

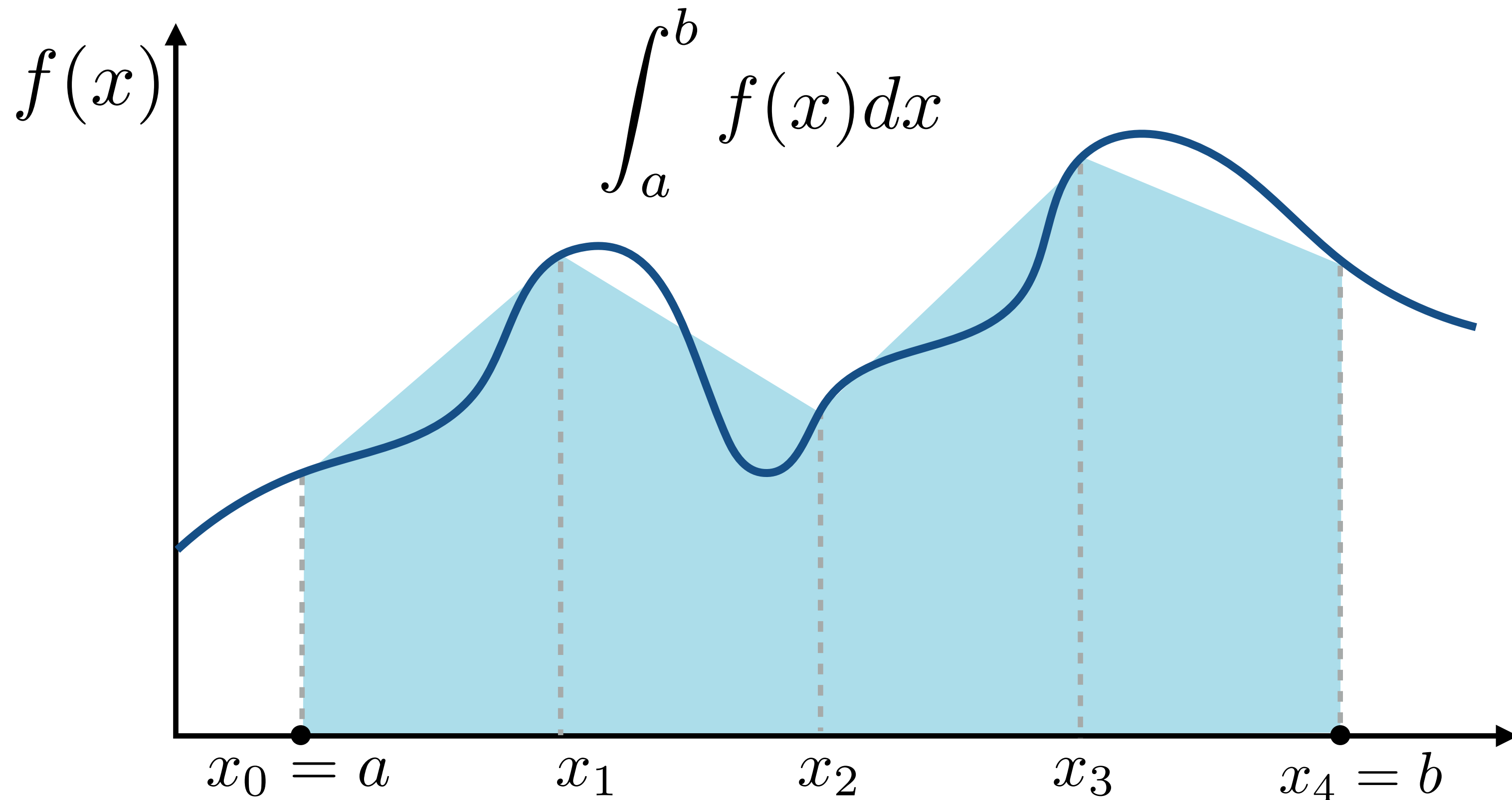
**Lecture 12:**

# **Monte Carlo Integration**

---

**Computer Graphics and Imaging**  
**UC Berkeley CS184/284A**

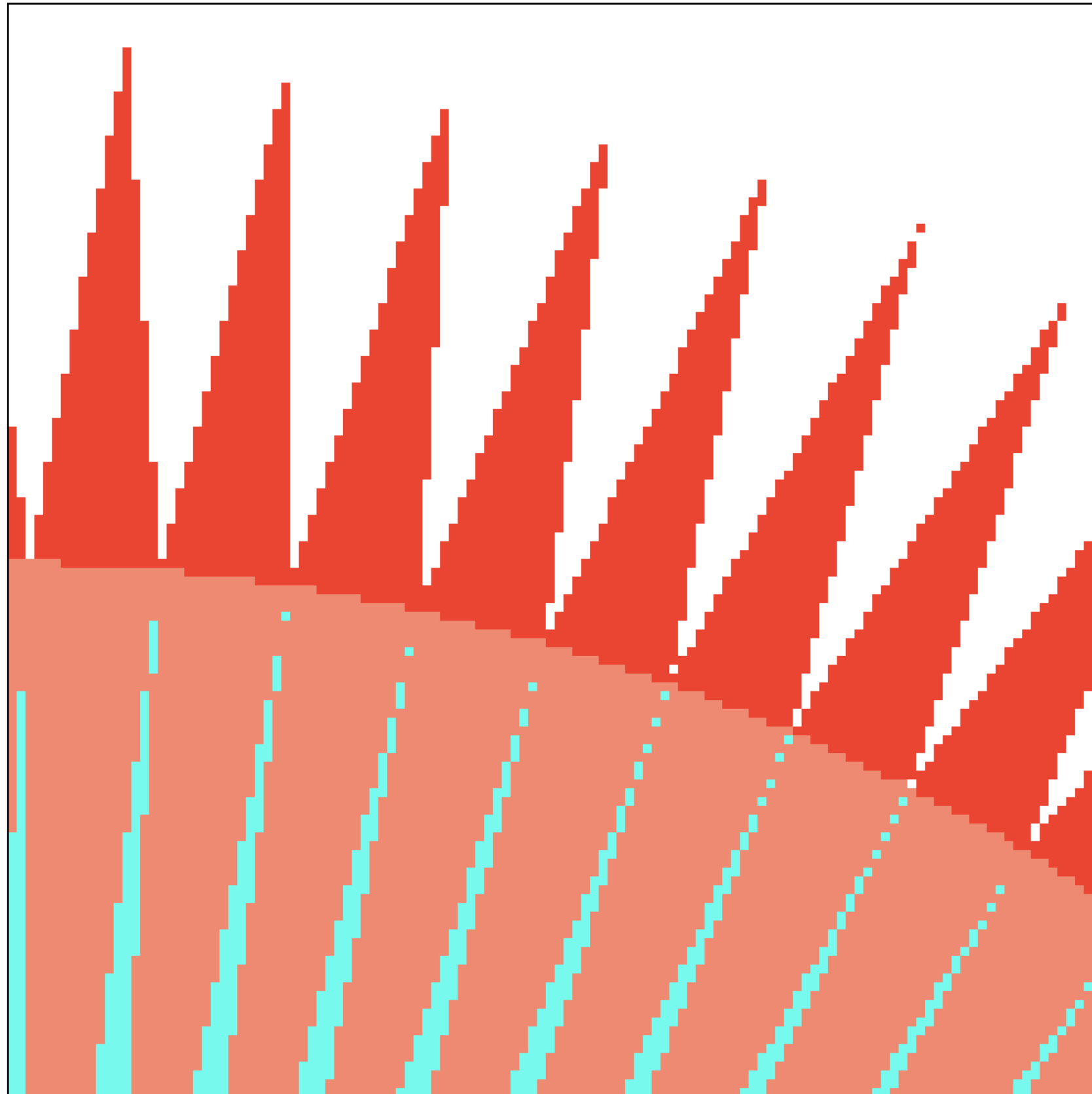
# Reminder: Quadrature-Based Numerical Integration



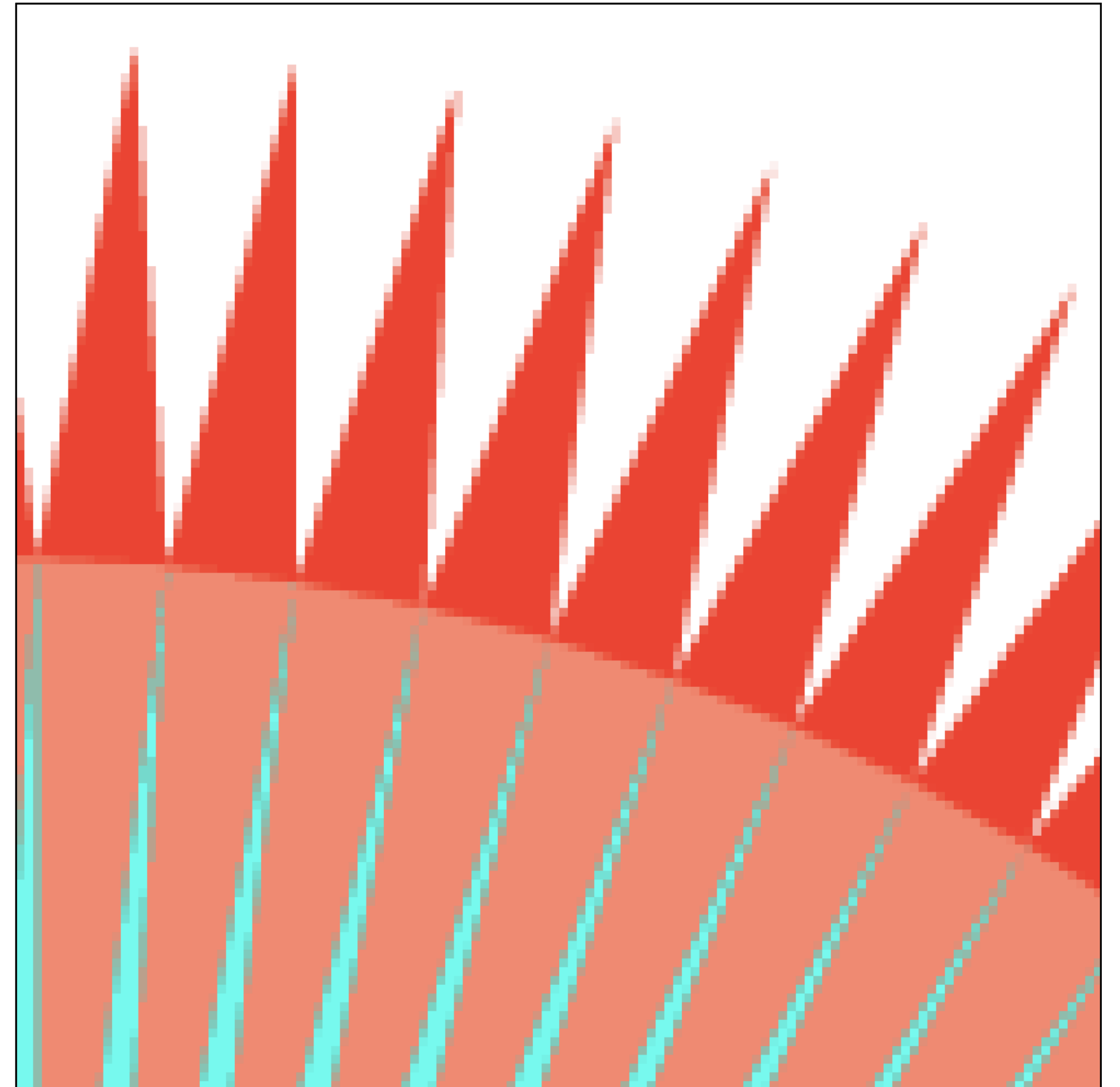
E.g. trapezoidal rule - estimate integral assuming function is piecewise linear

# **Multi-Dimensional Integrals (Rendering Examples)**

# 2D Integral: Recall Antialiasing By Area Sampling



Point sampling



Area sampling

Integrate over 2D area of pixel



# 2D Integral: Irradiance from the Environment

Computing flux per unit area on surface, due to incoming light from all directions.

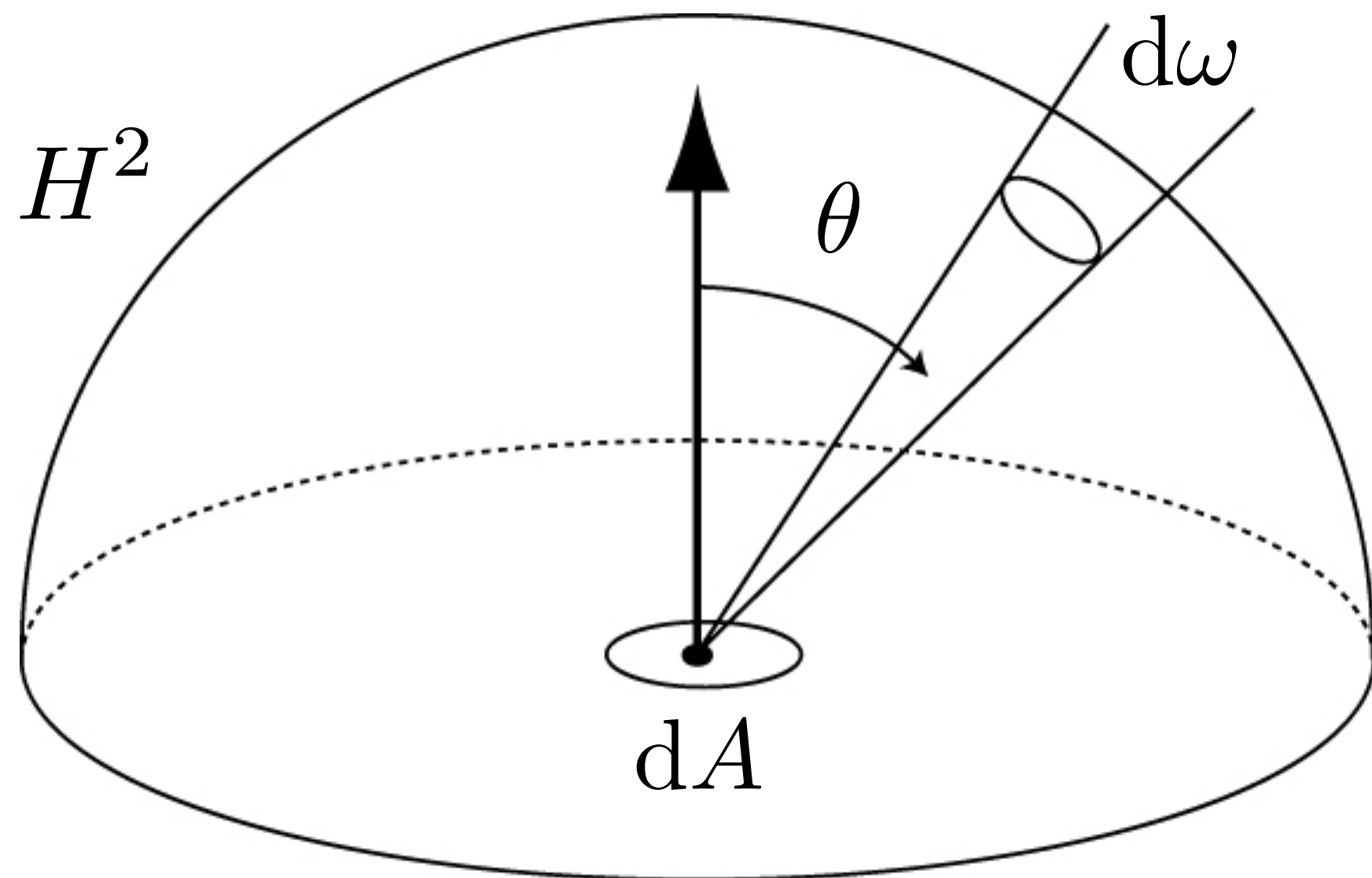
$$E(p) = \int_{H^2} L_i(p, \omega) \cos \theta \, d\omega$$

← Contribution to irradiance from light arriving from direction  $\omega$



Light meter

Hemisphere:  $H^2$





# 3D Integral: Motion Blur



Integrate over  
area of pixel  
and over  
exposure time.

Cook et al. "1984"



# 5D Integral: Real Camera Pixel Exposure

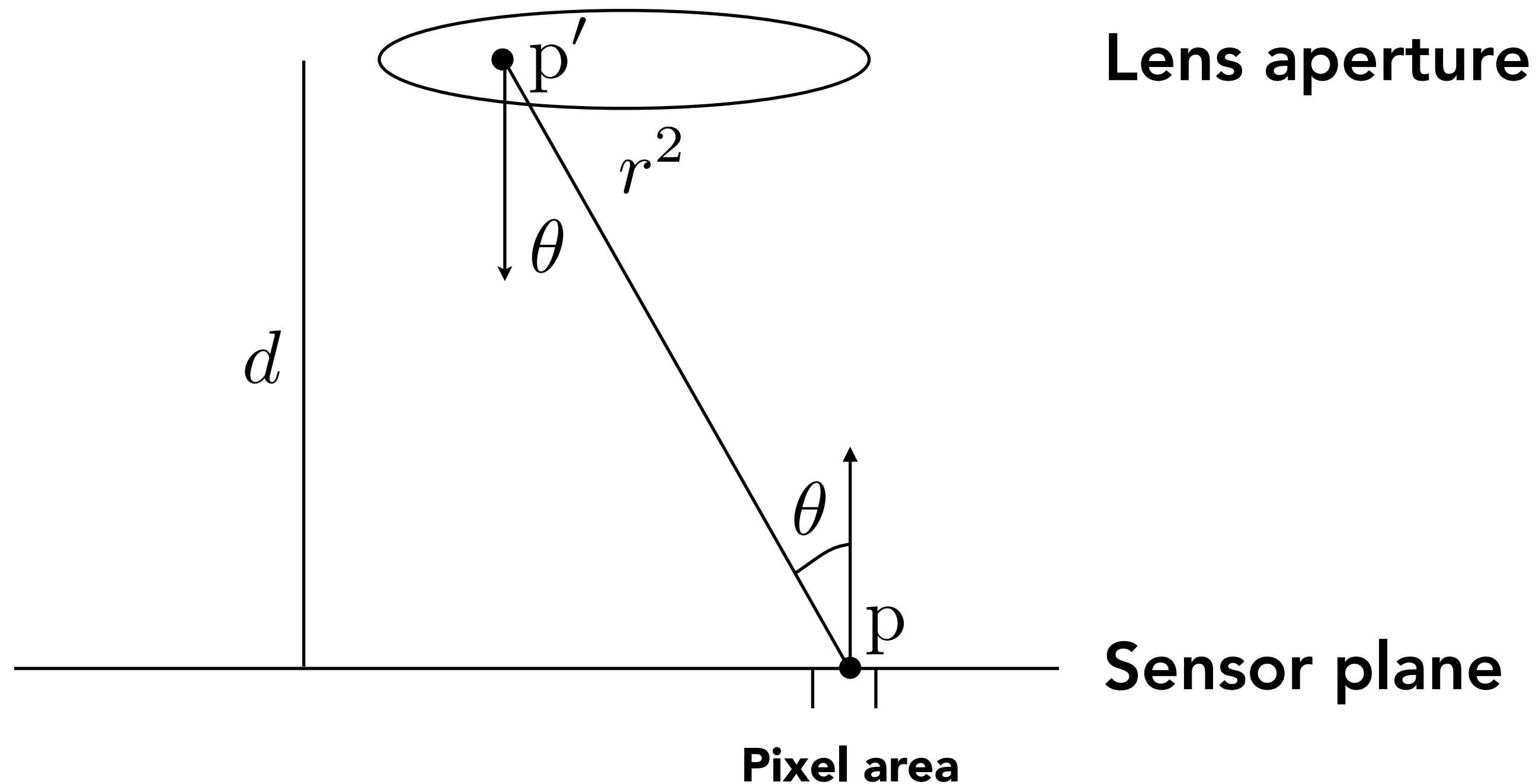


Credit: lycheng99, <http://flic.kr/p/x4DEZh>

Integrate over 2D lens pupil, 2D pixel, and over exposure time



# 5D Integral: Real Camera Pixel Exposure



$$Q_{\text{pixel}} = \frac{1}{d^2} \int_{t_0}^{t^1} \int_{A_{\text{lens}}} \int_{A_{\text{pixel}}} L(p' \rightarrow p, t) \cos^4 \theta dp dp' dt$$

# **The Curse of Dimensionality**

# High-Dimensional Integration

Complete set of samples:

$$N = \underbrace{n \times n \times \cdots \times n}_d = n^d$$

- "Curse of dimensionality"

