

Note: A tilde $\tilde{}$ indicates a point or vector in homogeneous form.

(3a) (2 points) Which of these is the camera's view direction:

A. $\mathbf{v} = -\mathbf{d}$

B. $\mathbf{v} = \mathbf{d}/||-\mathbf{d}||$

C. $\mathbf{v} = -\mathbf{d} \times \mathbf{c}$

D. $\mathbf{v} = \mathbf{c} - \mathbf{d}$

(3b) (3 points) Which of these is the camera's right vector:

A. $\mathbf{r} = \mathbf{v} \times \hat{\mathbf{y}}$

B. $\mathbf{r} = \mathbf{v}/||\mathbf{v}|| \times \hat{\mathbf{y}}$

C. $\mathbf{r} = \mathbf{v} \times \hat{\mathbf{y}}/||\mathbf{v} \times \hat{\mathbf{y}}||$

D. $\mathbf{r} = \mathbf{v} \times \hat{\mathbf{y}}/||\mathbf{v}||$

(3c) (3 points) Which of these is the **camera's** (not the world's) up vector:

A. $\mathbf{u} = \mathbf{r} \times \mathbf{v}$

B. $\mathbf{u} = \mathbf{r} \times \mathbf{v}/||\mathbf{c}||$

C. $\mathbf{u} = \hat{\mathbf{y}} \times \mathbf{v}/||\mathbf{v}||$

D. $\mathbf{u} = \hat{\mathbf{y}}/||\hat{\mathbf{y}}||$

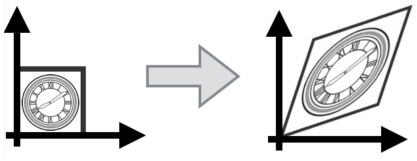
(3d) (3 points) Which of these is a matrix that transforms homogeneous vectors from the camera's right-handed coordinate system to the world's right-handed coordinate system:

A. $\begin{pmatrix} \tilde{r}_x & \tilde{u}_x & \tilde{v}_x & \tilde{c}_x \\ \tilde{r}_y & \tilde{u}_y & \tilde{v}_y & \tilde{c}_y \\ \tilde{r}_z & \tilde{u}_z & \tilde{v}_z & \tilde{c}_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$ **B.** $\begin{pmatrix} \tilde{r}_x & \tilde{u}_x & -\tilde{v}_x & \tilde{c}_x \\ \tilde{r}_y & \tilde{u}_y & -\tilde{v}_y & \tilde{c}_y \\ \tilde{r}_z & \tilde{u}_z & -\tilde{v}_z & \tilde{c}_z \\ \tilde{r}_w & \tilde{u}_w & -\tilde{v}_w & \tilde{c}_w \end{pmatrix}$ C. $\begin{pmatrix} \tilde{r}_x & \tilde{u}_x & -\tilde{v}_x & \tilde{c}_x \\ \tilde{r}_y & \tilde{u}_y & -\tilde{v}_y & \tilde{c}_y \\ \tilde{r}_z & \tilde{u}_z & -\tilde{v}_z & \tilde{c}_z \\ 1 & 1 & 1 & 0 \end{pmatrix}$

(3e) (3 points) When working with homogeneous vectors, which of these transformation matrices produces a same result (location in 3D space) as the 4×4 identity matrix?

A. $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ C. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ **D.** $\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

(3f) (3 points) Which 2D transformation sequence will produce the result depicted here:



A. Rotate(45)· Scale(1,2) · Rotate(45)

B. Scale(2,2)

C. Rotate(-45)· Scale(1,2) · Rotate(-45)

D. Rotate(135)· Scale(2,1) · Rotate(-135)

4. (Total : 12 points) Blinn-Phong Reflection Model

You have a 3D scene as follows:

- The camera/eye at the location $\mathbf{e}$ is pointed in the direction $\mathbf{v}$ with Y-up.
- There is one sphere with center $\mathbf{c}$ and radius r.
- The sphere has diffuse constant k_d and specular constant k_s.
- The surface's specular-power exponent is k.
- There is no ambient light in this scene.
- There is a point light located at $\mathbf{p}$ with intensity I_p.
- There is a directional light pointing in the direction $\mathbf{d}$ with intensity I_d.
- The point light falls off with $1/\text{distance}^2$.
- The directional light does not falloff with distance.
- Both lights illuminate the sphere.
- For simplicity, you may assume a gray world so that colors are just a single gray-scale value, not an RGB triple.