Numerical integration error:

$$\text{Error} \sim \frac{1}{n} = \frac{1}{N^{1/d}}$$

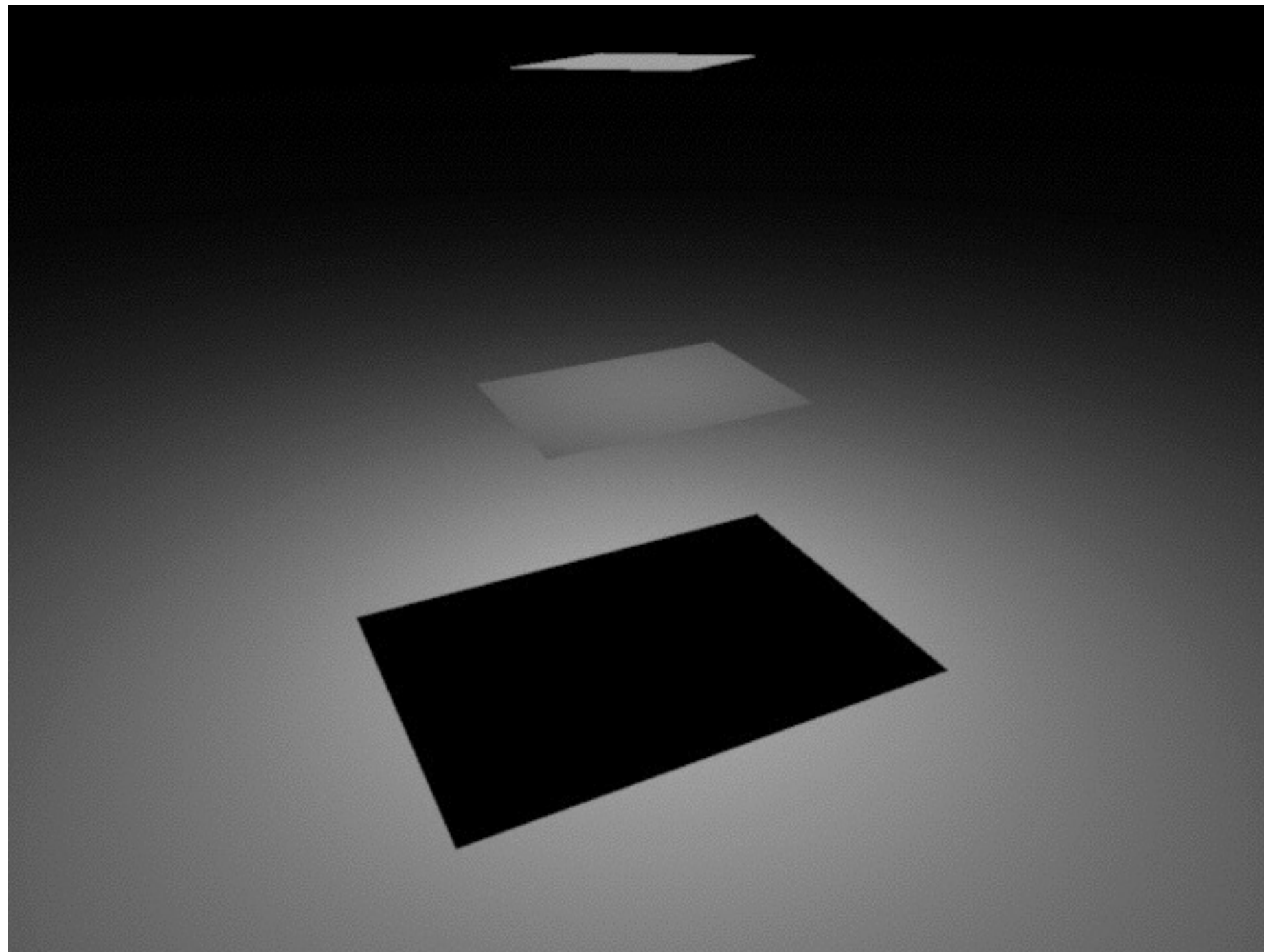
Random sampling error:

$$\text{Error} = \text{Variance}^{1/2} \sim \frac{1}{\sqrt{N}}$$

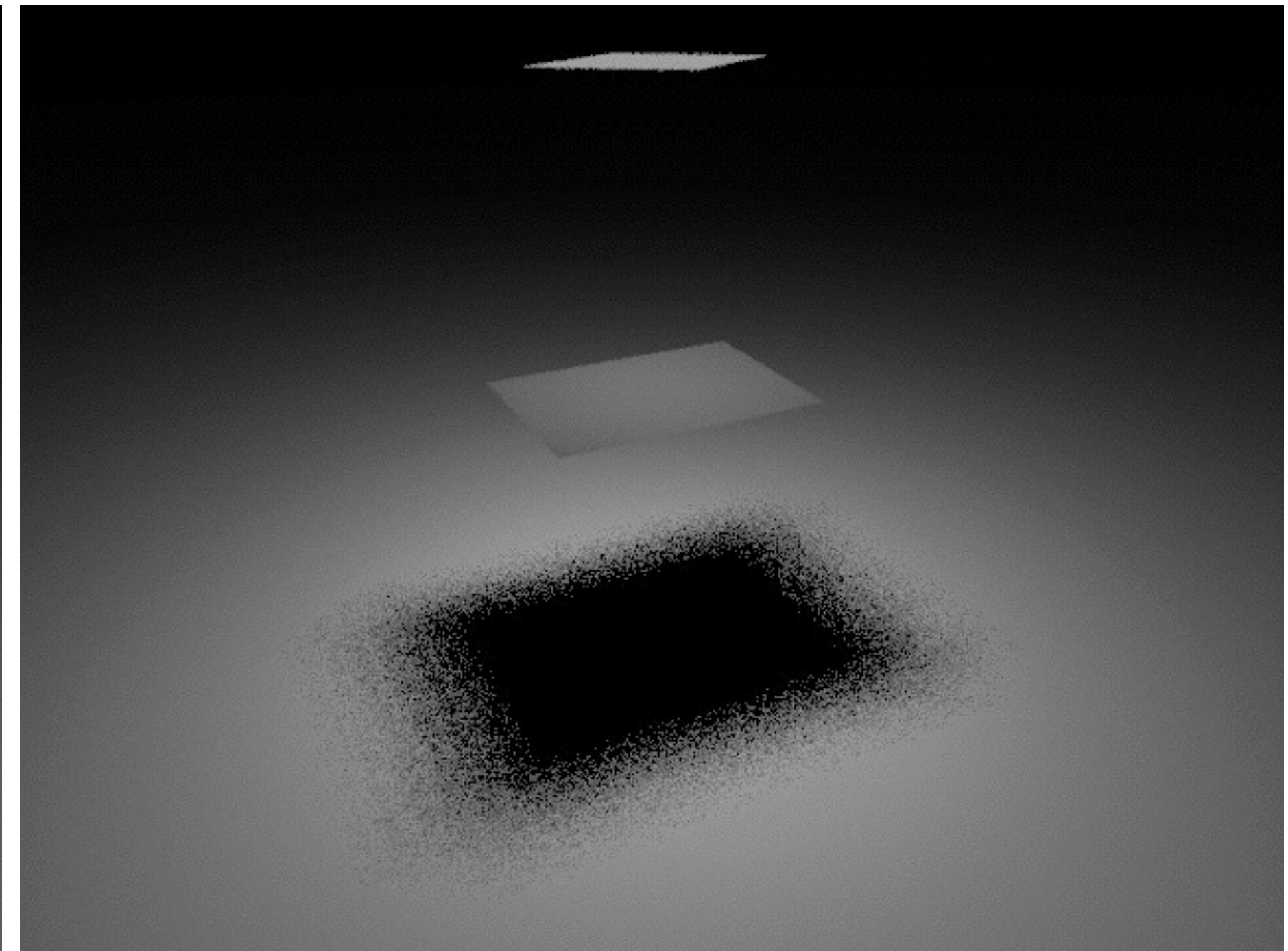
In high dimensions, Monte Carlo integration requires fewer samples than quadrature-based numerical integration

Global illumination = infinite-dimensional integrals

# Example: Discrete vs Monte Carlo - Shadows

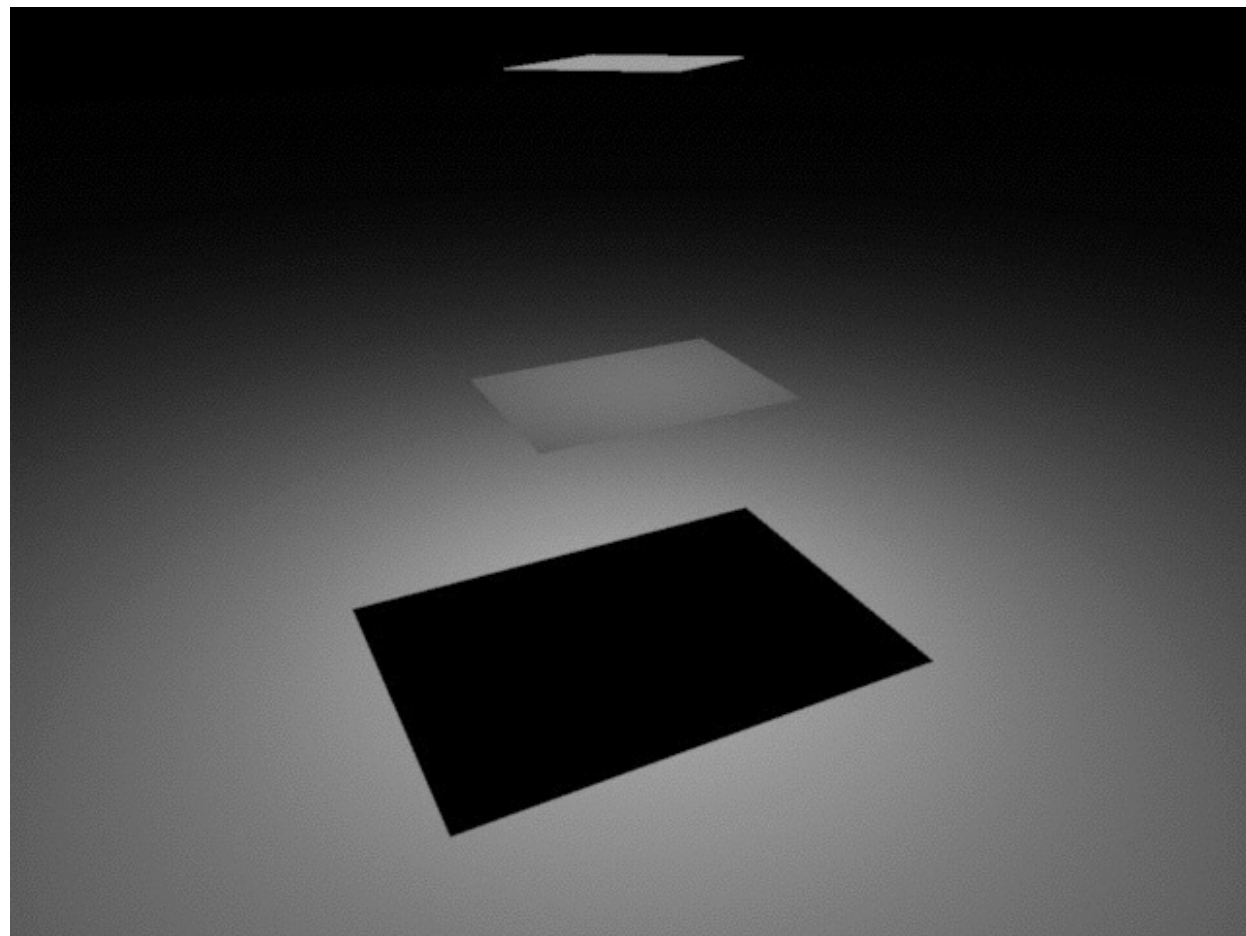


1 sample per pixel  
Sample center of light

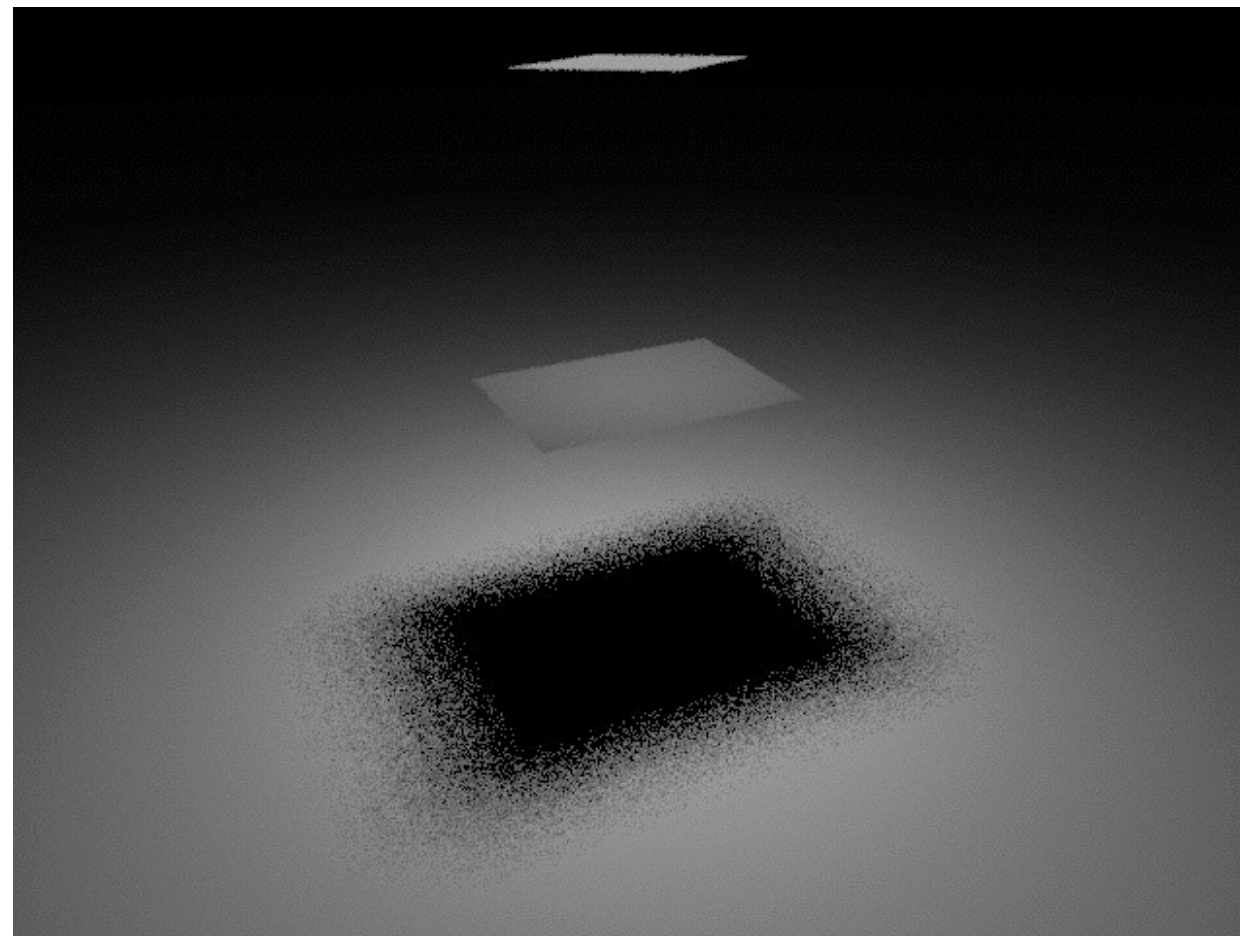


1 sample per pixel  
Sample random point on light

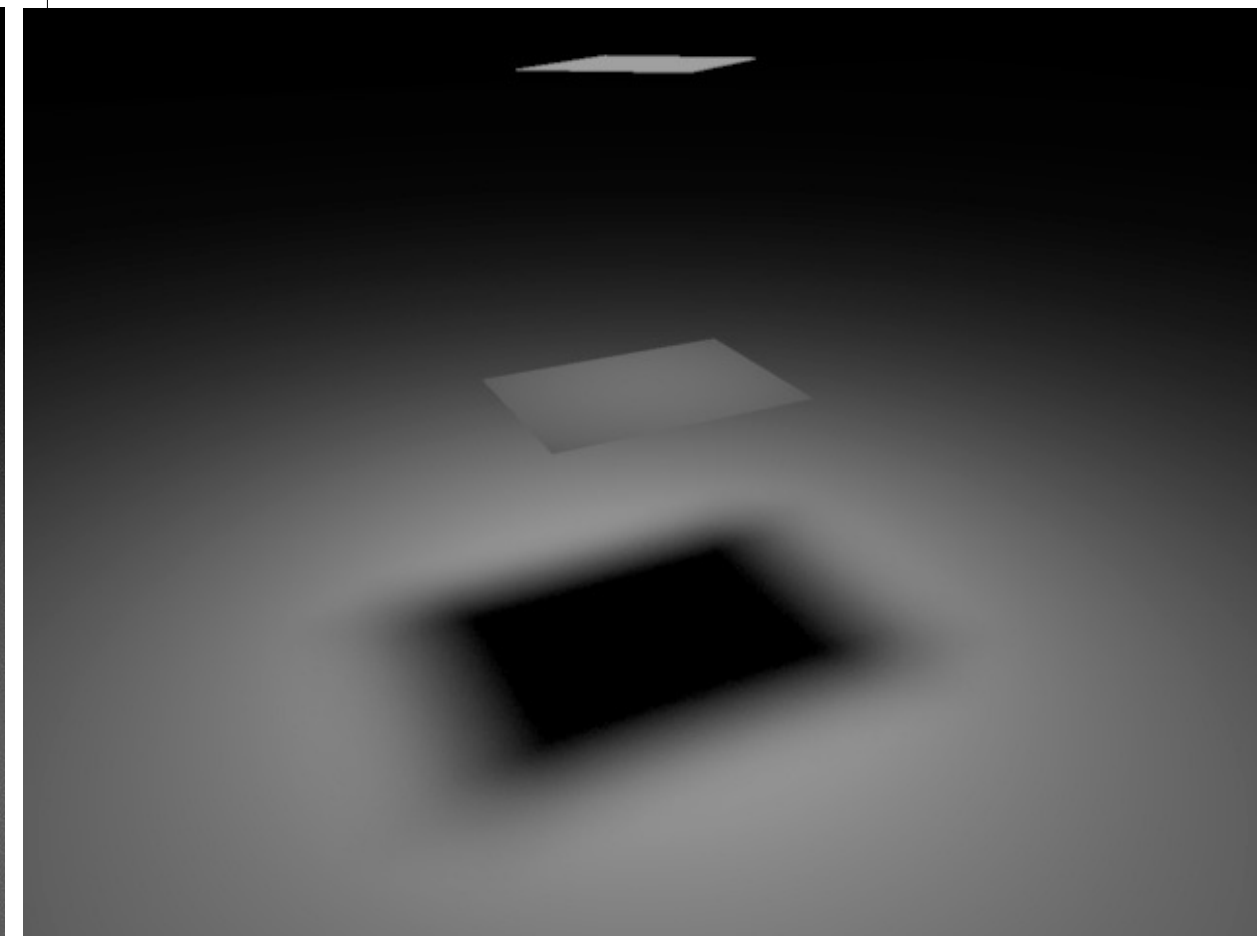
# Example: Discrete vs Monte Carlo - Shadows



Sample center  
of light



Sample random  
point on light



True answer



# Overview: Monte Carlo Integration

Idea: estimate integral based on random sampling of function

Advantages:

- General and relatively simple method
- Requires only function evaluation at any point
- Works for very general functions, including discontinuities
- Efficient for high-dimensional integrals — avoids “curse of dimensionality”

Disadvantages:

- Noise. Integral estimate is random, only correct “on average”
- Can be slow to converge — need a lot of samples

# Probability Review

# Random Variables

$X$  random variable. Represents a distribution of potential values

$X \sim p(x)$  probability density function (PDF). Describes relative probability of a random process choosing value  $x$

**Example: uniform PDF: all values over a domain are equally likely**

**e.g. A six-sided die**

$X$  takes on values 1, 2, 3, 4, 5, 6

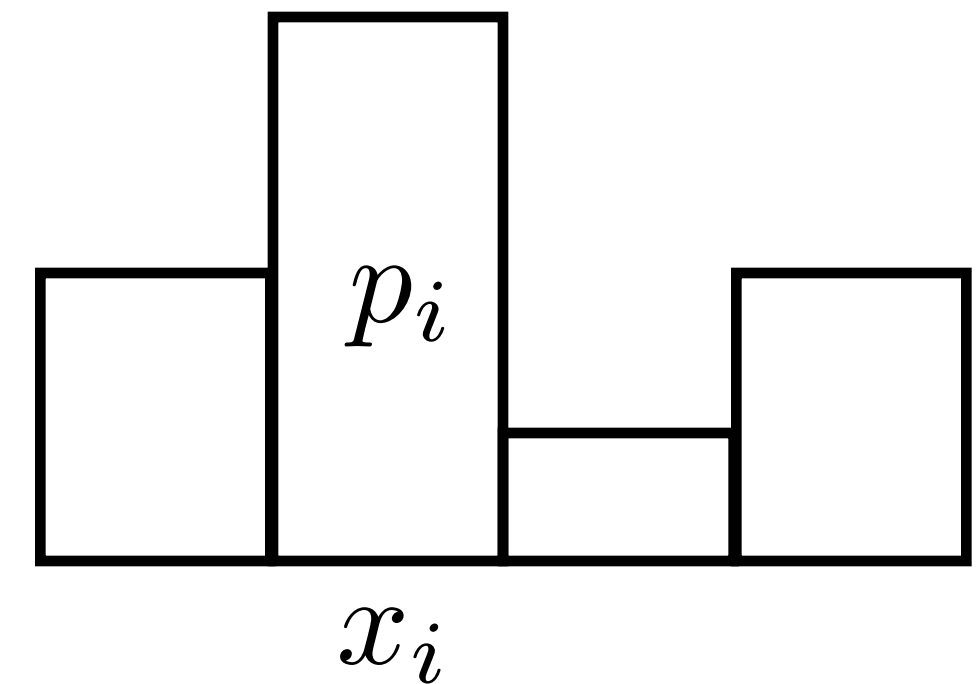
$$p(1) = p(2) = p(3) = p(4) = p(5) = p(6)$$



# Probability Distribution Function (PDF)

$n$  discrete values  $x_i$

With probability  $p_i$



Requirements of a probability distribution:

$$p_i \geq 0$$

$$\sum_{i=1}^n p_i = 1$$

Six-sided die example:  $p_i = \frac{1}{6}$



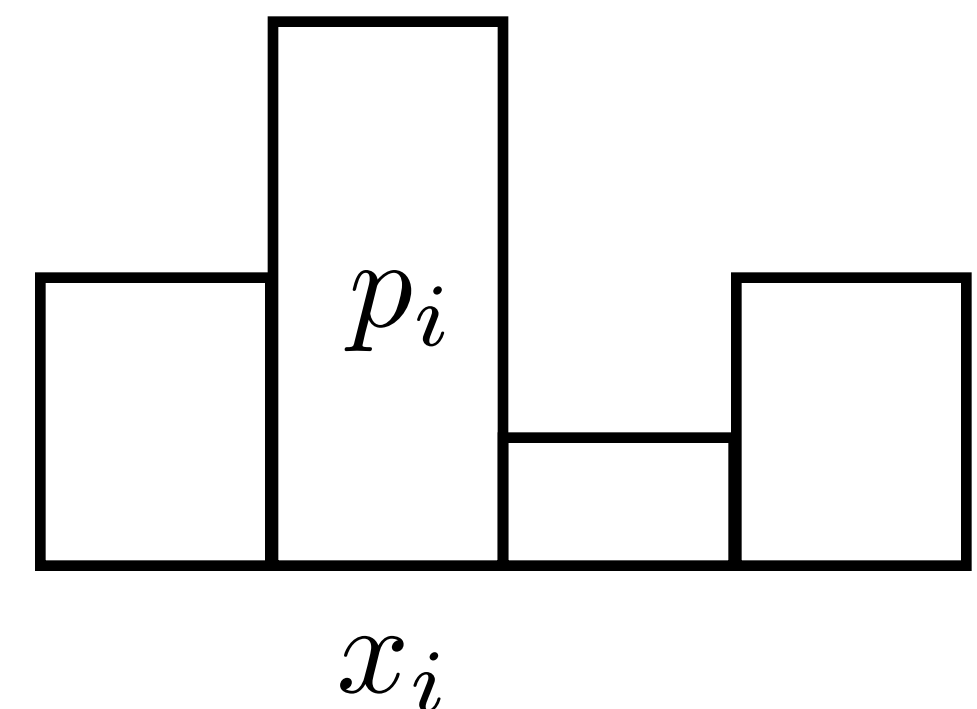
Think:  $p_i$  is the probability that a random measurement of  $X$  will yield the value  $x_i$

$X$  takes on the value  $x_i$  with probability  $p_i$

# Expected Value of a Random Variable

The average value that one obtains if repeatedly drawing samples from the random distribution.

$X$  drawn from distribution with  
 $n$  discrete values  $x_i$   
with probabilities  $p_i$



Expected value of  $X$ :  $E[X] = \sum_{i=1}^n x_i p_i$

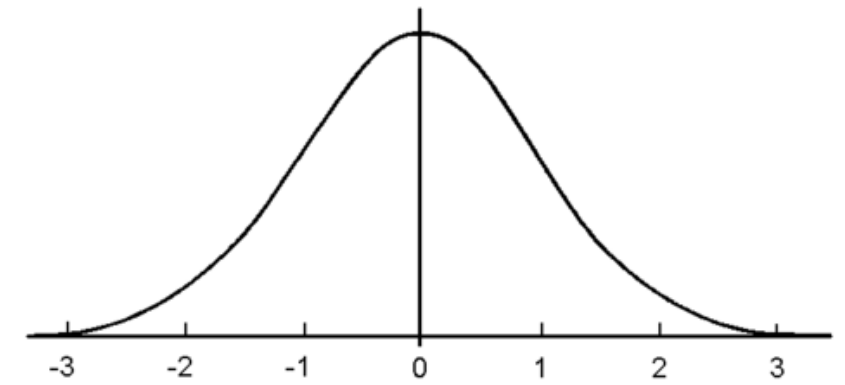
Die example:  $E[X] = \sum_{i=1}^6 \frac{i}{6}$



$$= (1 + 2 + 3 + 4 + 5 + 6) / 6 = 3.5$$

# Continuous Probability Distribution Function

$$X \sim p(x)$$



A random variable  $X$  that can take any of a continuous set of values, where the relative probability of a particular value is given by a continuous probability density function  $p(x)$ .

Conditions on  $p(x)$ :  $p(x) \geq 0$  and  $\int p(x) dx = 1$

Expected value of  $X$ :  $E[X] = \int x p(x) dx$

# Function of a Random Variable

A function  $Y$  of a random variable  $X$  is also a random variable:

$$X \sim p(x)$$

$$Y = f(X)$$

Expected value of a function of a random variable:

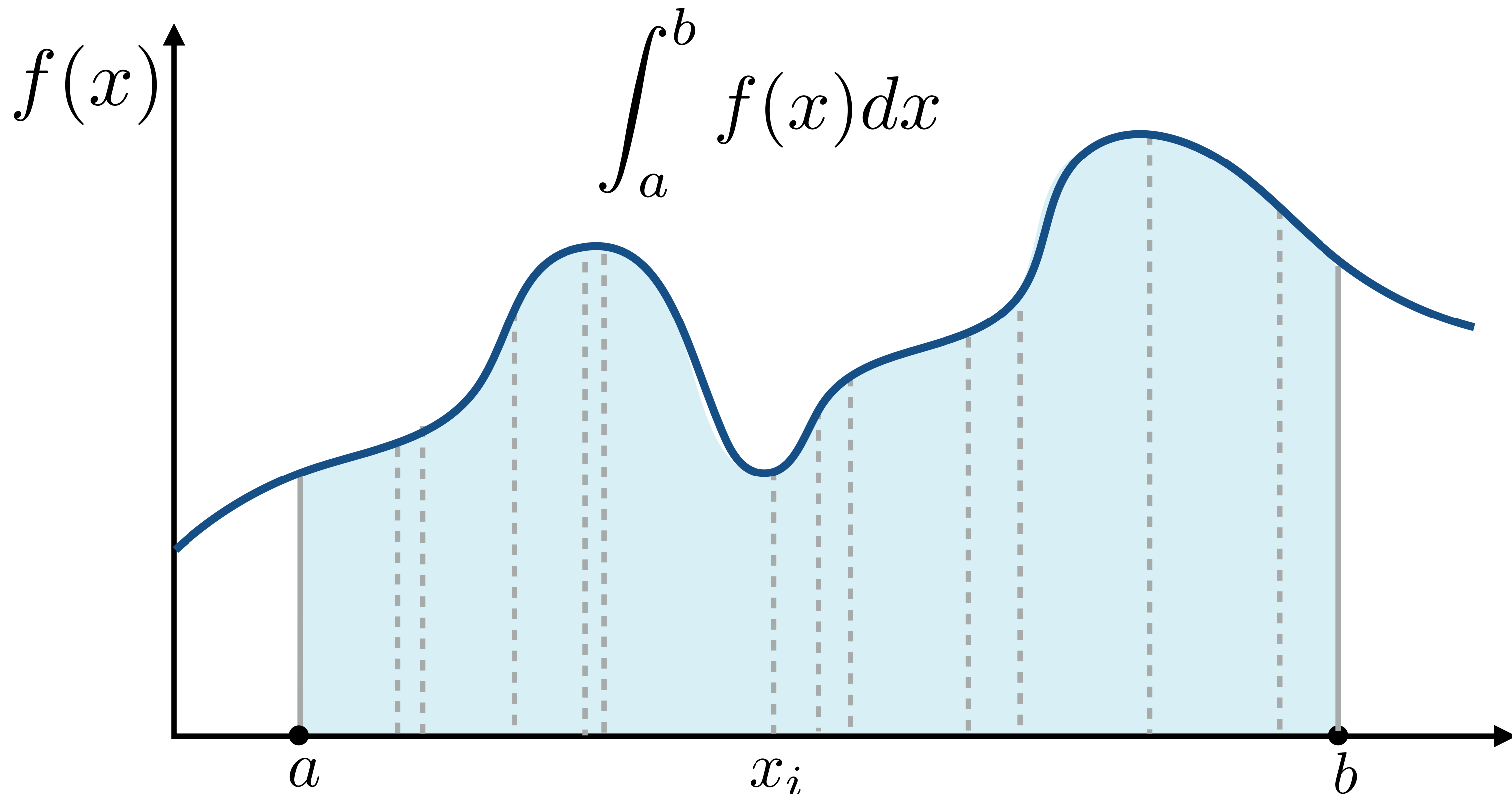
$$E[Y] = E[f(X)] = \int f(x) p(x) dx$$

# Monte Carlo Integration



# Monte Carlo Integration

Simple idea: estimate the integral of a function by averaging random samples of the function's value.



# Monte Carlo Integration

Let us define the Monte Carlo estimator for the definite integral of given function  $f(x)$

Definite integral

$$\int_a^b f(x) dx$$

Random variable

$$X_i \sim p(x)$$

**Note:  $p(x)$  must be nonzero for all  $x$  where  $f(x)$  is nonzero**

Monte Carlo estimator

$$F_N = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}$$

# Example: Basic Monte Carlo Estimator

The basic Monte Carlo estimator is a simple special case where we sample with a uniform random variable

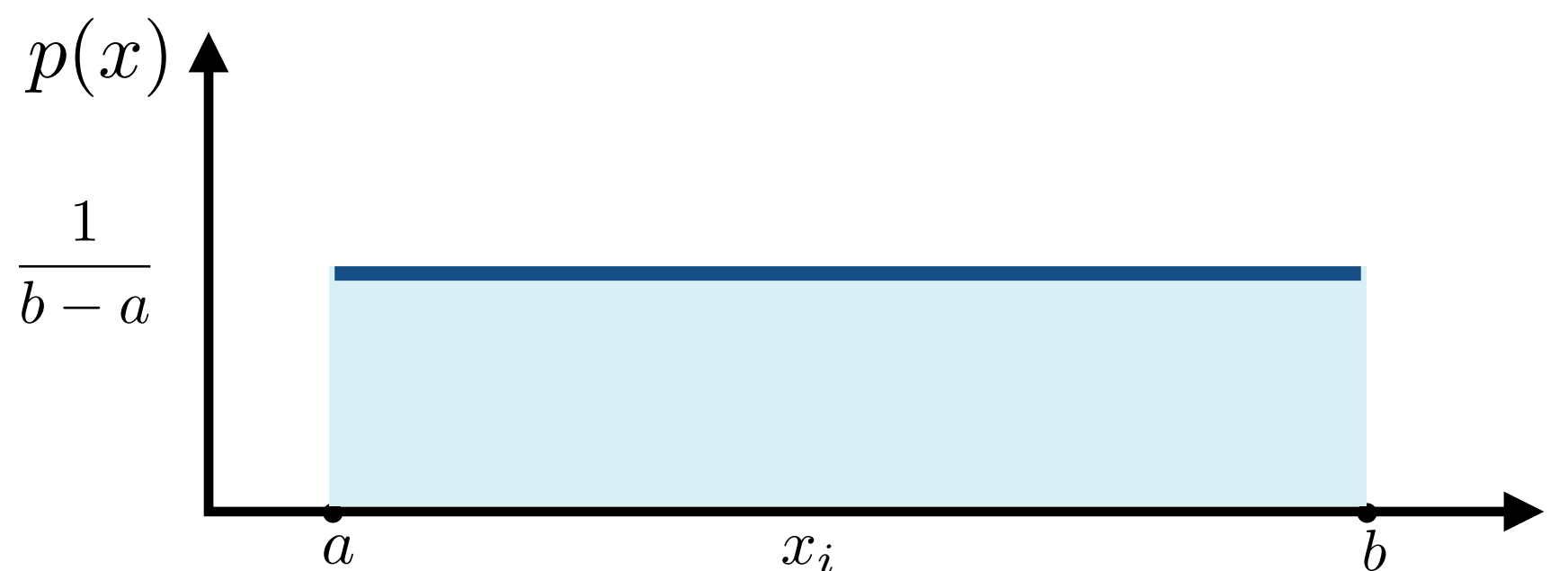
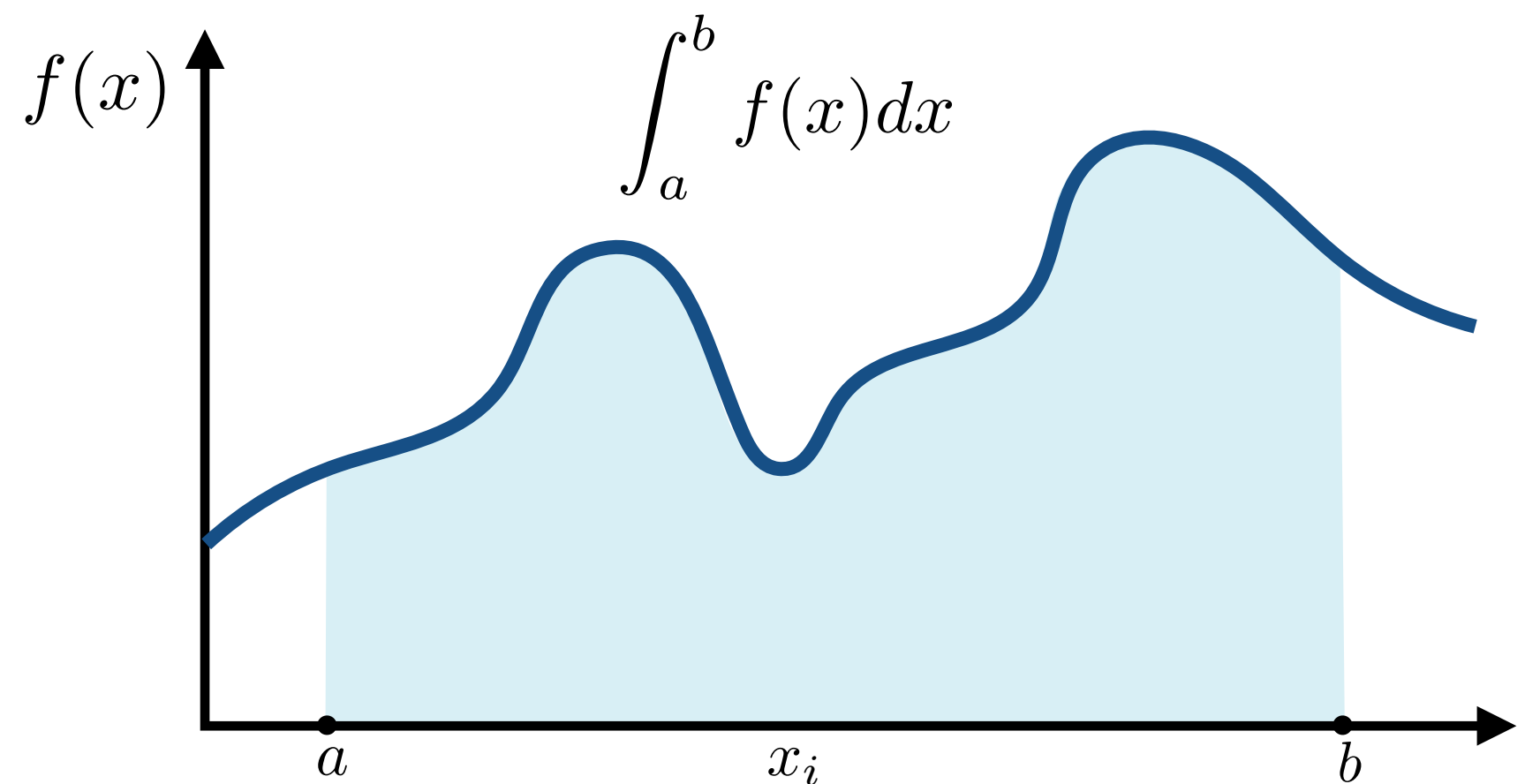
Uniform random variable

$$X_i \sim p(x) = C \text{ (constant)}$$

$$\int_a^b p(x) dx = 1$$

$$\implies \int_a^b C dx = 1$$

$$\implies C = \frac{1}{b - a}$$



# Example: Basic Monte Carlo Estimator

The basic Monte Carlo estimator is a simple special case where we sample with a uniform random variable

Basic Monte Carlo estimator (derivation)

$$\begin{aligned} F_N &= \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)} && \text{(MC Estimator)} \\ &= \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{1/(b-a)} \\ &= \frac{b-a}{N} \sum_{i=1}^N f(X_i) \end{aligned}$$

# Example: Basic Monte Carlo Estimator

Let us define the Monte Carlo estimator for the definite integral of given function  $f(x)$

Definite integral

$$\int_a^b f(x) dx$$

Uniform random variable

$$X_i \sim p(x) = \frac{1}{b - a}$$

Basic Monte Carlo estimator

$$F_N = \frac{b - a}{N} \sum_{i=1}^N f(X_i)$$

# Unbiased Estimator

**Definition:** A randomized integral estimator is *unbiased* if its expected value is the desired integral.

**Fact:** the general and basic Monte Carlo estimators are unbiased (proof on next slide)

**Why do we want unbiased estimators?**

# Proof That Monte Carlo Estimator Is Unbiased

$$\begin{aligned} E[F_N] &= E \left[ \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)} \right] \\ &= \frac{1}{N} \sum_{i=1}^N E \left[ \frac{f(X_i)}{p(X_i)} \right] \\ &= \frac{1}{N} \sum_{i=1}^N \int_a^b \frac{f(x)}{p(x)} p(x) dx \\ &= \frac{1}{N} \sum_{i=1}^N \int_a^b f(x) dx \\ &= \int_a^b f(x) dx \end{aligned}$$

Properties of expected values:

$$E \left[ \sum_i Y_i \right] = \sum_i E[Y_i]$$
$$E[aY] = aE[Y]$$

**The expected value of the Monte Carlo estimator is the desired integral.**

# Variance of a Random Variable

## Definition

$$\begin{aligned} V[Y] &= E[(Y - E[Y])^2] \\ &= E[Y^2] - E[Y]^2 \end{aligned}$$

Variance decreases linearly with number of samples

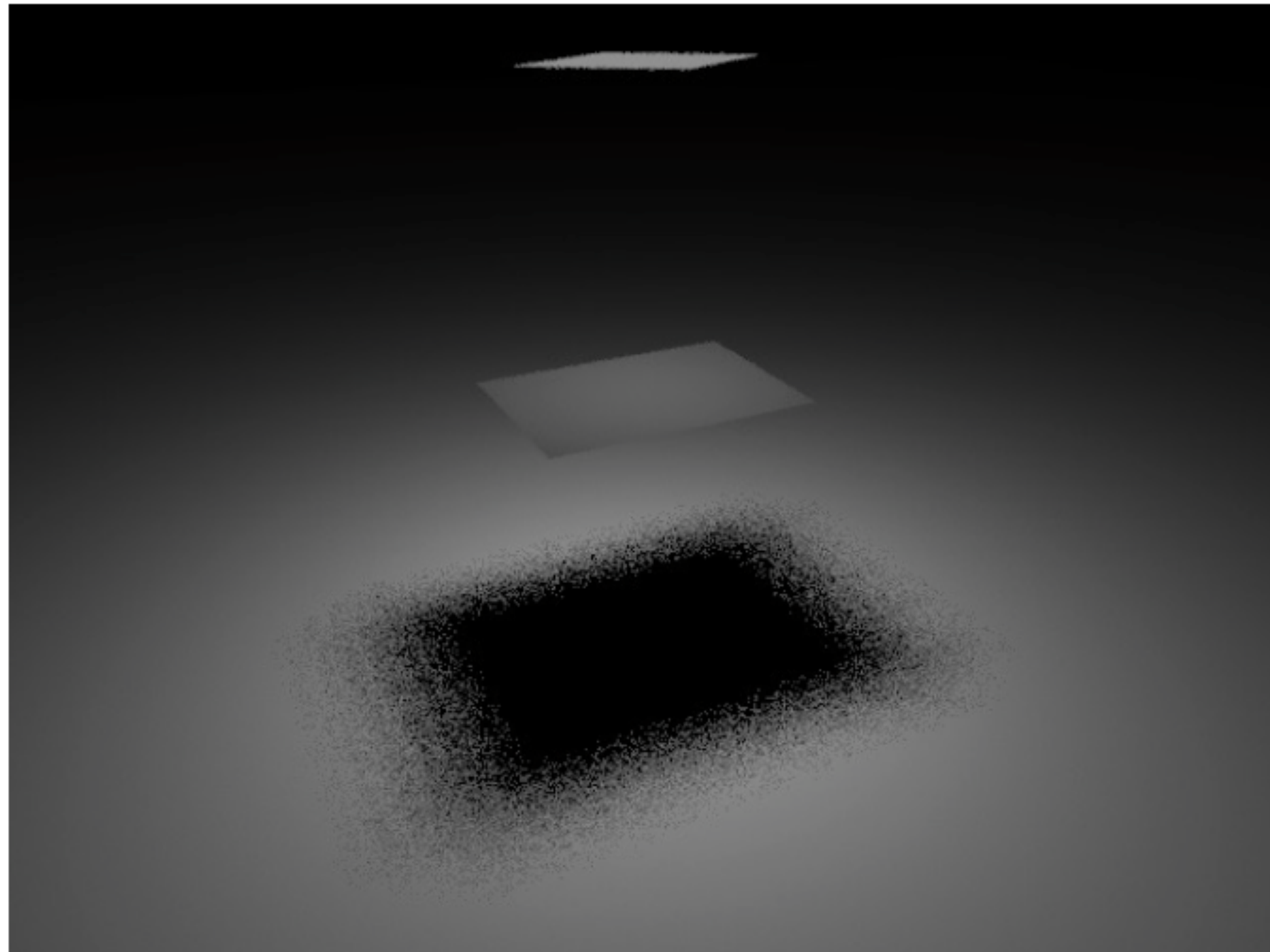
$$V\left[\frac{1}{N} \sum_{i=1}^N Y_i\right] = \frac{1}{N^2} \sum_{i=1}^N V[Y_i] = \frac{1}{N^2} N V[Y] = \frac{1}{N} V[Y]$$

## Properties of variance

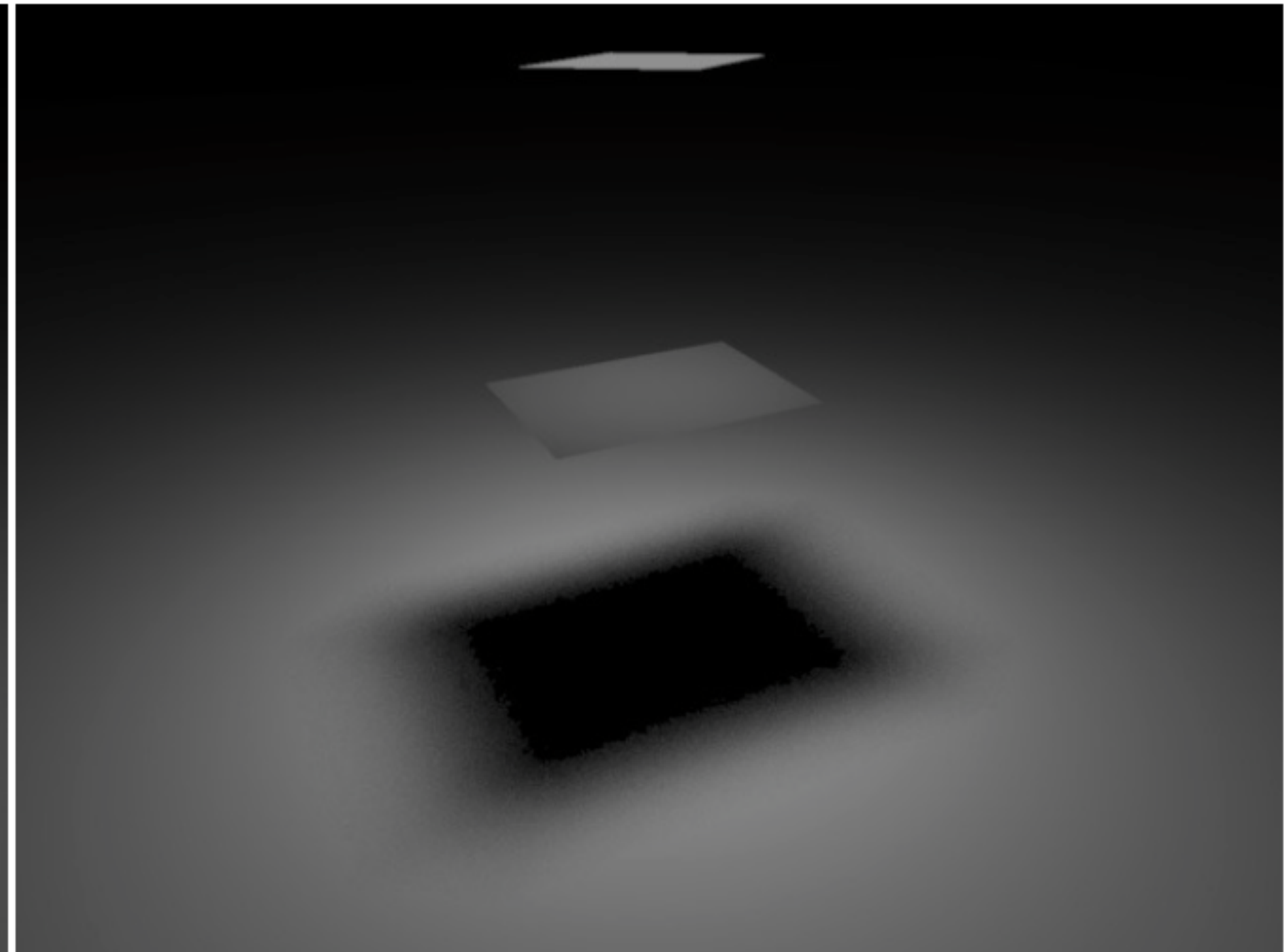
$$V\left[\sum_{i=1}^N Y_i\right] = \sum_{i=1}^N V[Y_i] \qquad V[aY] = a^2 V[Y]$$



# More Random Samples Reduces Variance



1 shadow ray



16 shadow rays

# Definite Integral Can Be N-Dimensional

Example in 3D:

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z) dx dy dz$$

**Uniform 3D random variable\***

$$X_i \sim p(x, y, z) = \frac{1}{x_1 - x_0} \frac{1}{y_1 - y_0} \frac{1}{z_1 - z_0}$$