Notes: Pay attention to what each variable means according to the definitions here. A diagram may be helpful. Not everything in the description is relevant to the final answers. A hat $\hat{}$ indicates a normalized vector. Do not assume a vector is normalized unless there is a hat over it or a formula explicitly includes normalization.

(4a) (3 points) If a point $\mathbf{s}$ is on the surface of the sphere, write an equation for the unit surface normal $\hat{\mathbf{n}}$ at $\mathbf{s}$:

Solution: $\hat{\mathbf{n}} = (\mathbf{s} - \mathbf{c})/||(\mathbf{s} - \mathbf{c})||$ or $\hat{\mathbf{n}} = \text{normalize}(\mathbf{s} - \mathbf{c})$

(4b) (9 points) Assuming that the point $\mathbf{s}$ is visible to the camera, write out a formula for computing B_s the reflected brightness of surface point $\mathbf{s}$ as seen by the camera according to the Blinn-Phong reflection model. If you'd like, you may define auxiliary variables, but be sure keep them unique. Note: The function $\text{zmax}(x) = \max(0, x)$ may be useful. You can assume surfaces are one-sided so that normal flipping is not required.

UPLOAD PHOTO **On a sheet of your own paper, write your answer. Take a photo of your page and upload to Gradescope Exam 1.**

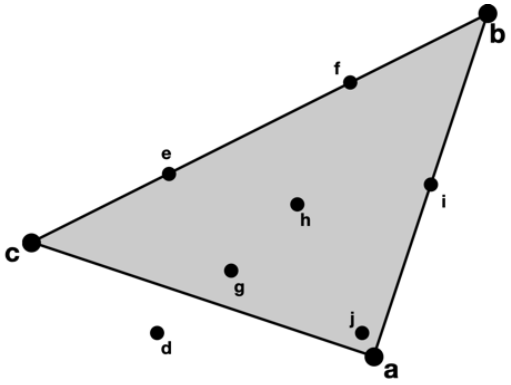
Solution:

$$
\begin{aligned}
\hat{\mathbf{w}} &= \text{normalize}(\mathbf{e} - \mathbf{s}) \\
\hat{\mathbf{l}}_p &= \text{normalize}(\mathbf{p} - \mathbf{s}) \\
\hat{\mathbf{l}}_d &= \text{normalize}(-\mathbf{d}) \\
\hat{\mathbf{h}}_p &= \text{normalize}(\hat{\mathbf{l}}_p + \hat{\mathbf{w}}) \\
\hat{\mathbf{h}}_d &= \text{normalize}(\hat{\mathbf{l}}_d + \hat{\mathbf{w}}) \\
B_s &= \left(k_d(\hat{\mathbf{l}}_p \cdot \hat{\mathbf{n}}) + k_s(\hat{\mathbf{h}}_p \cdot \hat{\mathbf{n}})^k\right) I_p/||\mathbf{p} - \mathbf{s}||^2 + \\
&\quad \left(k_d(\hat{\mathbf{l}}_d \cdot \hat{\mathbf{n}}) + k_s(\hat{\mathbf{h}}_d \cdot \hat{\mathbf{n}})^k\right) I_d
\end{aligned}
$$

5. (Total : 18 points) Geometry

(5a) (4 points) Barycentric Coordinates. The list below specifies barycentric coordinates (α, β, and γ) of eight points defined with respect to the shown triangle. Indicate which point corresponds to which coordinates. The triangle's vertices are ordered a, b, c which correspond to α, β, and γ respectively.

	α	β	γ
1 :	0.3333	0.3333	0.333
2 :	0.9	0.05	0.05
3 :	0	0.3	0.7
4 :	0	0.7	0.3
5 :	0.5	0.5	0
6 :	0.45	0.1	0.45
7 :	0	0	1
8 :	0.55	−0.1	0.55



5a.i. #1 = **h**

5a.ii. #2 = **j**

5a.iii. #3 = **e**

5a.iv. #4 = **f**

5a.v. #5 = **i**

5a.vi. #6 = **g**

5a.vii. #7 = **c**

5a.viii. #8 = **d**

(5b) Bounds

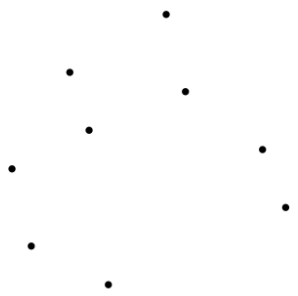
5b.i. (2 points) Draw the axis-aligned bounding box around the indicated points on the left.