**Basic 3D MC estimator\***

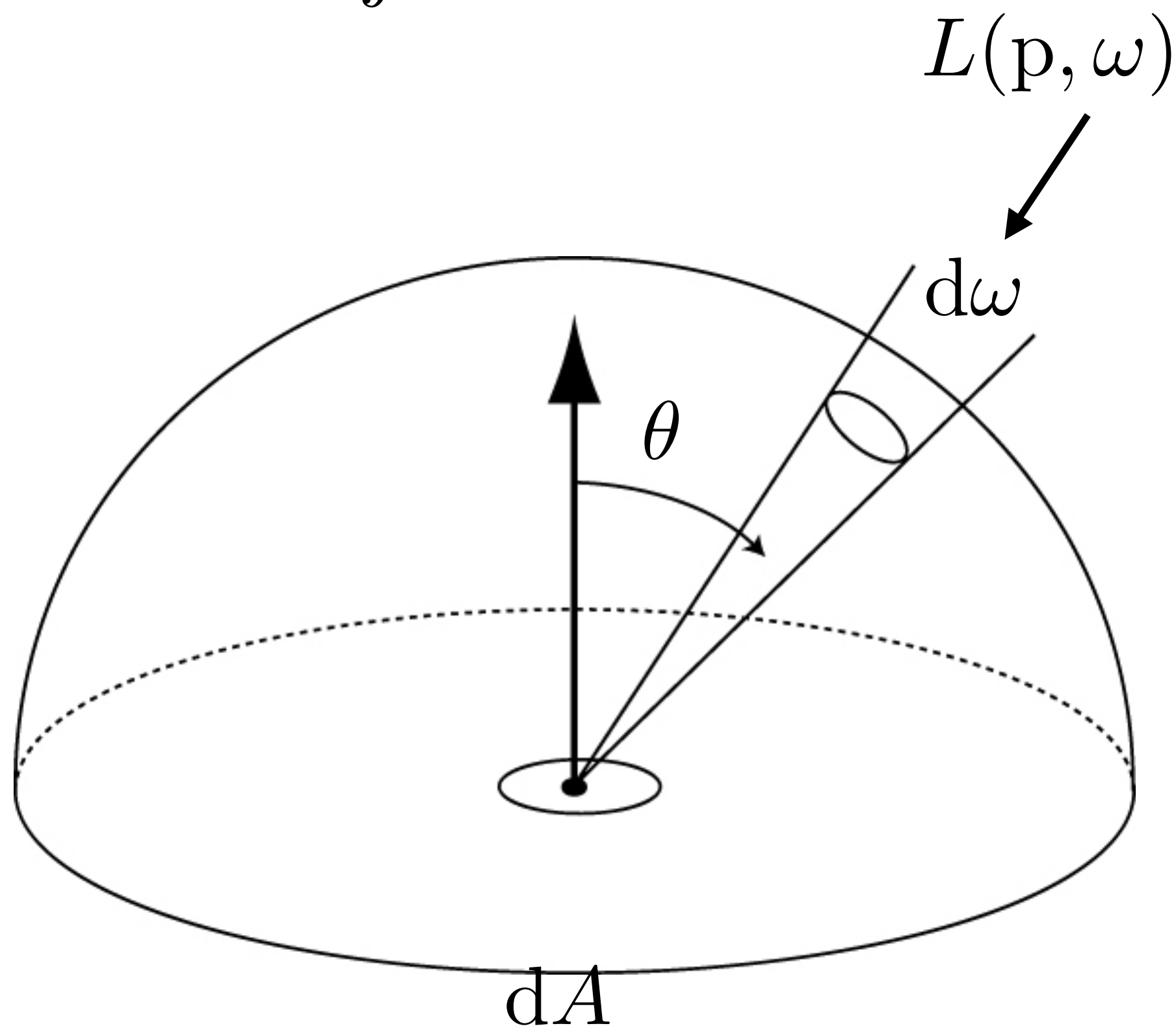
$$F_N = \frac{(x_1 - x_0)(y_1 - y_0)(z_1 - z_0)}{N} \sum_{i=1}^N f(X_i)$$

\* Generalizes to arbitrary N-dimensional PDFs

# **Example: Monte Carlo Estimate Of Direct Lighting Integral**

# Direct Lighting (Irradiance) Estimate

$$E(p) = \int L(p, \omega) \cos \theta \, d\omega$$



**Idea: sample directions over hemisphere uniformly in solid angle**

**Estimator:**

$$X_i \sim p(\omega) \quad p(\omega) = \frac{1}{2\pi}$$

$$Y_i = f(X_i)$$

$$Y_i = L(p, \omega_i) \cos \theta_i$$

$$F_N = \frac{2\pi}{N} \sum_{i=1}^N Y_i$$

# Direct Lighting (Irradiance) Estimate

Sample directions over hemisphere uniformly in solid angle

$$E(p) = \int L(p, \omega) \cos \theta \, d\omega$$

Given surface point  $p$

Initialize Monte Carlo estimator  $F_N$  to 0

For each of  $N$  samples:

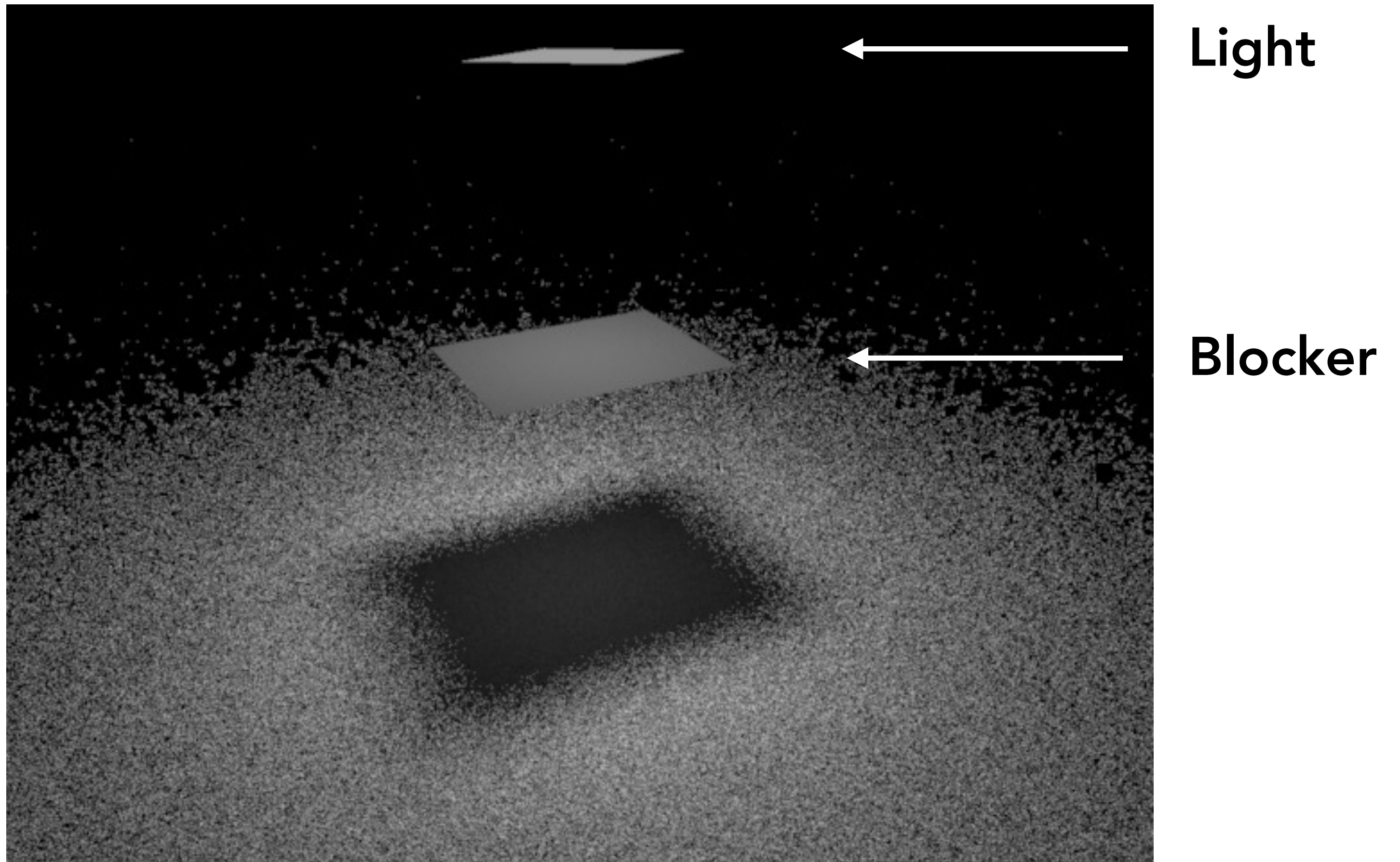
A ray tracer evaluates radiance along a ray

Generate random direction:  $\omega_i$

Compute incoming radiance  $L_i$  arriving at  $p$  from direction  $\omega_i$

Increment the Monte Carlo estimator:  $F_N := F_N + \frac{2\pi}{N} L_i \cos \theta_i$

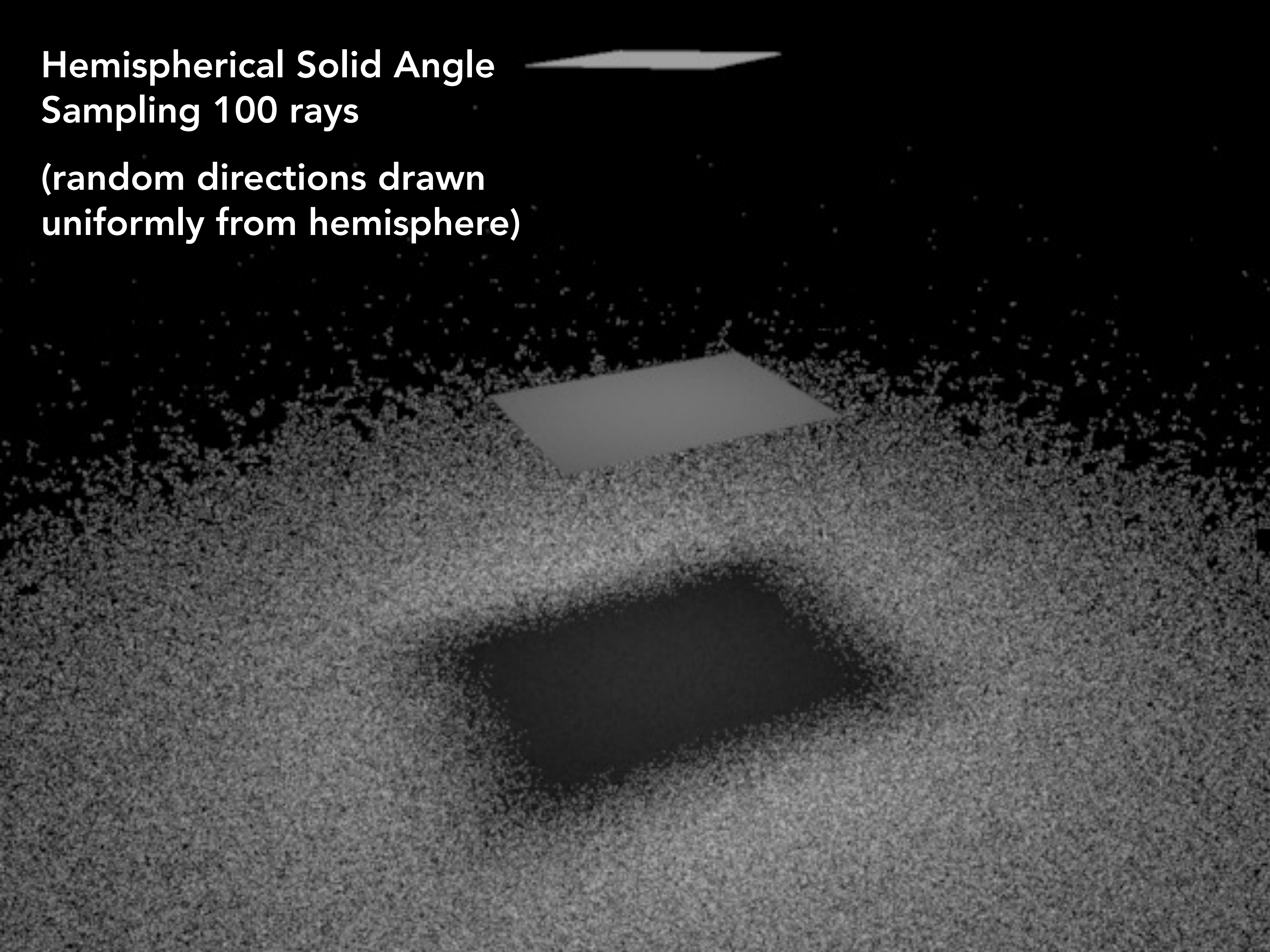
# Direct Lighting - Solid Angle Sampling



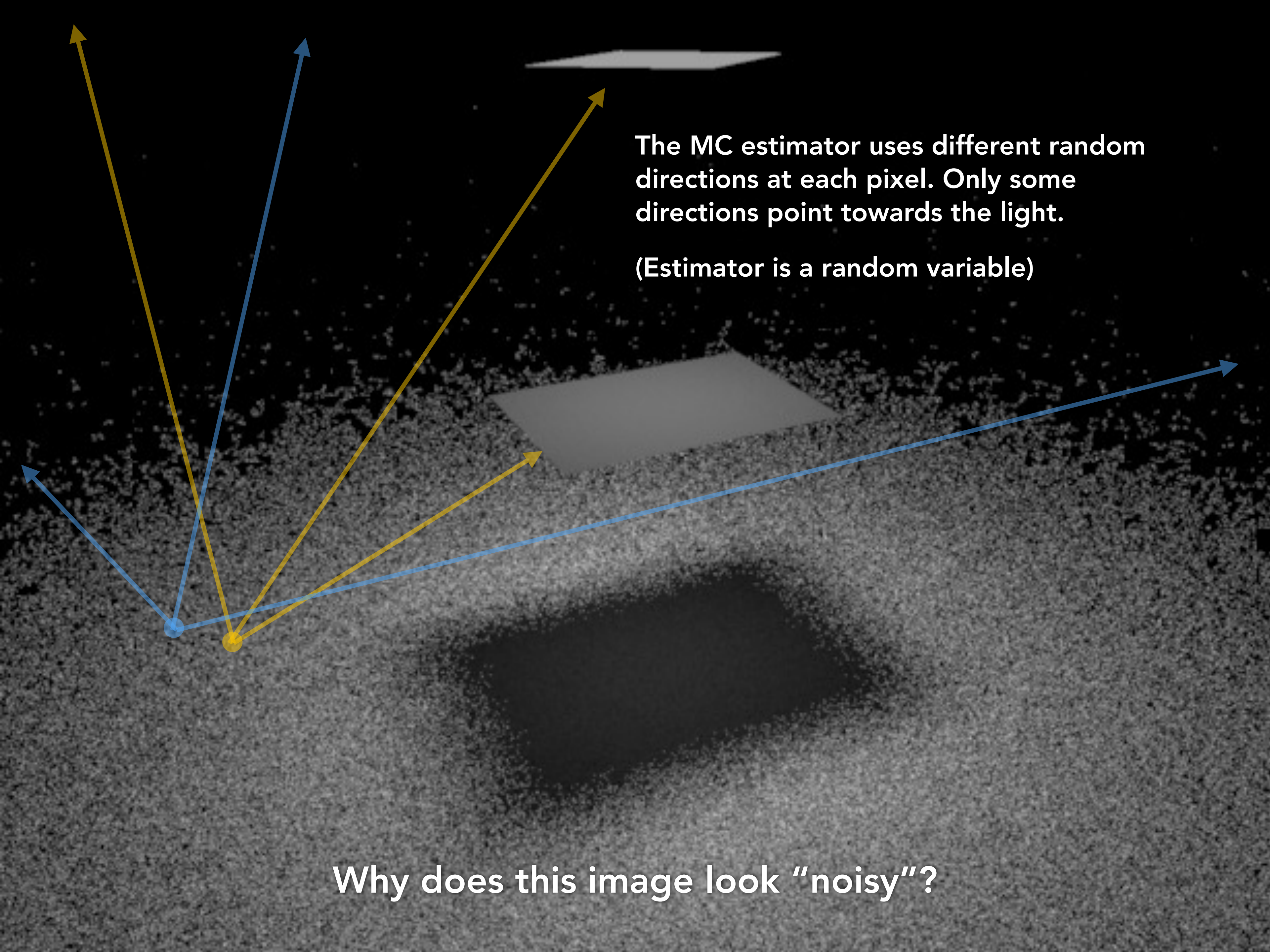
Trace 100 rays per pixel



**Hemispherical Solid Angle  
Sampling 100 rays**  
(random directions drawn  
uniformly from hemisphere)







The MC estimator uses different random directions at each pixel. Only some directions point towards the light.

(Estimator is a random variable)

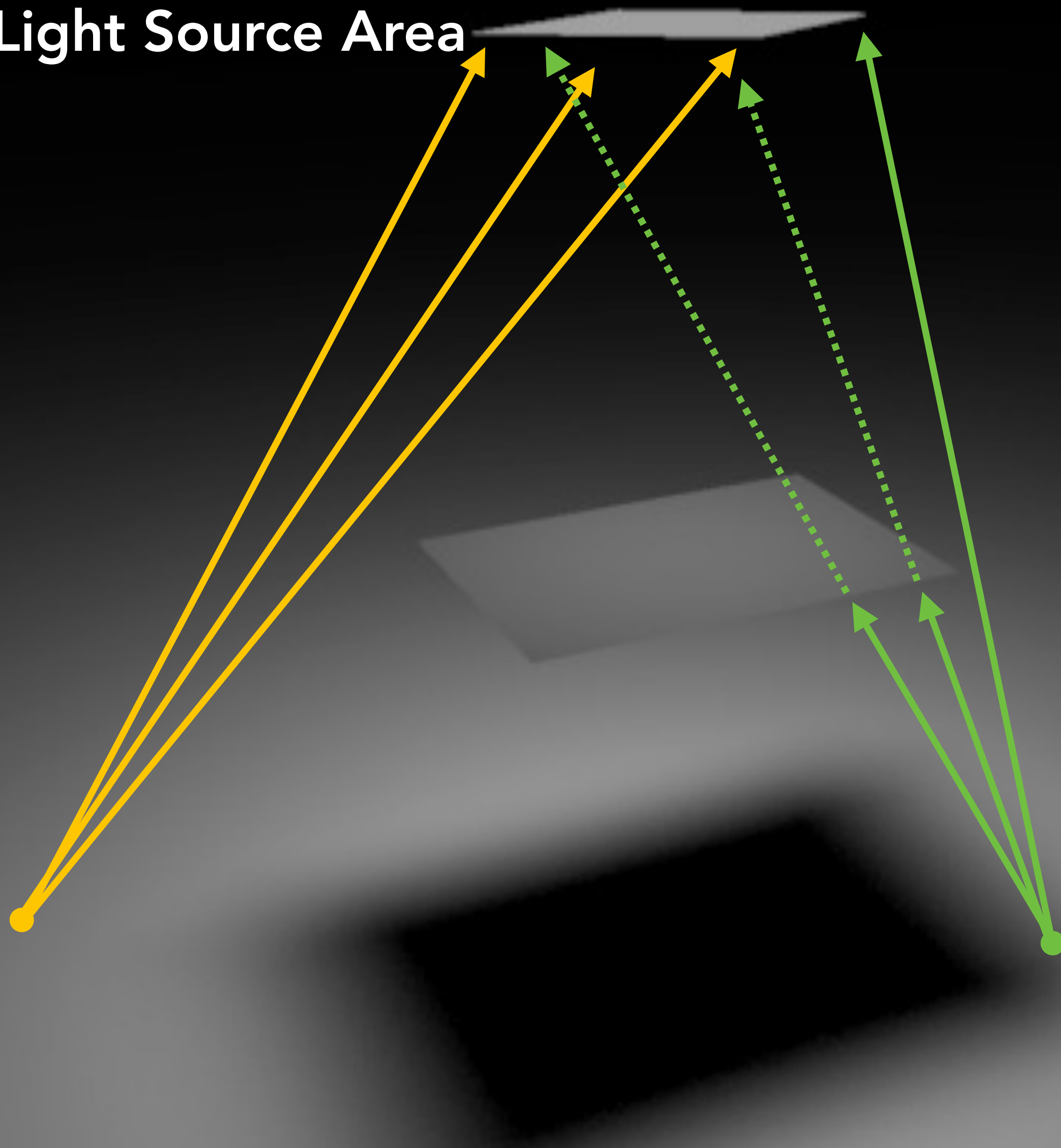
Why does this image look "noisy"?



**Observation: incoming radiance is zero  
for most directions in this scene**

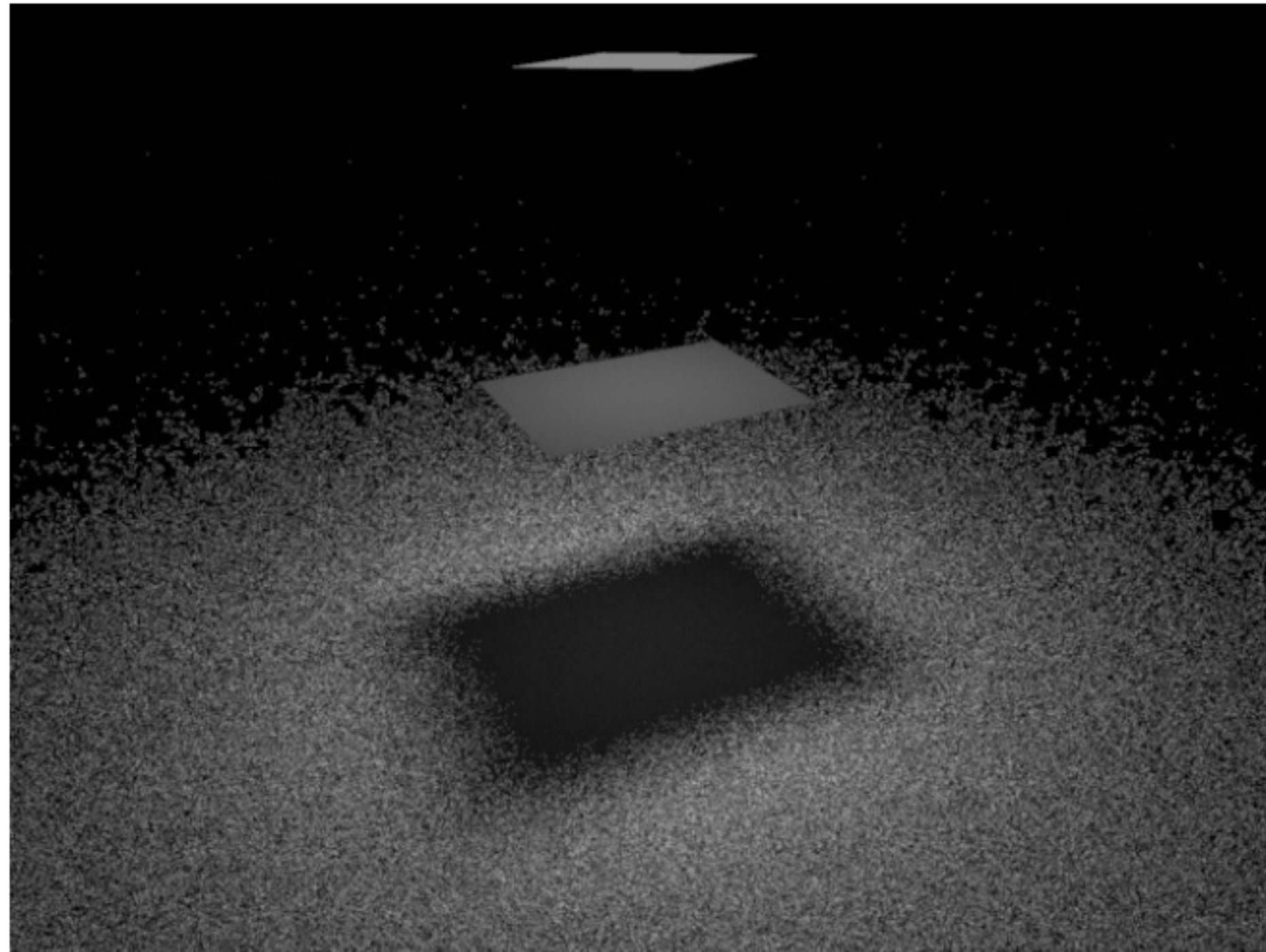
**Idea: integrate only over the area of the light  
(directions where incoming radiance could be non-zero)**

Sampling Light Source Area  
100 rays



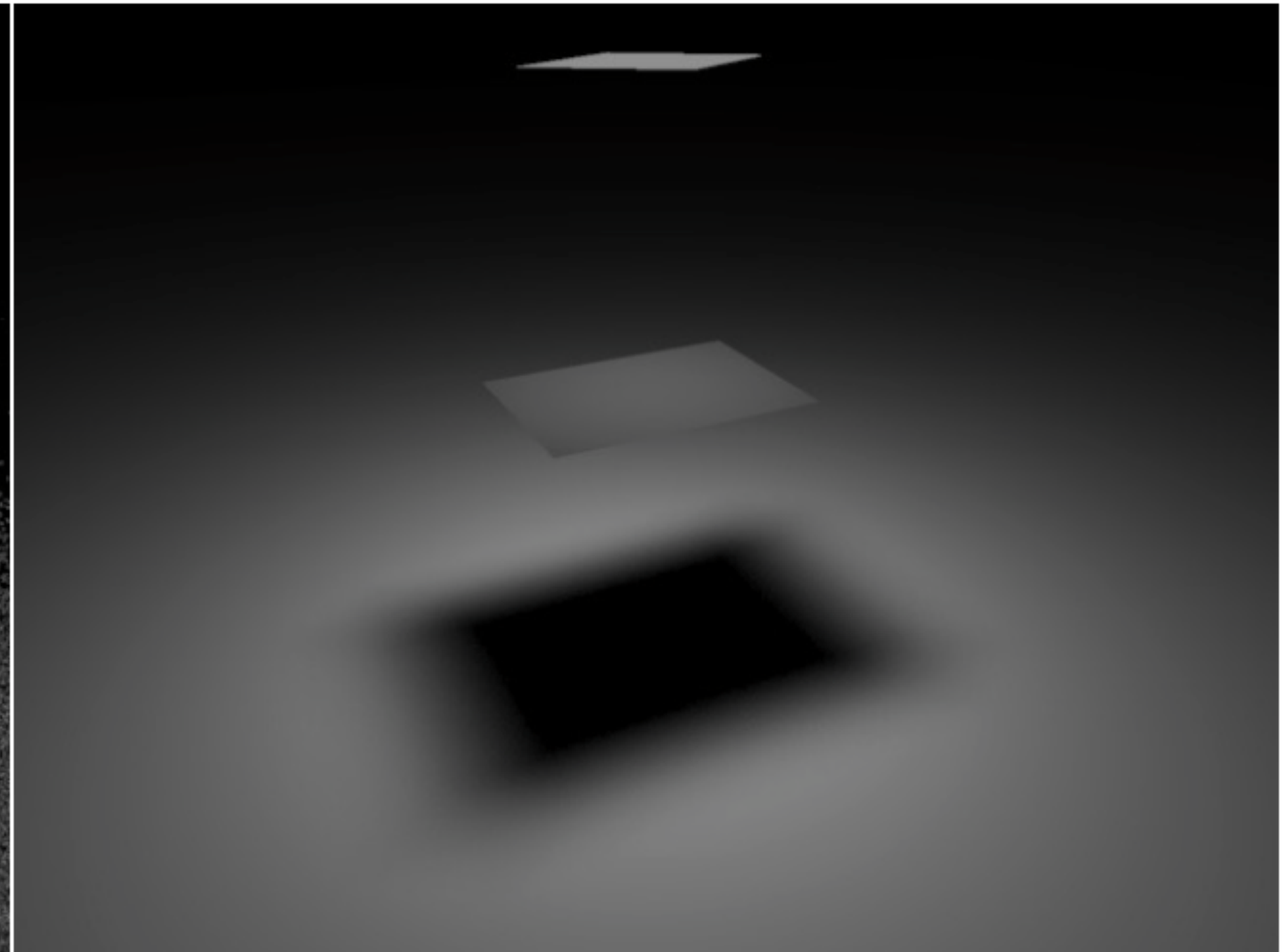
If no occlusion is present, all directions chosen in computing estimate "hit" the light source.  
(Choice of direction only matters if portion of light is occluded from surface point  $p$ .)

# Solid Angle Sampling vs Light Area Sampling



## Sampling solid angle

100 random directions on hemisphere



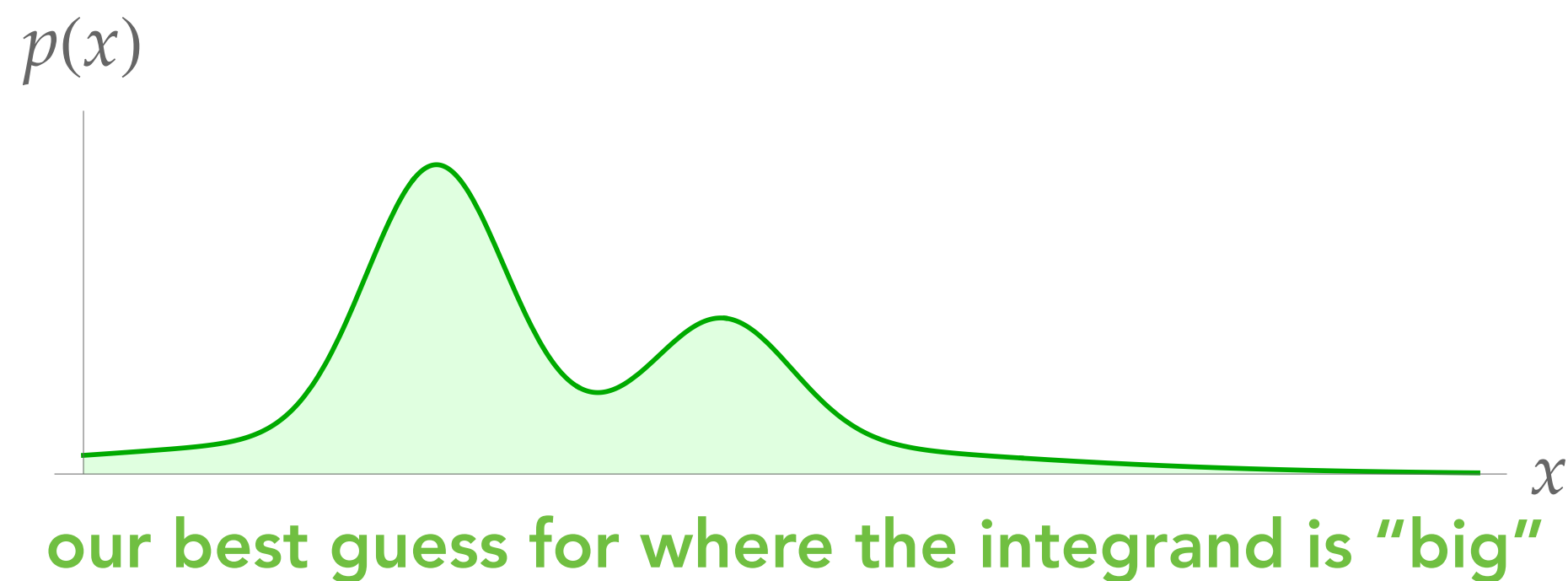
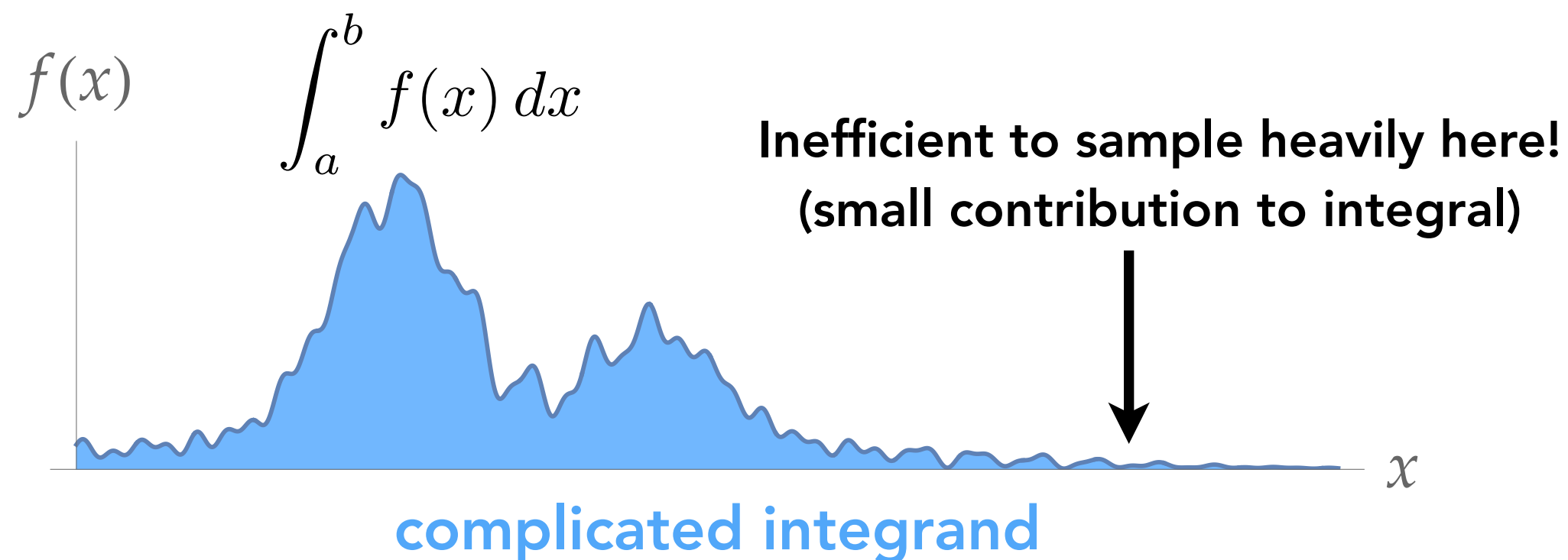
## Sampling light source area

100 random points on area of light source

# Importance Sampling

# Importance Sampling

Simple idea: sample the integrand according to how much we expect it to contribute to the integral.



**Note:**  $p(x)$  must be non-zero where  $f(x)$  is non-zero

Basic Monte Carlo:

$$\frac{b-a}{N} \sum_{i=1}^N f(X_i)$$

( $x_i$  are sampled *uniformly*)

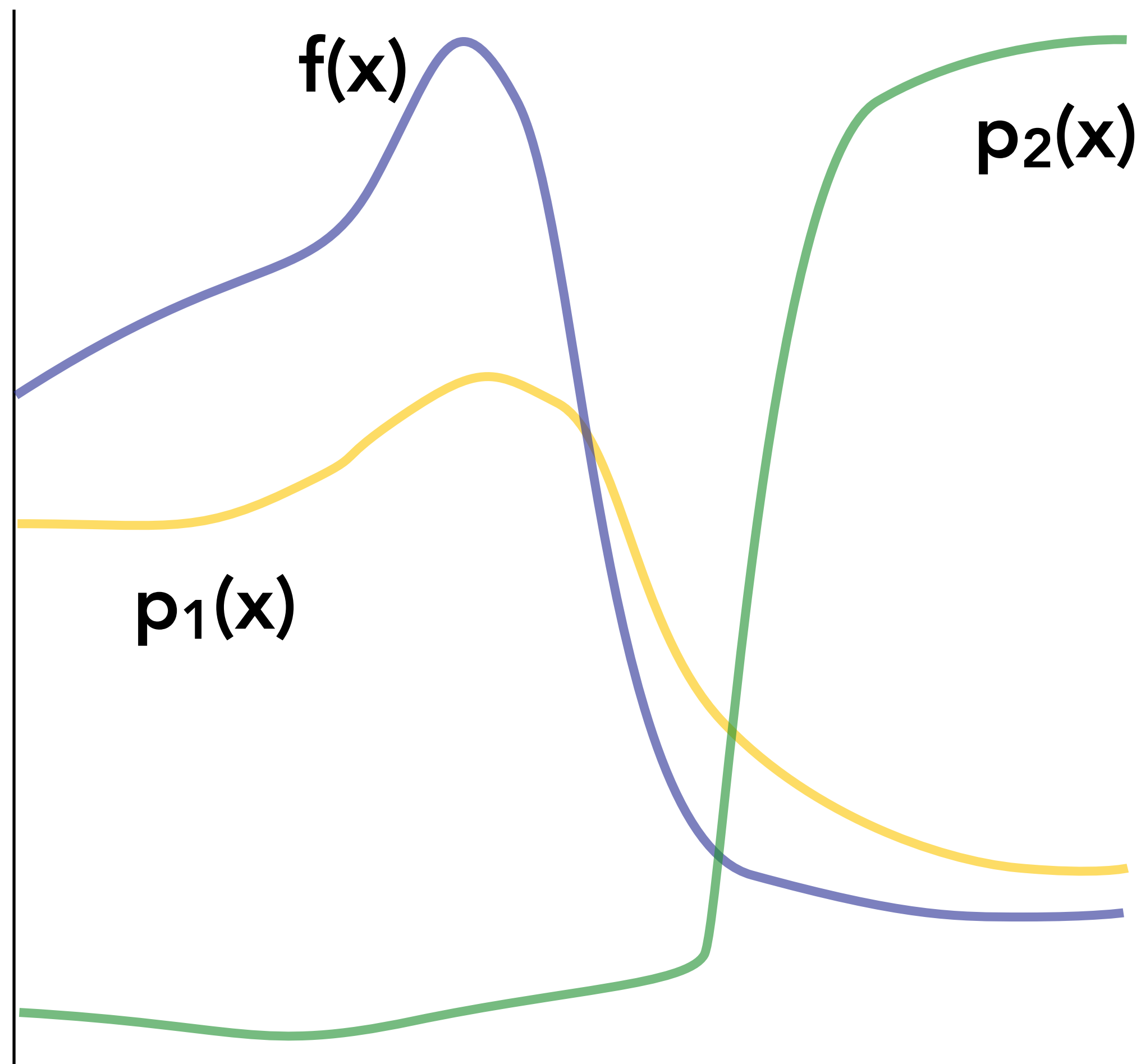
Importance-Sampled Monte Carlo:

$$\frac{1}{n} \sum_{i=1}^n \frac{f(x_i)}{p(x_i)}$$

( $x_i$  are sampled proportional to  $p$ )

"If I sample  $x$  less frequently, each sample should count for more."

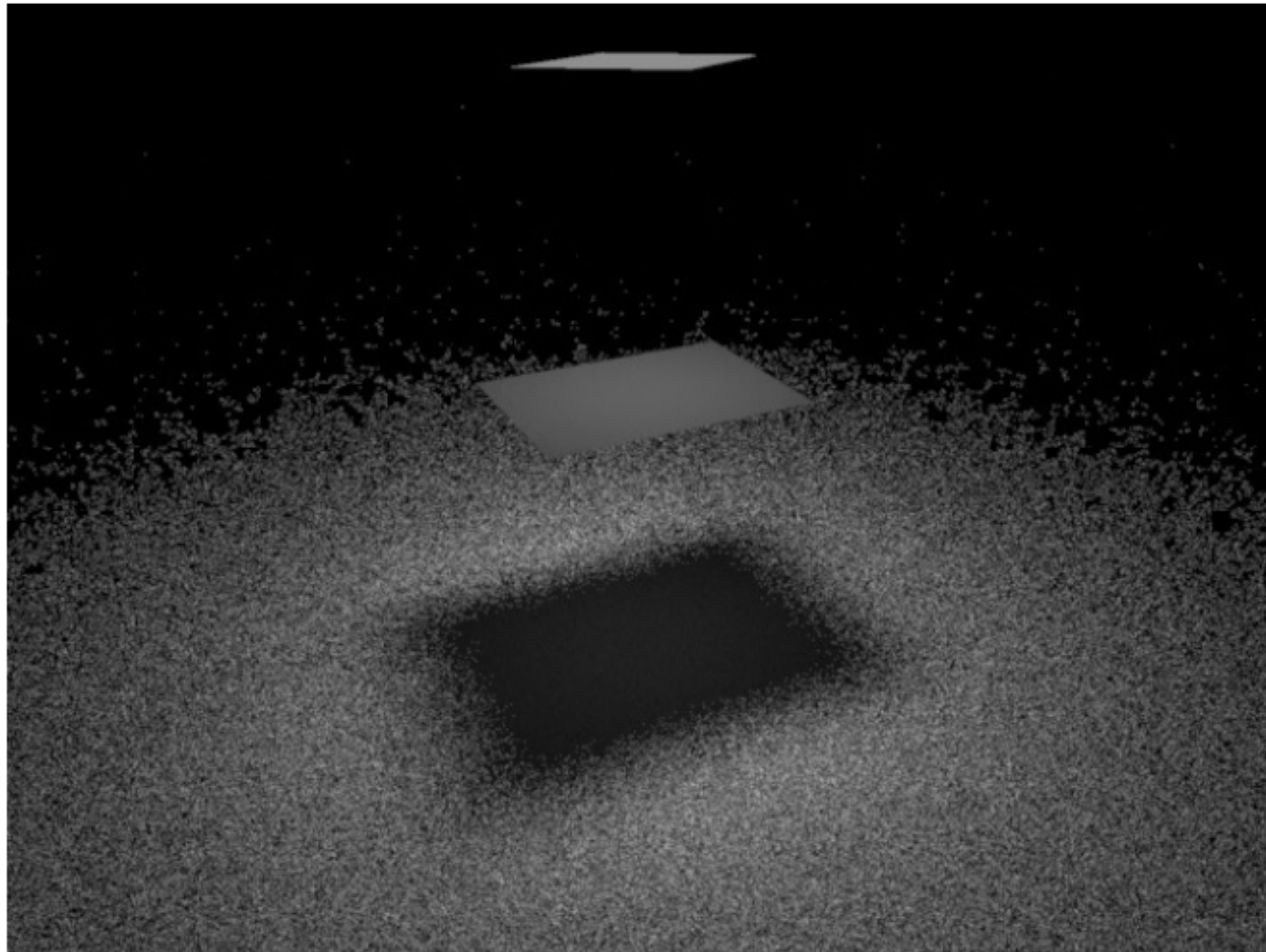
# Effect of Sampling Distribution "Fit"



What is the behavior of  $f(x)/p_1(x)$ ?  $f(x)/p_2(x)$ ?  
How does this impact the variance of the estimator?