UPLOAD PHOTO **On a sheet of your own paper, write your answer. Take a photo of your page and upload to Gradescope Exam 1.**

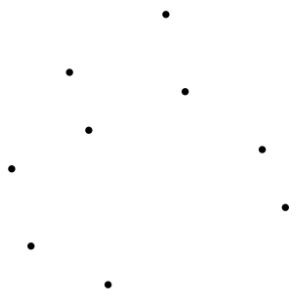
5b.ii. (2 points) Draw the convex hull around the indicated points on the right.

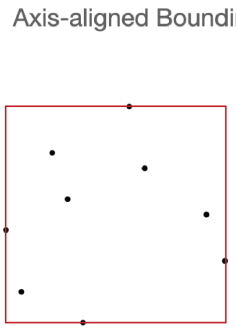
UPLOAD PHOTO **On a sheet of your own paper, copy the diagram above and draw your answer. Take a photo and upload to Gradescope Exam 1.**

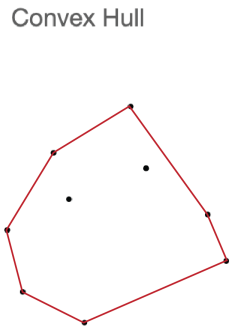
Axis-aligned Bounding Box



Convex Hull



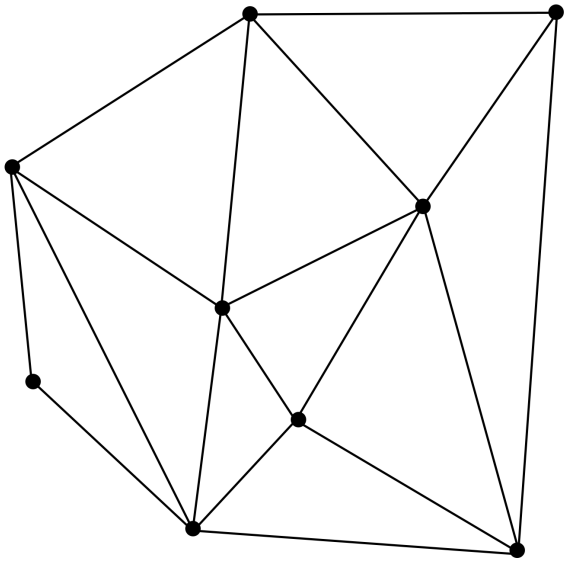




Solution:

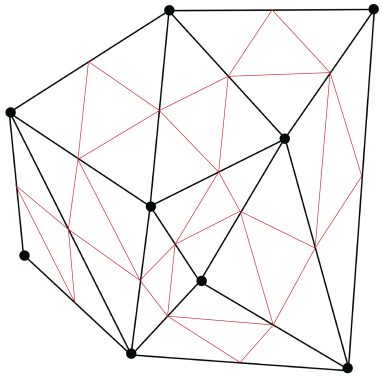
(5c) Subdivision

5c.i. (5 points) Consider the following mesh. Draw in one level of **Loop subdivision**.
Note: Please make sure your submission is clear and readable. A ruler might be useful.

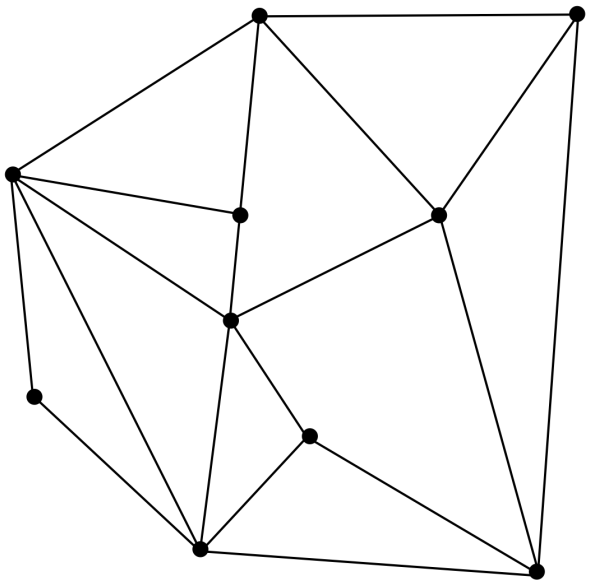


UPLOAD PHOTO **On a sheet of your own paper, copy the diagram above and draw your answer. Take a photo and upload to GRADESCOPE EXAM 1.**