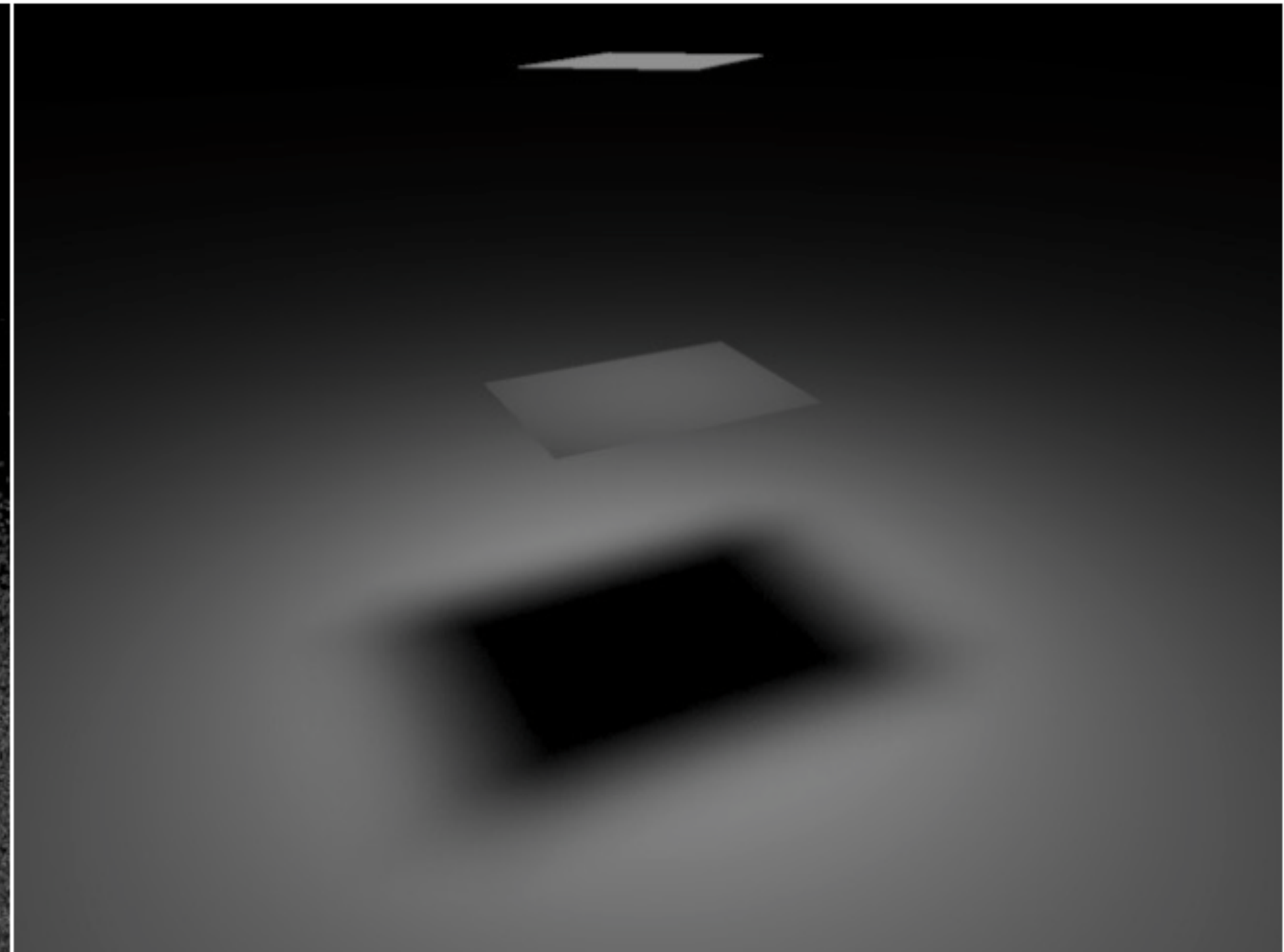


# Solid Angle Sampling vs Light Area Sampling



## Sampling solid angle

100 random directions on hemisphere

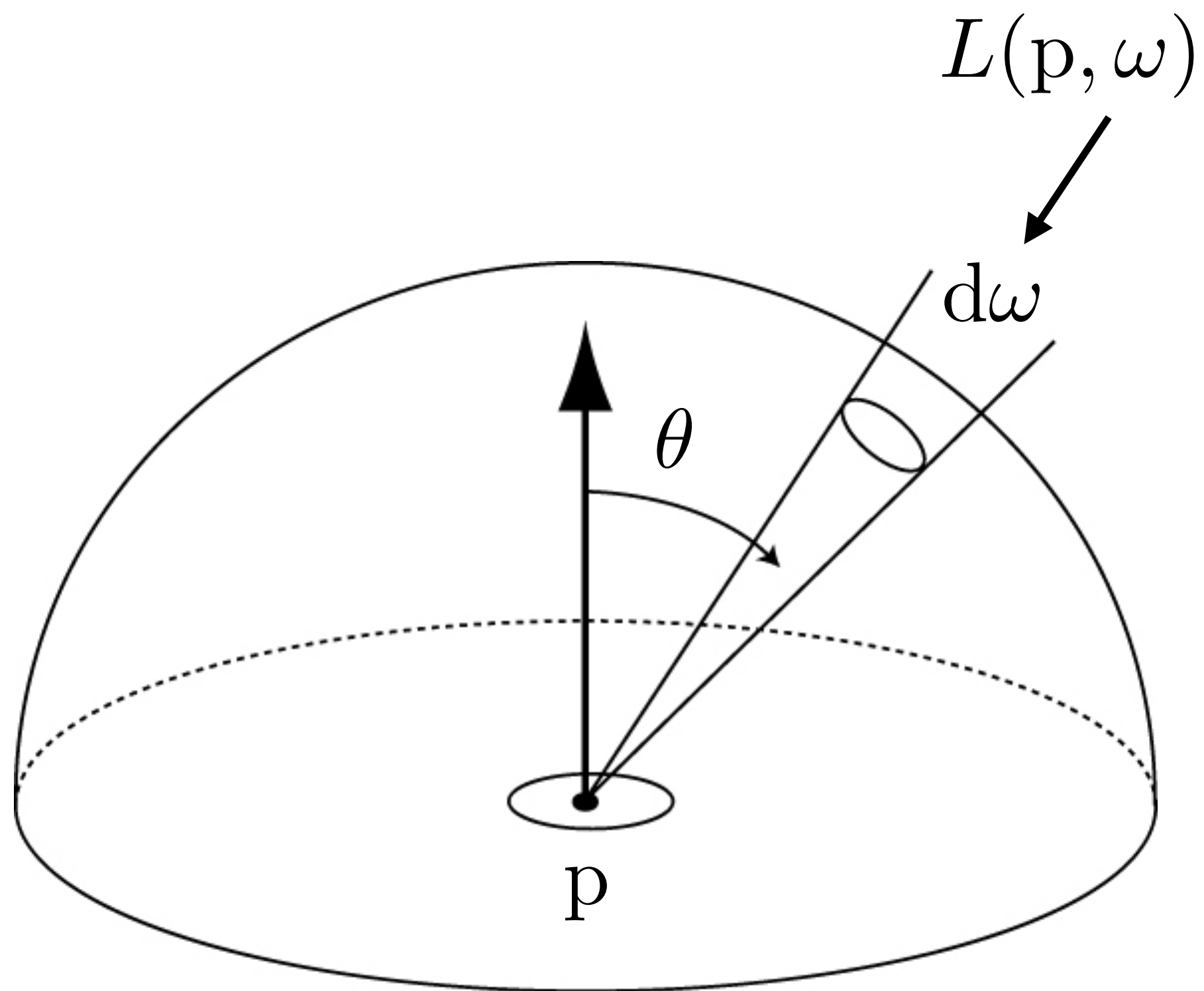


## Sampling light source area

100 random points on area of light source

# Changing Basis of Integration: Sampling Hemisphere

$$E(p) = \int L(p, \omega) \cos \theta \, d\omega$$





# Changing Basis of Integration: Sampling Light Source Area

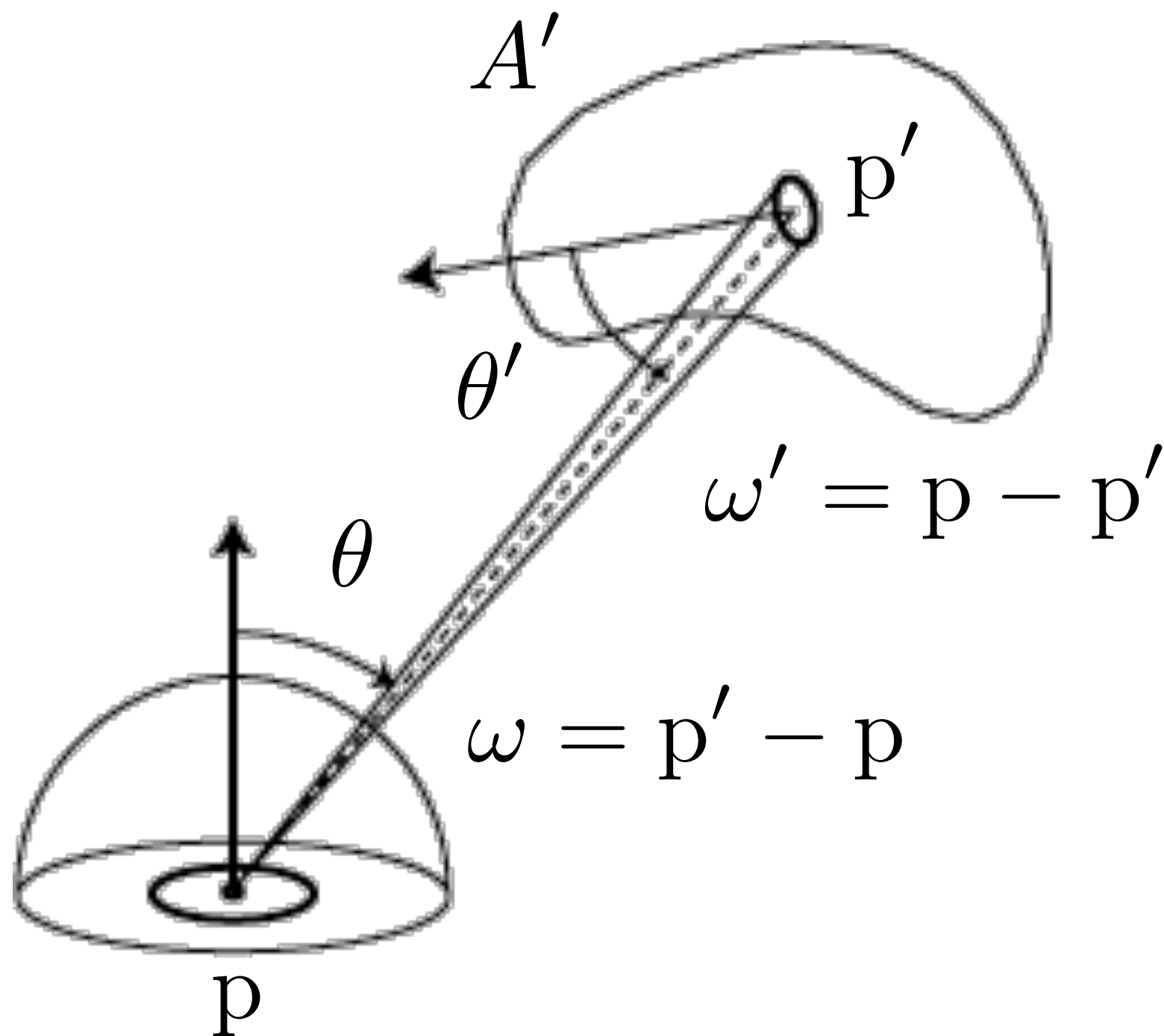
$$E(p) = \int_{A'} L_o(p', \omega') V(p, p') \frac{\cos \theta \cos \theta'}{|p - p'|^2} dA'$$

Change of variables to integral over area of light

$$dw = \frac{dA' \cos \theta'}{|p' - p|^2}$$

Binary visibility function:  
1 if  $p'$  is visible from  $p$ , 0 otherwise  
(accounts for light occlusion)

Outgoing radiance from light point  $p'$ , in direction  $w'$  towards  $p$



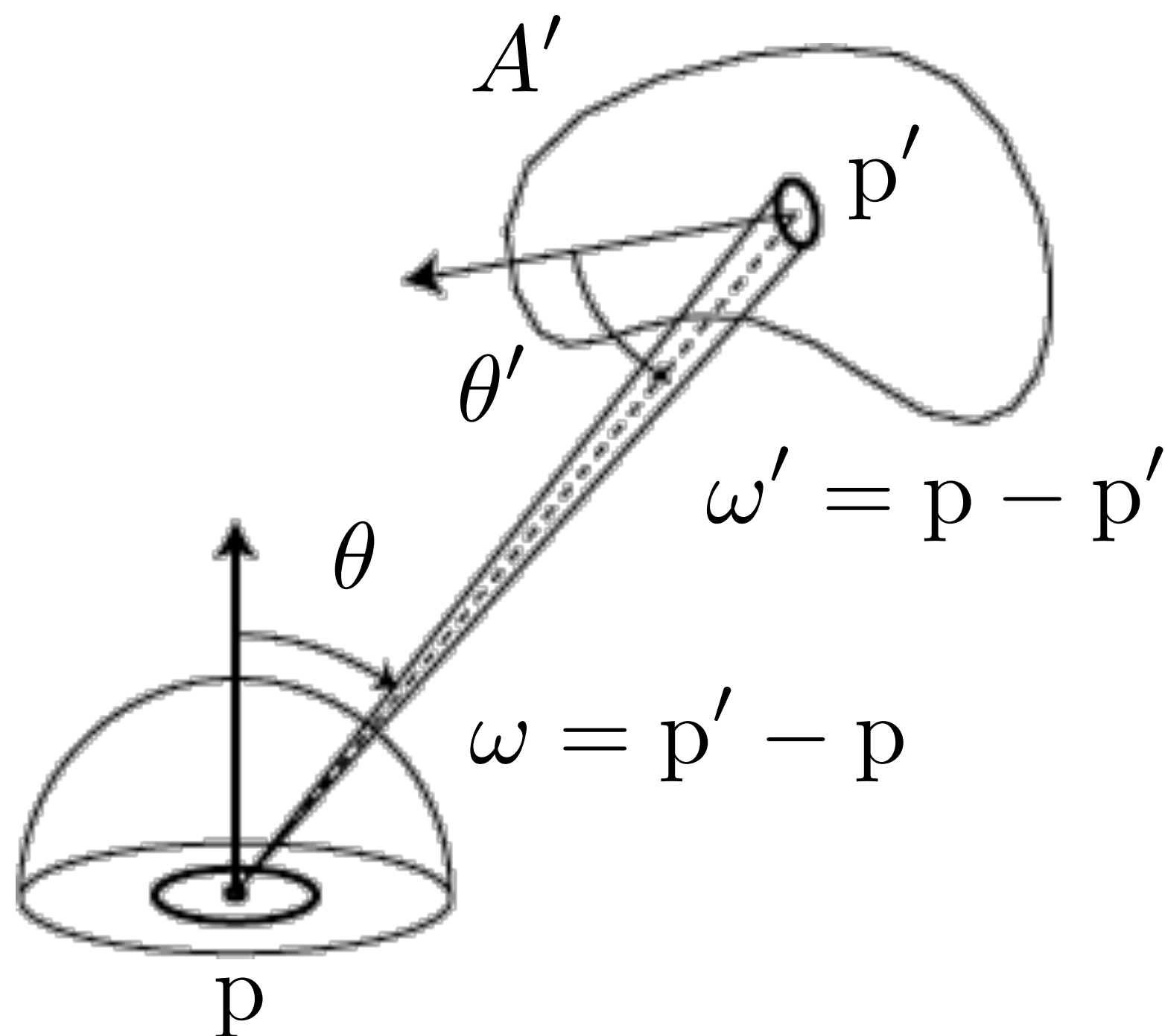
# Monte Carlo Estimate by Sampling Light Source Area

$$E(p) = \int_{A'} L_o(p', \omega') V(p, p') \frac{\cos \theta \cos \theta'}{|p - p'|^2} dA'$$

Randomly sample light source area  $A'$  (assume uniformly over area)

$$\int_{A'} p(p') dA' = 1$$

$$p(p') = \frac{1}{A'}$$

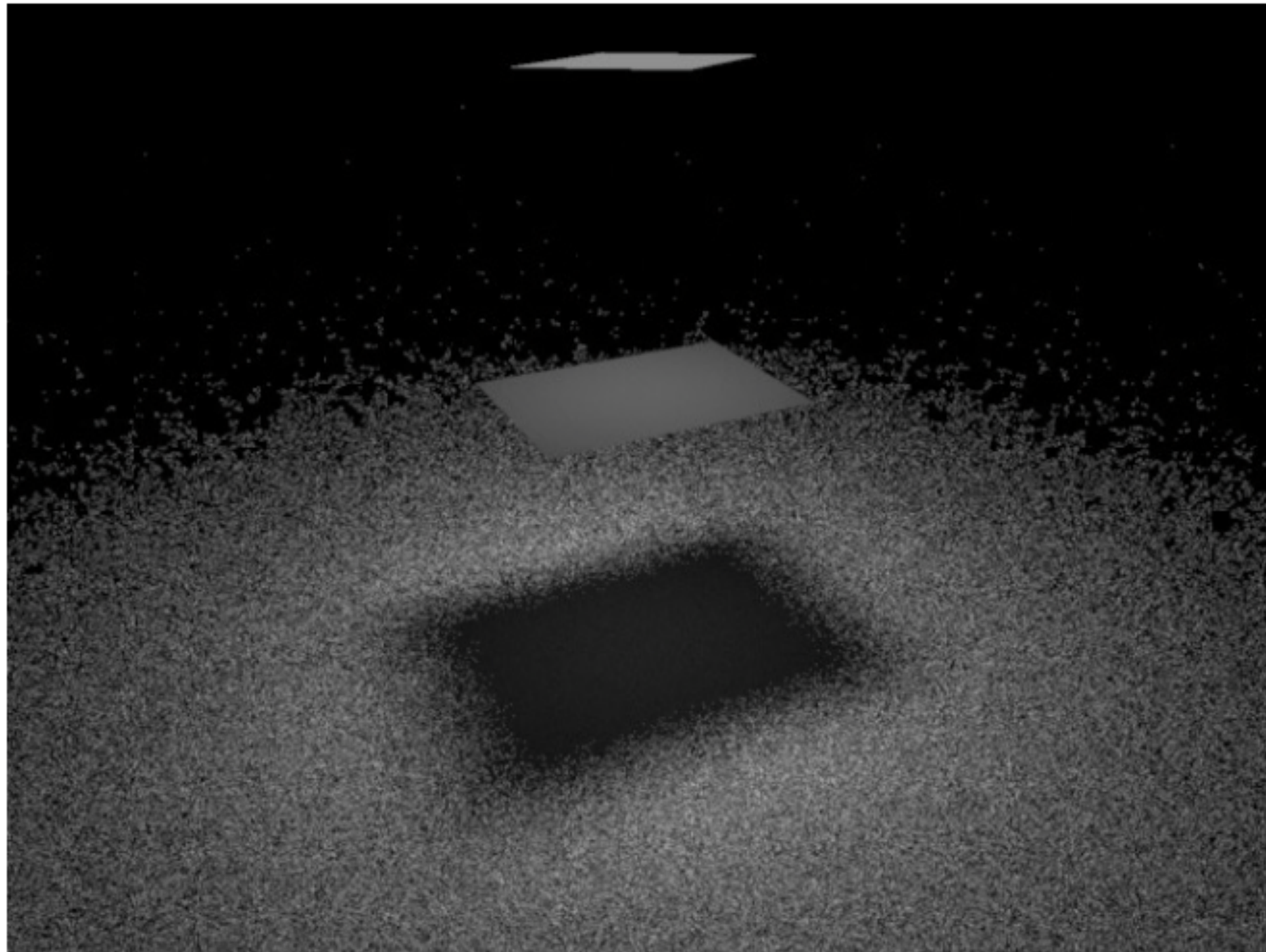


**Monte Carlo Estimator**

$$F_N = \frac{A'}{N} \sum_{i=1}^N Y_i$$

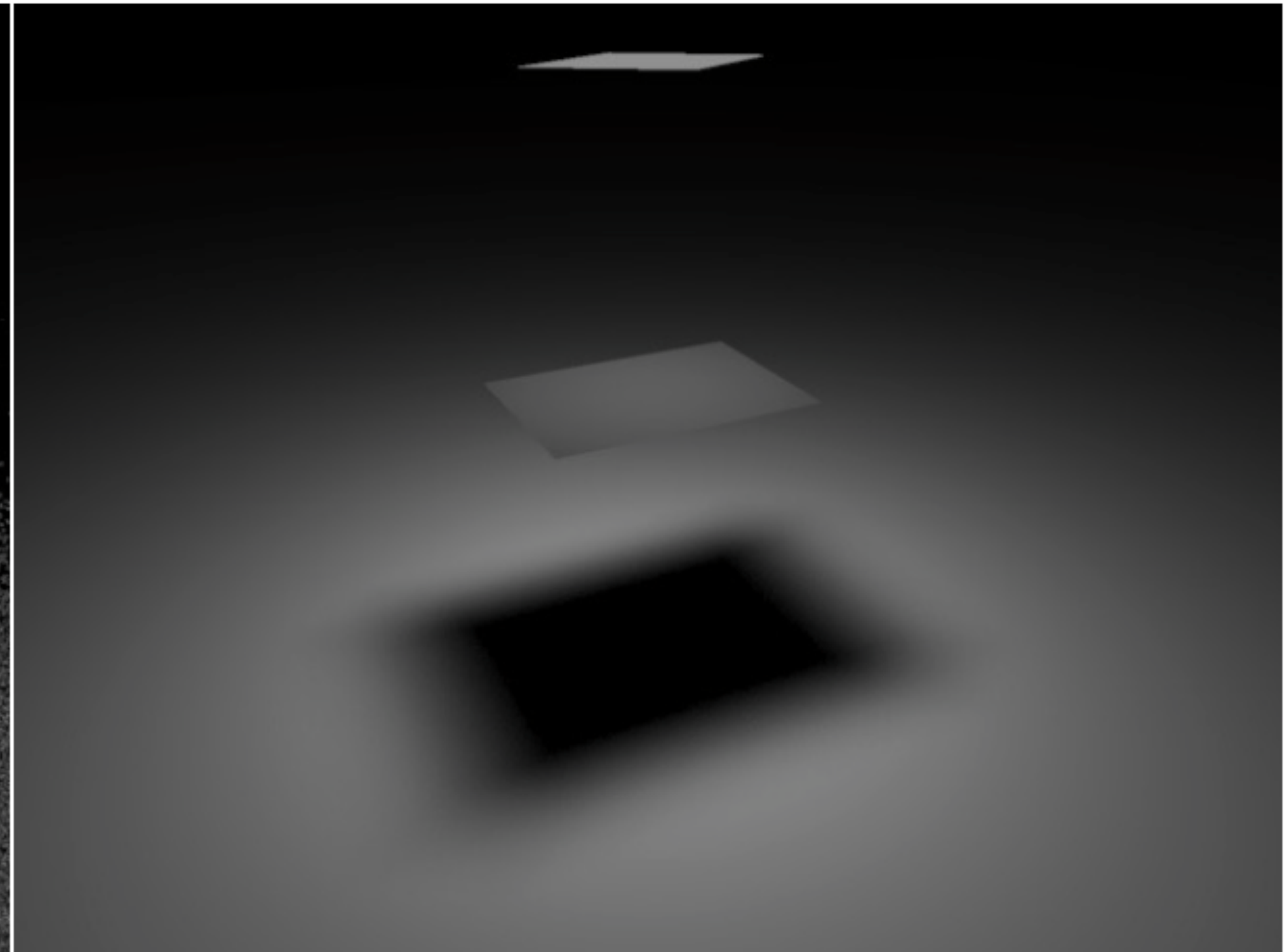
$$Y_i = L_o(p'_i, \omega'_i) V(p, p'_i) \frac{\cos \theta_i \cos \theta'_i}{|p - p'_i|^2}$$

# Solid Angle Sampling vs Light Area Sampling



**Sampling solid angle**

**100 random directions on hemisphere**



**Sampling light source area**

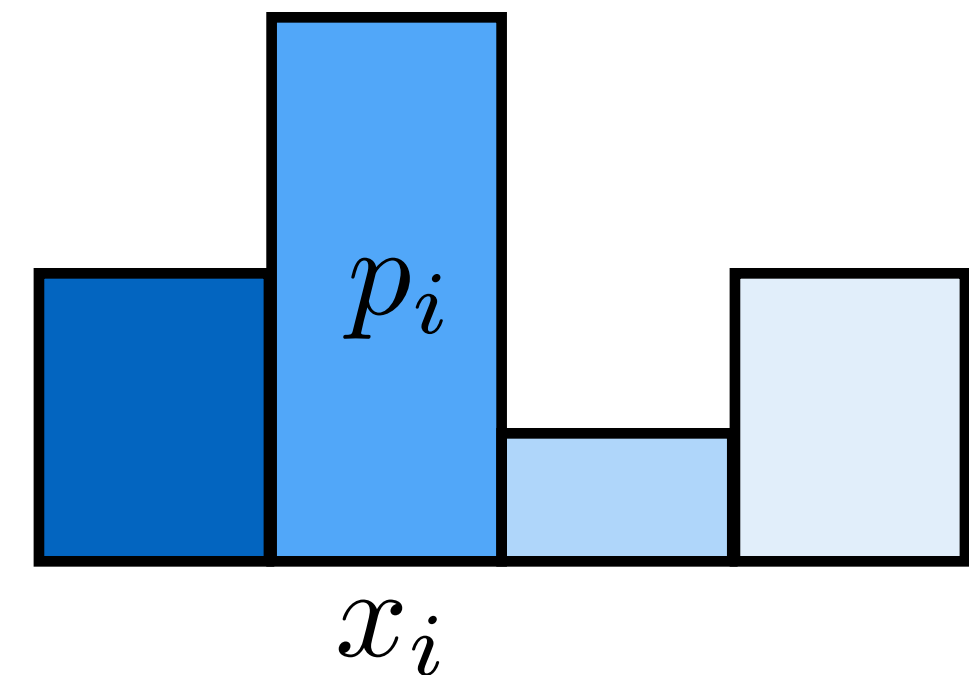
**100 random points on area of light source**

**How to Draw Samples From a  
Desired Probability Distribution?  
One Approach: Inversion Method**

# Task: Draw A Random Value From a Given PDF

## Task:

Given a PDF for a discrete random variable, probability  $p_i$  for each value  $x_i$ ,



Draw a random value  $X$  from this PDF.