Solution:

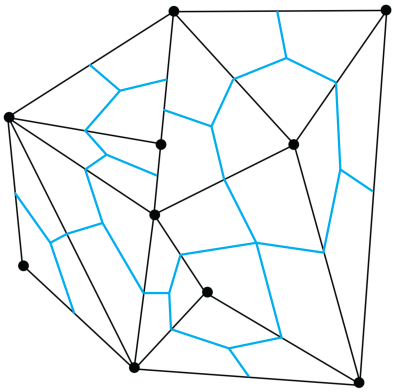


5c.ii. (5 points) Consider the following mesh. Draw in one level of **Catmull-Clark subdivision**.
Note: Please make sure your submission is clear and readable. A ruler might be useful.



UPLOAD PHOTO **On a sheet of your own paper, copy the diagram above and draw your answer. Take a photo and upload to GRADESCOPE EXAM 1.**

Solution:



6. (Total : 17 points) Acceleration Structures

(6a) As part of a 2D object collision simulation, you need to write code to determine whether the boundaries of two shapes intersect. The boundary for each shape is given as a set of line segments.

6a.i. (3 points) First, you try the simple, exhaustive approach. Write a function that checks for any intersection by iterating over each line segment in one shape's boundary and testing for intersection against each line segment in the other shape.

```
struct Point2D {
    float x, y;
};

// A line segment defined by two endpoints
struct LineSegment2D {
    Point2D x0;
    Point2D x1;
}

// A helper function that you can use. This function tests whether
// line segments a and b intersect.  You do NOT need to implement this
// function.
bool lineSegmentsIntersect(const LineSegment2D &a,
                           const LineSegment2D &b) {
    /* Implementation Hidden */
}

bool doShapeBoundariesIntersect(const vector<LineSegment2D> &shape1,
                                const vector<LineSegment2D> &shape2) {

    /* Write your code answer in the box below. */
```

}

UPLOAD PHOTO **Take a photo or screenshot of your code for this box and upload it to Gradescope Exam 1.**

Solution:

```
bool doShapeBoundariesIntersect(const vector<LineSegment2D> &shape1,
                                const vector<LineSegment2D> &shape2) {

    for (int i = 0; i < shape1.size(); ++i) {
        for (int j = 0; j < shape2.size(); ++j) {
            if (lineSegmentsIntersect(shape1[i], shape2[j]) return true;
        }
    }
    return false;
}
```

6a.ii. (10 points) Next, you decide to try an implementation using a Bounding Volume Hierarchy. For this part, assume that each shape is provided to you within a BVH data structure, as follows:

```
// An axis-aligned box
struct Box {
    Point2D lowerLeft, upperRight;
};

struct BVHNode {
    Box boundingBox;
    boolean isLeafNode;

    // Only contain valid child pointers if not a leaf node
    BVHNode* left;
    BVHNode* right;

    // Only contains line segments if a leaf node
    vector<LineSegment2D> lineSegments;
};
```

You are also given the following helper functions that you can use if you wish.

```
// A helper function that you can use. This function tests whether
// a LineSegment2D and a Box intersect.  You do NOT need to implement this
// function.
bool lineSegmentAndBoxIntersect(const LineSegment2D &a, const Box &b) {
    /* Implementation Hidden */
}


// A helper function that you can use. This function tests whether
// two Boxes intersect.  You do NOT need to implement this function.
bool boxesIntersect(const Box &a, const Box &b) {
    /* Implementation Hidden */
}
```

Now, write code to compute whether the boundaries of the two shapes intersect, where each shape is provided as a BVH. You may call this function recursively and use any helper functions you want, including `doShapeBoundariesIntersect`. (Continued on next page.)

```
bool doBVHIntersect(BVHNode *shape1, BVHNode *shape2) {

    /* Write your code answers in the boxes below. */
```

}

UPLOAD PHOTO **Take a photo or screenshot of your code for this box and upload it to GRADESCOPE EXAM 1.**

Solution:

```
bool doBVHIntersect(BVHNode *shape1, BVHNode *shape2) {

    if (!boxesIntersect(shape1->boundingBox, shape2->boundingBox))
        return false;

    if (!shape1->isLeafNode) {
```

```
            return doBVHIntersect(shape1->left, shape2)
                || doBVHIntersect(shape1->right, shape2);
        } else if (!shape2->isLeafNode) {
            return doBVHIntersect(shape1, shape2->left)
                || doBVHIntersect(shape1, shape2->right);
        }

        // both must be leaf nodes
        return doShapeBoundariesIntersect(shape1->lineSegments, shape2->lineSegments);



    }
```

6a.iii. (2 points) Under what situations will the exhaustive implementation be slower than the BVH-based implementation? Describe a simple example.

Solution:
The exhaustive implementation will be slower for two detailed polygons which are far enough apart that their bounding boxes do not overlap. The BVH method will immediately exit, but the naive method will need to iterate through all edges.

6a.iv. (2 points) Under what situations will the exhaustive implementation be faster than the BVH-based implementation? Describe a simple example.

Solution:
The exhaustive implementation will be faster for two simple polygons which are already superimposed. Many of the BVH boxes already overlap, which will eliminate the early-exits.

7. (Total : 14 points) Photometry

(7a) Solid Angle

You are on a hike, look up and see a white shape in the sky. It's a UFO! This UFO has the shape of a perfectly flat disk with its flat side parallel to the ground. The disk has a diameter of 10 meters, and it is cruising at an altitude of 3 kilometers.

7a.i. (2 points) When the UFO flies directly over you, what is its solid angle from your perspective? (Approximate answers are allowed, with full credit for any answer within 1% of the solution.)

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Solution: It is small relative to distance, so we can assume that the entire disk is at the same distance. $A/r^2 = \pi(5\text{m})^2/(3000\text{m})^2 = \pi/360,000$ steradians.

7a.ii. (4 points) Later, the UFO has cruised so that it is directly over a point on the ground 4 kilometers away from you. It is still at the same altitude. What is the solid angle of the UFO from your perspective at this time? (Approximate answers are allowed, with full credit for any answer within 1% of the solution.)

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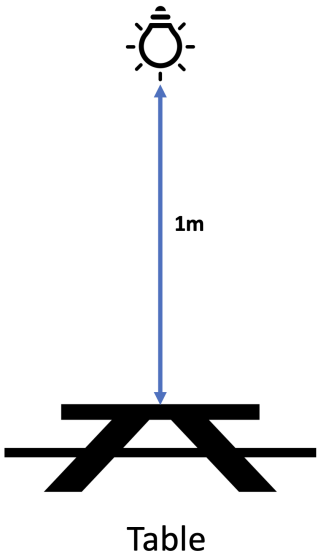
Solution: Now the distance has increased to 5 kilometers. Also, the disk is now pointed at an angle to the viewing direction, with $\cos\theta = \frac{3}{5}$. $A\cos\theta/r^2 = \pi(5\text{m})^2\frac{3}{5}/(5000\text{m})^2 = 3\pi/5,000,000$ steradians.

(7b) Photometry

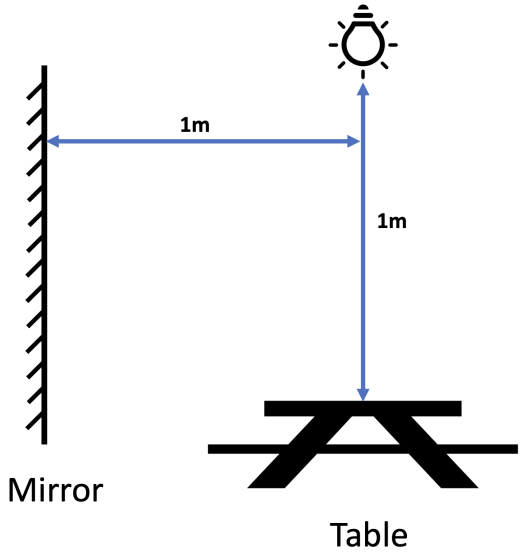
7b.i. (2 points) In your apartment, it feels too dark to read your book at the dining table. There is a point light source, L, positioned over the table that outputs 1000 lumens uniformly into the hemisphere below it. First, calculate the **illuminance** (including correct units) falling onto a book that is positioned 1 meter directly below the light.



UPLOAD PHOTO **On a sheet of your own paper, write your answer. Take a photo of your page and upload to GRADESCOPE EXAM 1.**

Solution:
The factors are 1000 lumens divided into 2π steradians of solid angle below the light, $1/r^2$ falloff where r is the distance from light source to point on the paper, and $\cos\theta$ due to Lambert's Law, where θ is the angle between the light and the paper's surface normal.
The illuminance is $\frac{1000}{2\pi} = \frac{500}{\pi} \approx 160$ lux. Units of lumens / meter^2 are also acceptable.

7b.ii. (3 points) You've read that the illuminance for comfortable reading should be at least 500 lux, so you try to brighten things up. First, you hang a mirror on the wall by the dining table, such that the book receives additional light reflected in the mirror. The wall where you hang the mirror is exactly 1 meter away from the bulb / book. You can assume the mirror is a perfect reflector. Calculate what the illuminance is now. Is it bright enough yet?



UPLOAD PHOTO **On a sheet of your own paper, write your answer. Take a photo of your page and upload to GRADESCOPE EXAM 1.**

Solution:
The additional light is due to the reflection of the light source. From the perspective of the book, the reflected light source appears to be at a distance of 2 meters horizontally (twice the distance to the mirror, due to reflection) and 1 meter vertically (a total distance of $\sqrt{5}$ meters). Also, the reflected light rays arrive at the book at an angle with $\cos\theta = \frac{1}{\sqrt{5}}$. The additional illuminance is $\frac{1000}{2\pi} * \frac{1}{\sqrt{5}}/5 \approx 14$ lumens.
Thus, the total illuminance is approximately 173 lux. It's still not bright enough!

7b.iii. (3 points) Next, you remove the mirror, and instead attach a lamp shade over the light source. The inside of the shade is reflective, and it directs all the light from the bulb downward, such that it uniformly fills a circle on the table that is 1 meter in diameter. Calculate the illuminance now. Is it bright enough now?
UPLOAD PHOTO **On a sheet of your own paper, write your answer. Take a photo of your page and upload to GRADESCOPE EXAM 1.**

Solution:
All the power now falls within the circle, which has an area of $\pi(0.5)^2$. The illuminance is 1000 lumens divided by this area, or approximately 1273 lux. It's bright enough now!

8. (Total : 12 points) Monte Carlo Integration

For each row in the table below, you are given a definite integral to evaluate with Monte Carlo integration, the probability density function (PDF) to draw random samples from and the number of samples to take. Your job is to write down an equation for the corresponding unbiased Monte Carlo estimator in the last column. The first two rows are completed as examples. In this question, H^2 denotes the hemisphere. Simplify your answer as much as possible, and show your work for partial credit.

(8a) (12 points) Monte Carlo Estimators

	Definite Integral	**Number of Samples**	**Random Sampling**	**Estimator**
	$\int_a^b f(x)\,dx$	1	Random X drawn uniformly at random from $[a, b]$	$(b-a)f(X)$
	$\int_a^b f(x)\,dx$	N	Random X_i drawn uniformly at random from $[a, b]$	$\frac{b-a}{N}\sum_{i=1}^{N} f(X_i)$
(a)	$\int_0^\pi f(x)\,dx$	1	Random X drawn from $[0, \pi]$ with probability proportional to $\sin x$.	
(b)	$\int_{H^2} L(\omega)\cos\theta\,d\omega$	N	Random ω_i drawn from H^2 with probability proportional to $\cos(\theta_i)$*	
(c)	$\int_a^b \int_c^d h(x, y)\,dx\,dy$	N	Random (X_i, Y_i) drawn within the domain of integration with probability proportional to function $g(X_i, Y_i)$.	

* Following convention in lecture, θ is the angle between ω and the north pole of the sphere. Tip: it was shown in lecture that $\int_{H^2} \cos(\omega) d\omega = \pi$.

UPLOAD PHOTO On a sheet of your own paper, write your answer for each row. Take a photo of your page and upload to GRADESCOPE EXAM 1.

Solution:

(a) $\frac{2f(x)}{\sin x}$

(b) $\frac{\pi}{N} \sum_{i=1}^{N} L(\omega_i)$

(c) $\frac{\int_a^b \int_c^d g(x,y)\, dx\, dy}{N} \sum_{i=1}^{N} \frac{h(X_i,Y_i)}{g(X_i,Y_i)}$