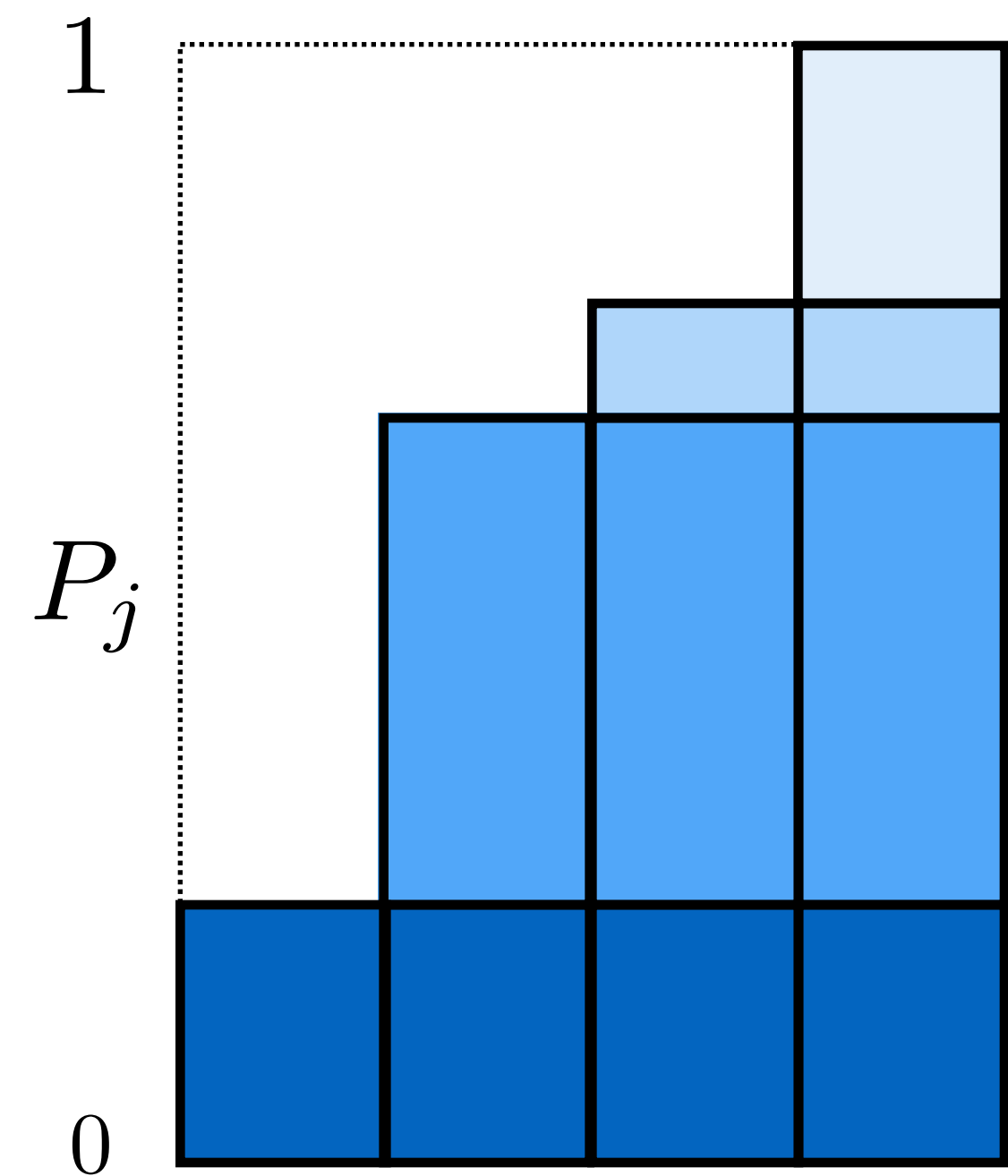
## Step 1:

Calculate cumulative PDF:  $P_j = \sum_{i=1}^j p_i$

Note: must have

$$0 \leq P_i \leq 1$$

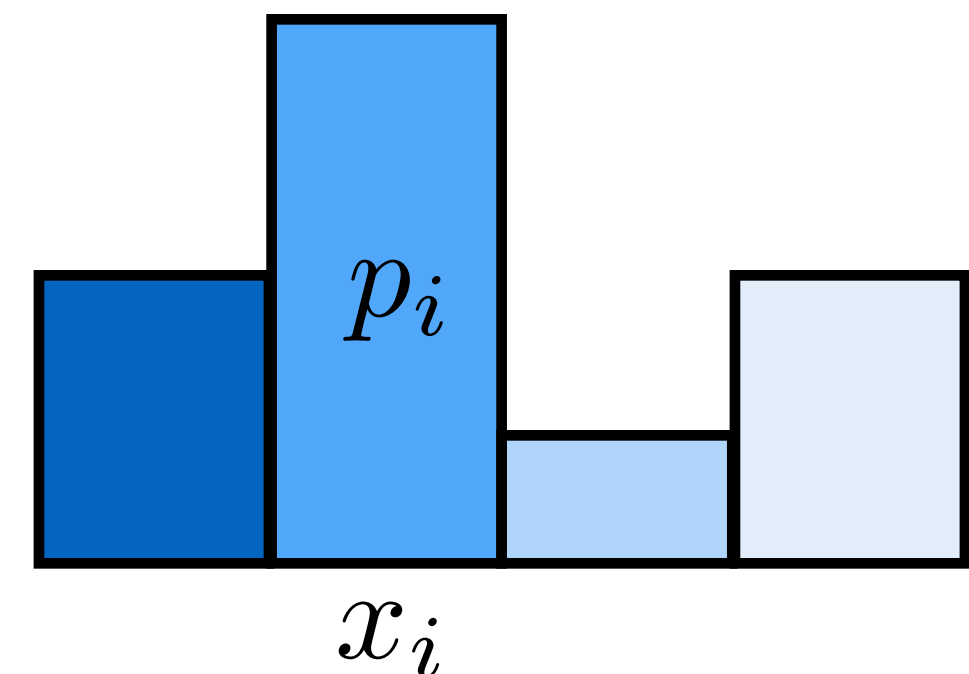
$$P_n = 1$$



# Task: Draw A Random Value From a Given PDF

Task:

Given a PDF for a discrete random variable, probability  $p_i$  for each value  $x_i$ ,

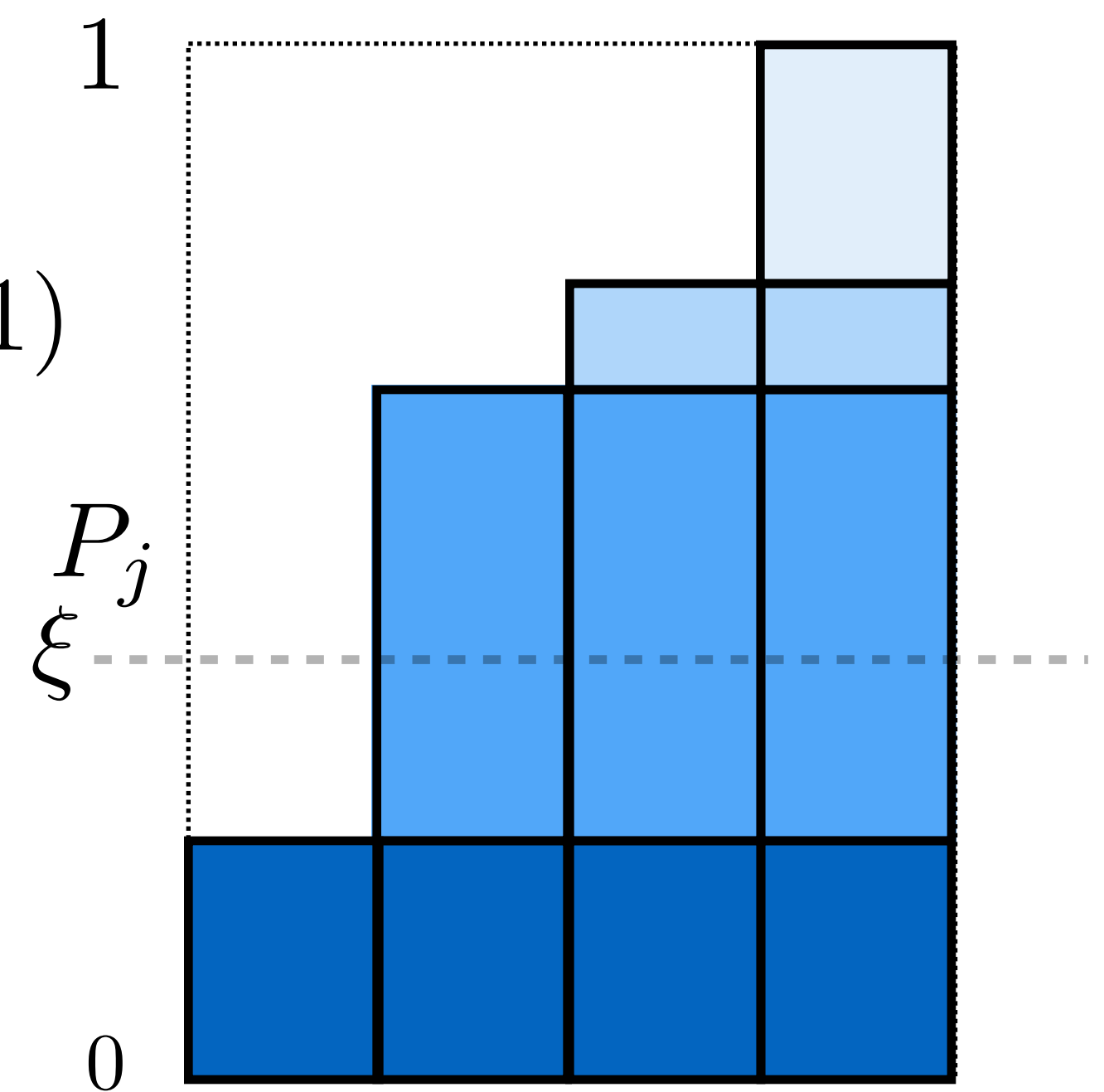


Draw a random value  $X$  from this PDF.

Step 2:

Given a uniform random variable  $\xi \in [0, 1)$

choose  $X = x_i$   
such that  $P_{i-1} < \xi \leq P_i$



How to compute? Binary search.

# Cumulative Density Function (CDF) - Continuous Case

**PDF**  $p(x)$

$$p(x) \geq 0$$

**CDF**  $P(x)$

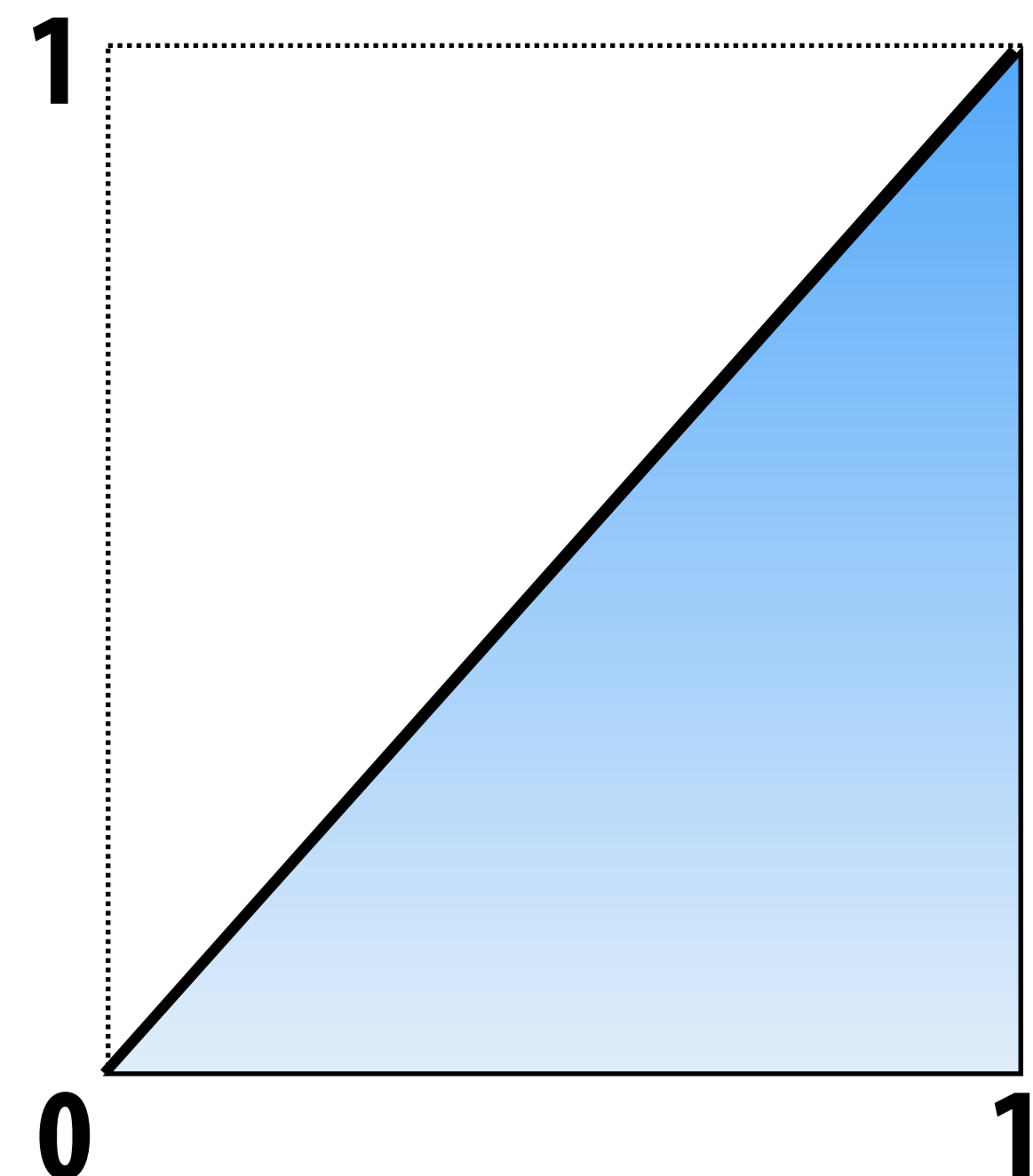
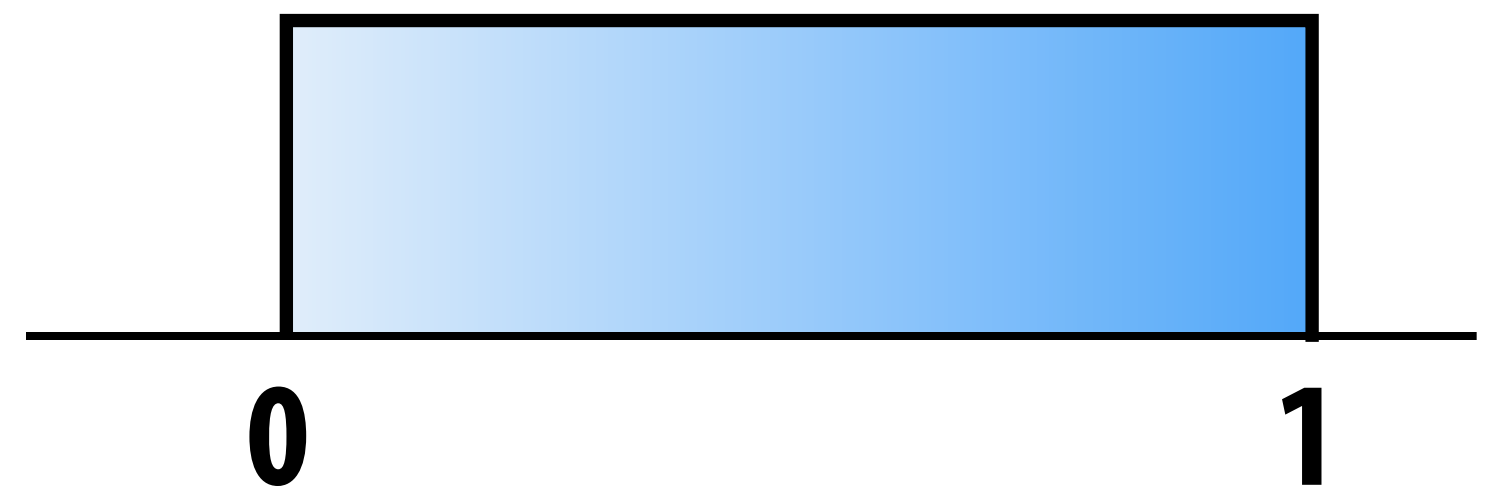
$$P(x) = \int_0^x p(x) dx$$

$$P(x) = \Pr(X < x)$$

$$P(1) = 1$$

$$\begin{aligned} \Pr(a \leq X \leq b) &= \int_a^b p(x) dx \\ &= P(b) - P(a) \end{aligned}$$

Uniform distribution  
on unit interval



# Sampling Continuous PDF Using Its CDF

Called the "inversion method"

Cumulative probability distribution function

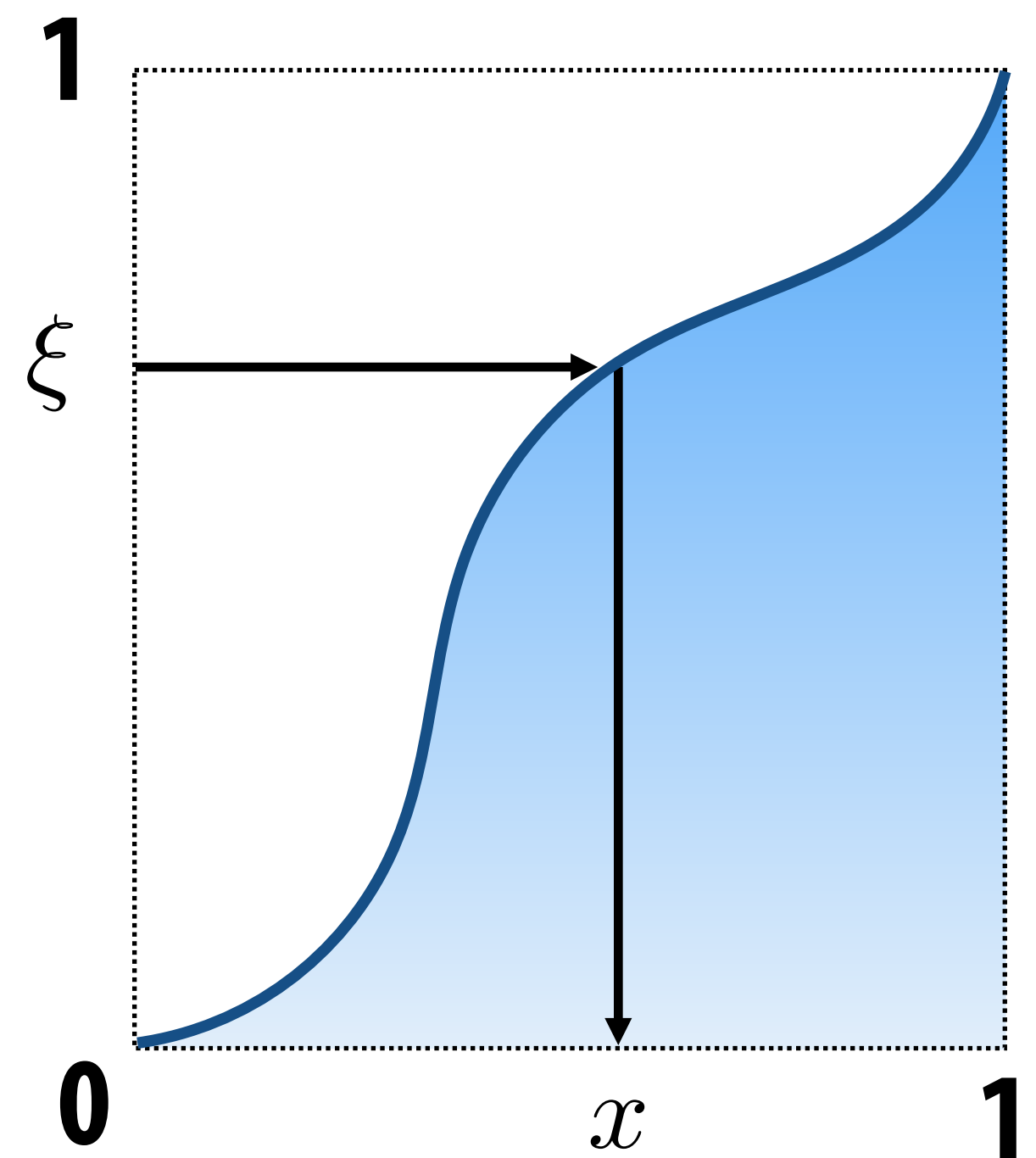
$$P(x) = \Pr(X < x)$$

Construction of samples:

$$\text{Solve for } x = P^{-1}(\xi)$$

Must know the formula for:

1. The integral of  $p(x)$  - CDF
2. The inverse function  $P^{-1}(x)$



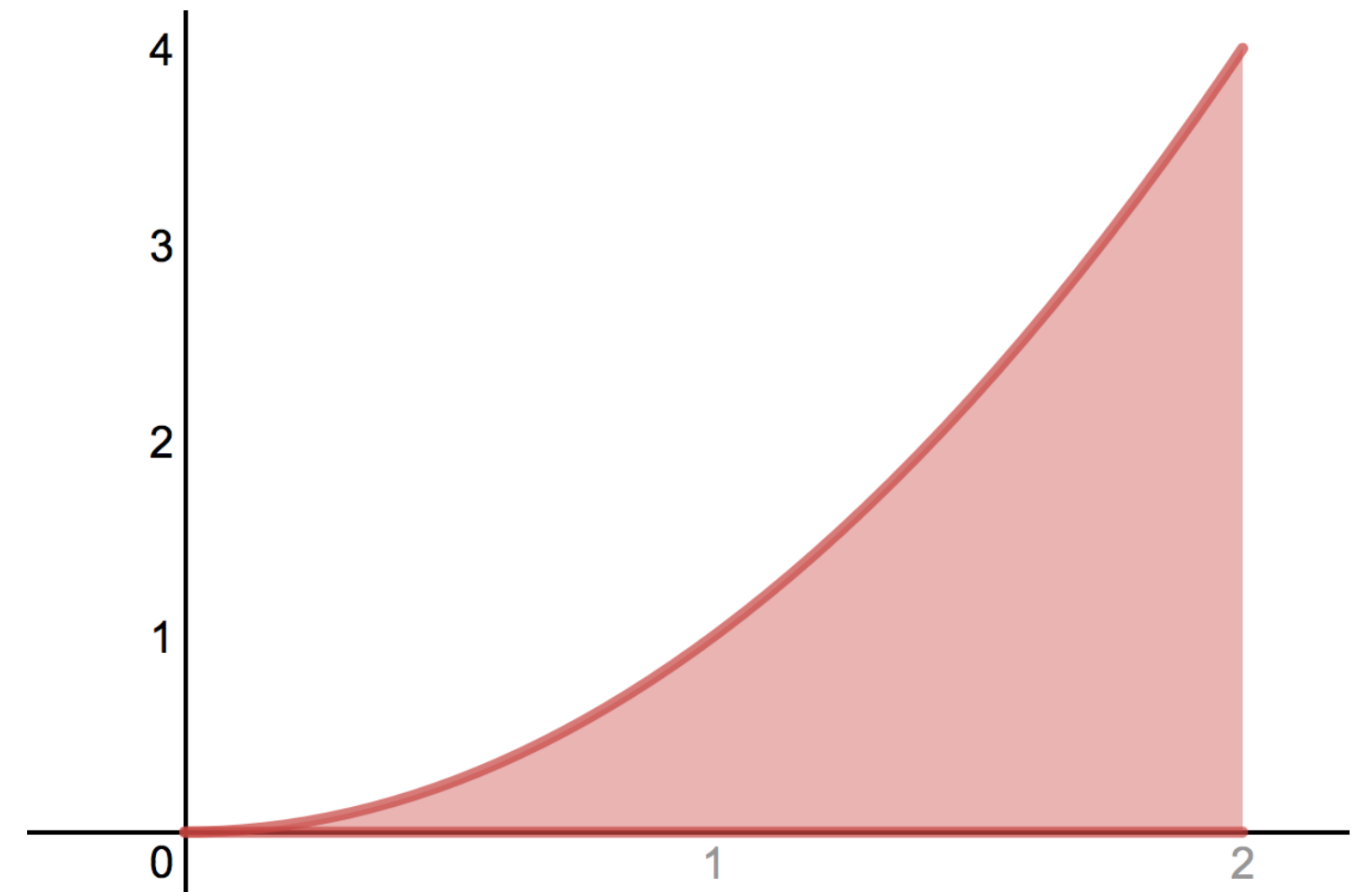


# Example: Sample Proportional to $x^2$

Given:

$$f(x) = x^2 \quad x \in [0, 2]$$

Want to sample  
according to this  
graph:

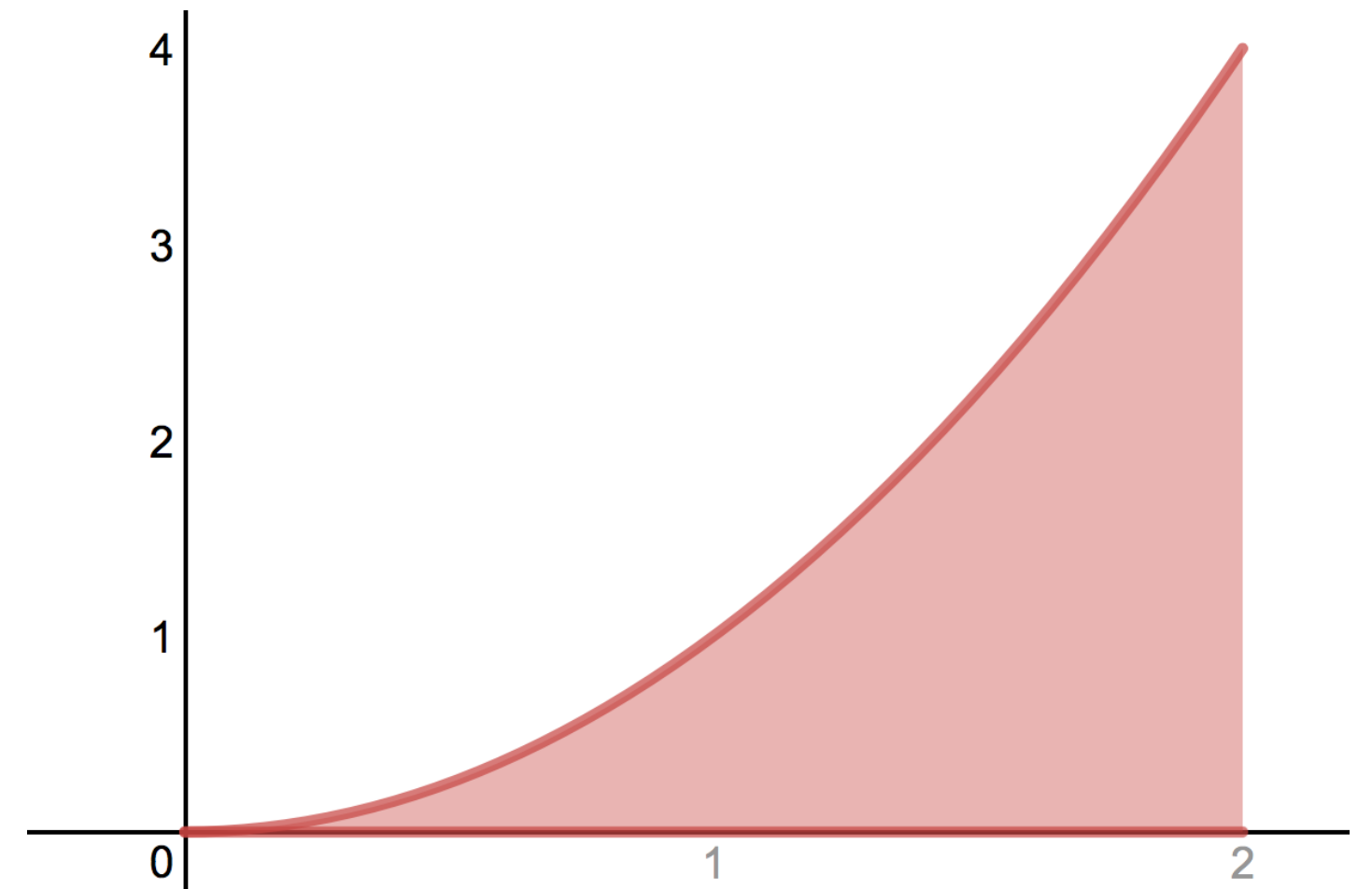


# Example: Sample Proportional to $x^2$

Given:

$$f(x) = x^2 \quad x \in [0, 2]$$

Want to sample  
according to this  
graph:



Step 0: compute PDF by normalizing

$$p(x) = c f(x) = c x^2$$

$$\text{Also } 1 = \int_0^2 p(x) dx = \int_0^2 c x^2 dx = \left. \frac{c x^3}{3} \right|_0^2 = \frac{8c}{3}$$

$$\implies c = \frac{3}{8}$$

$$\implies p(x) = \frac{3x^2}{8}$$

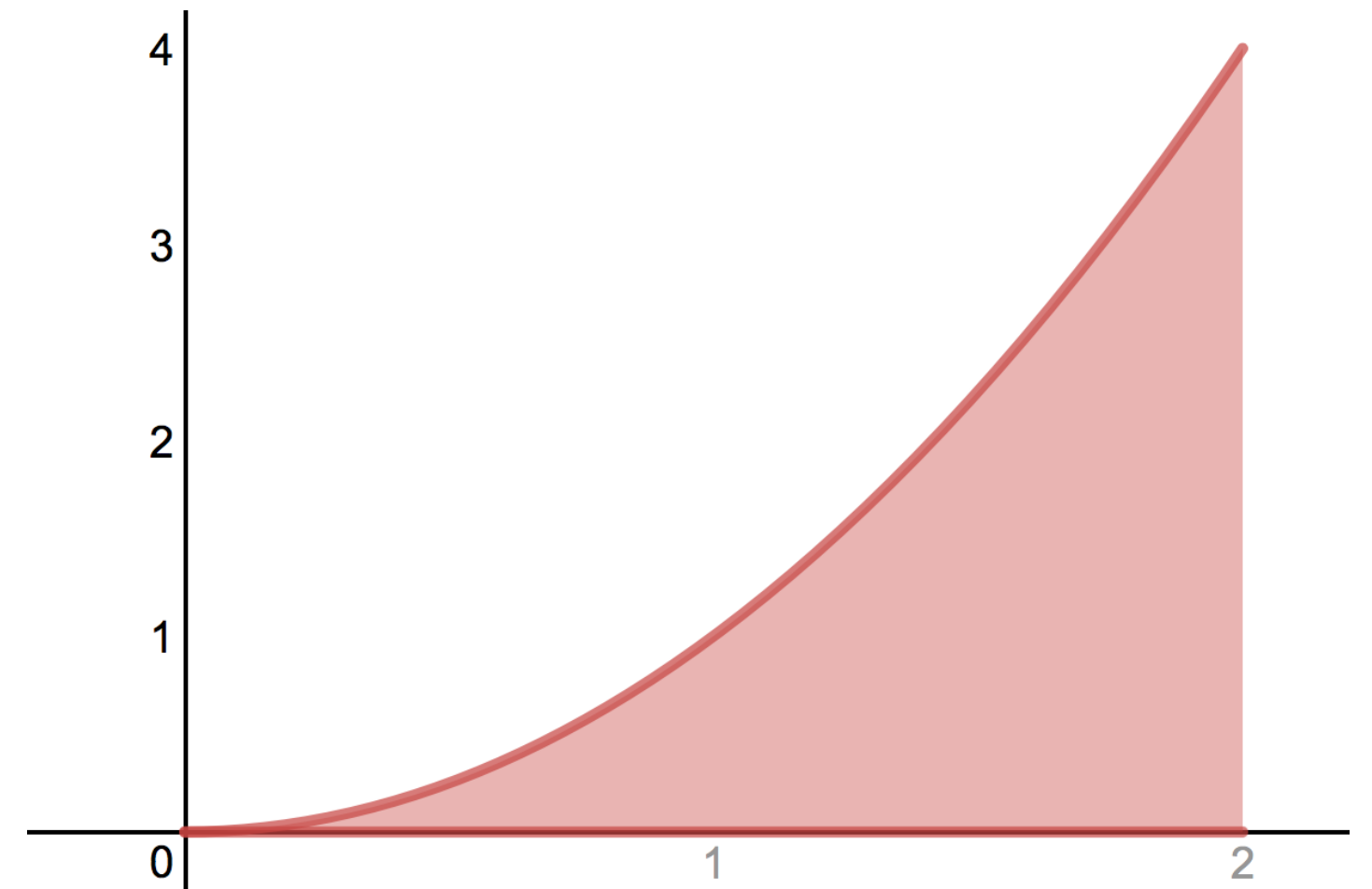
# Example: Sample Proportional to $x^2$

Given:

$$f(x) = x^2 \quad x \in [0, 2]$$

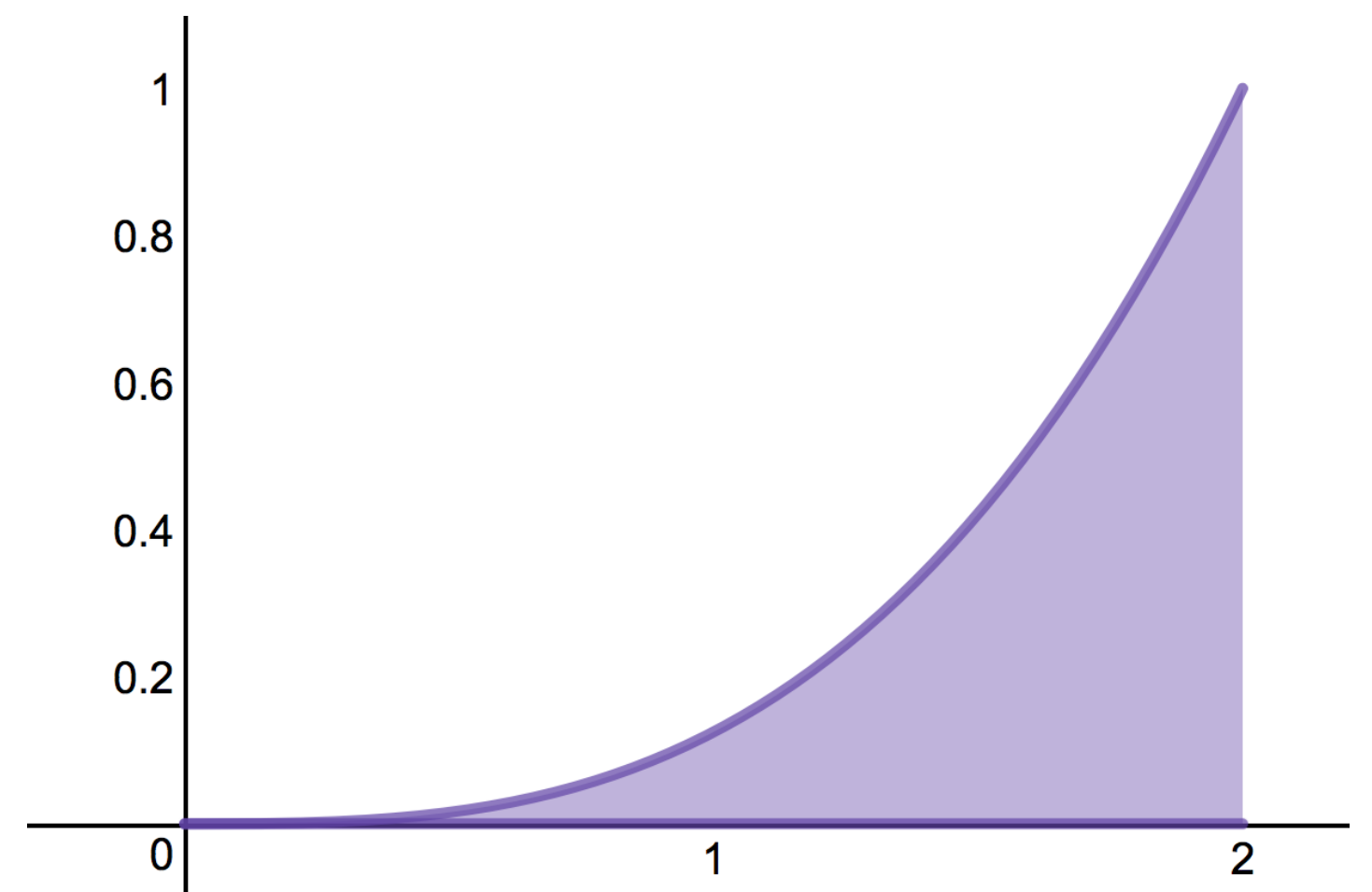
Want to sample according to this

$$\implies p(x) = \frac{3x^2}{8}$$



Step 1: Compute CDF:

$$\begin{aligned} P(x) &= \int_0^x p(x) dx \\ &= \frac{x^3}{8} \end{aligned}$$



# Example: Sample Proportional to $x^2$

Given:

$$f(x) = x^2 \quad x \in [0, 2]$$

$$p(x) = \frac{3}{8}x^2$$

$$P(x) = \frac{x^3}{8}$$

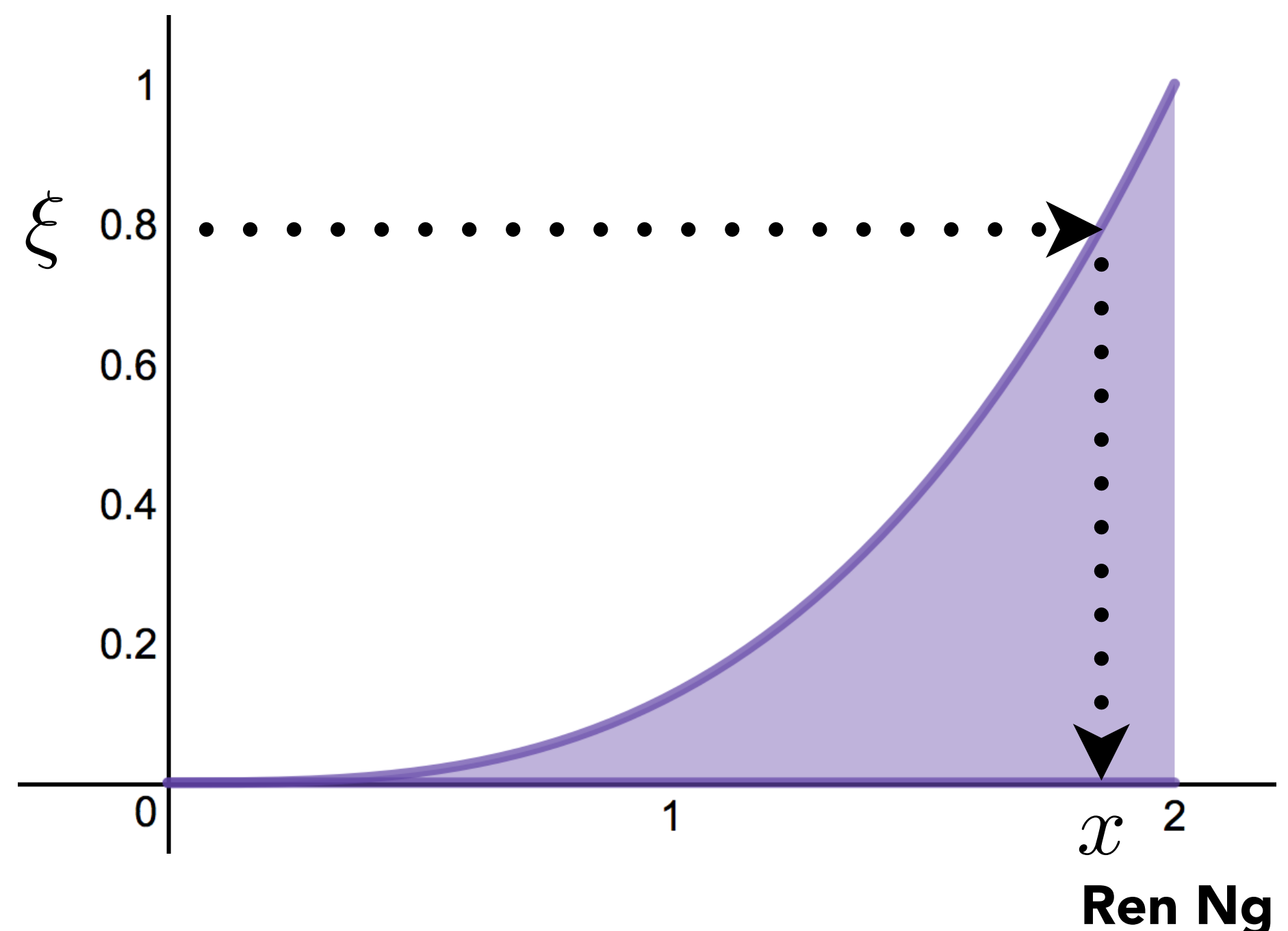
Step 2: Sample from  $p(x)$

$$\xi = P(x) = \frac{x^3}{8}$$

$$x = \sqrt[3]{8\xi}$$

## Applying the inversion method

Remember  $\xi$  is uniform random number in  $[0,1)$



# Things to Remember

## Monte Carlo integration

- Unbiased estimators
- Good for high-dimensional integrals
- Estimates are visually noisy and need many samples
- Importance sampling can reduce variance (noise) if probability distribution "fits" underlying function

## Sampling random variables

- Inversion method, rejection sampling
- Sampling in 1D, 2D, disks, hemispheres

# Acknowledgments

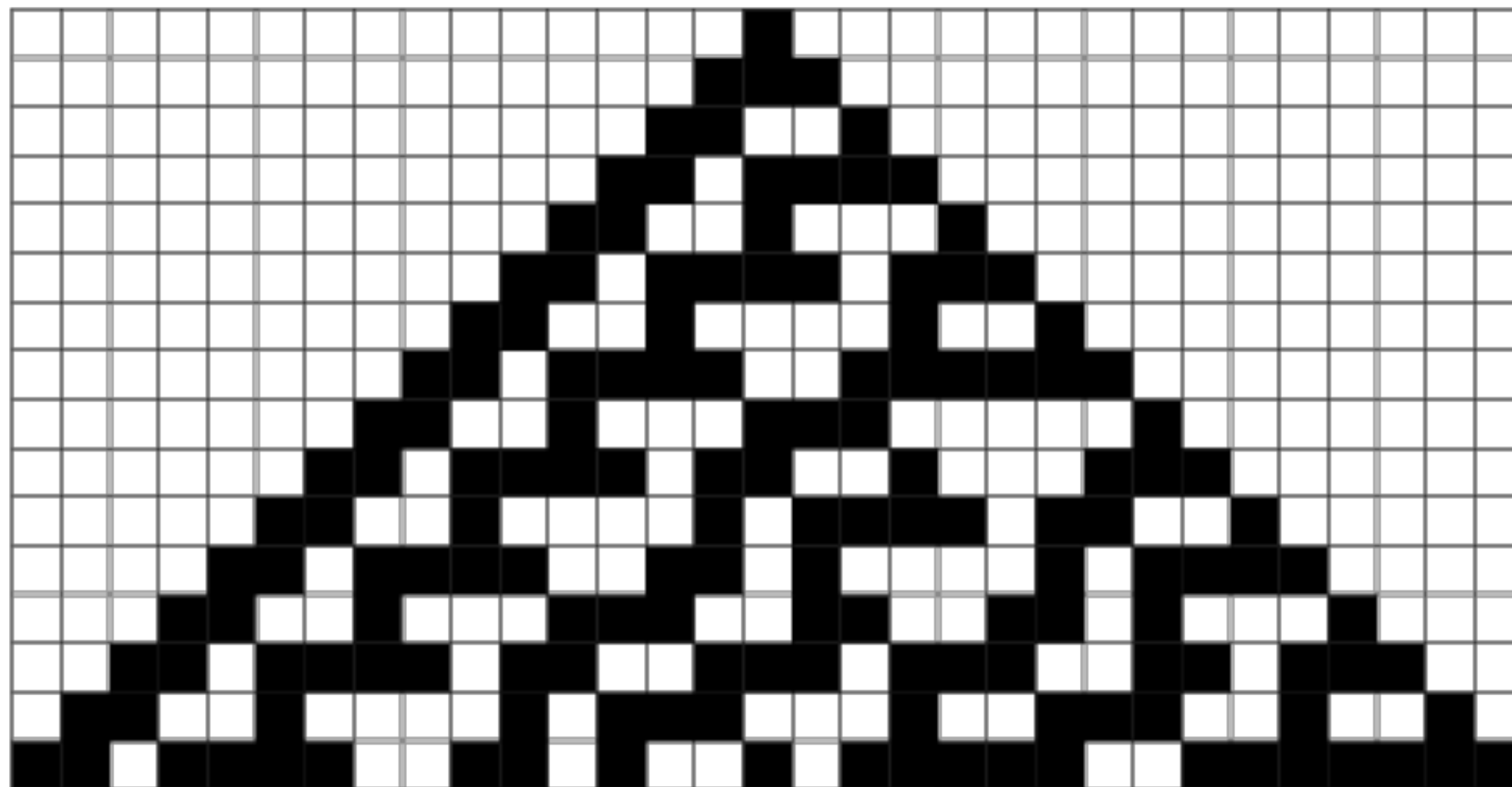
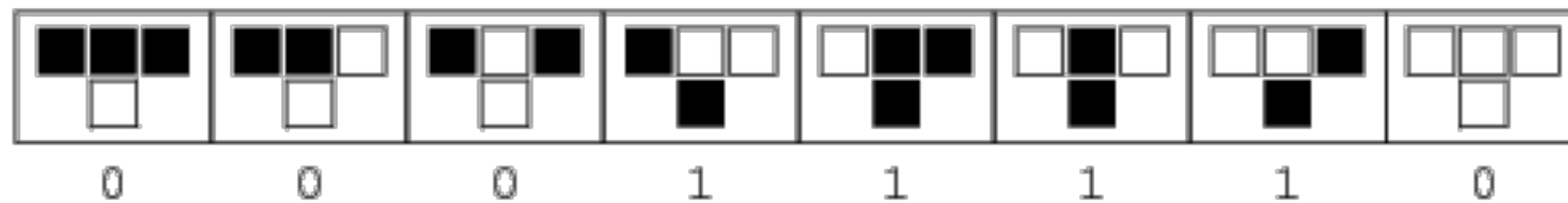
Many thanks to Kayvon Fatahalian, Matt Pharr, and Pat Hanrahan, who created the majority of these slides. Thanks also to Keenan Crane.

**Extra**

# Pseudo-Random Number Generation

Example: cellular automata #30

*rule 30*

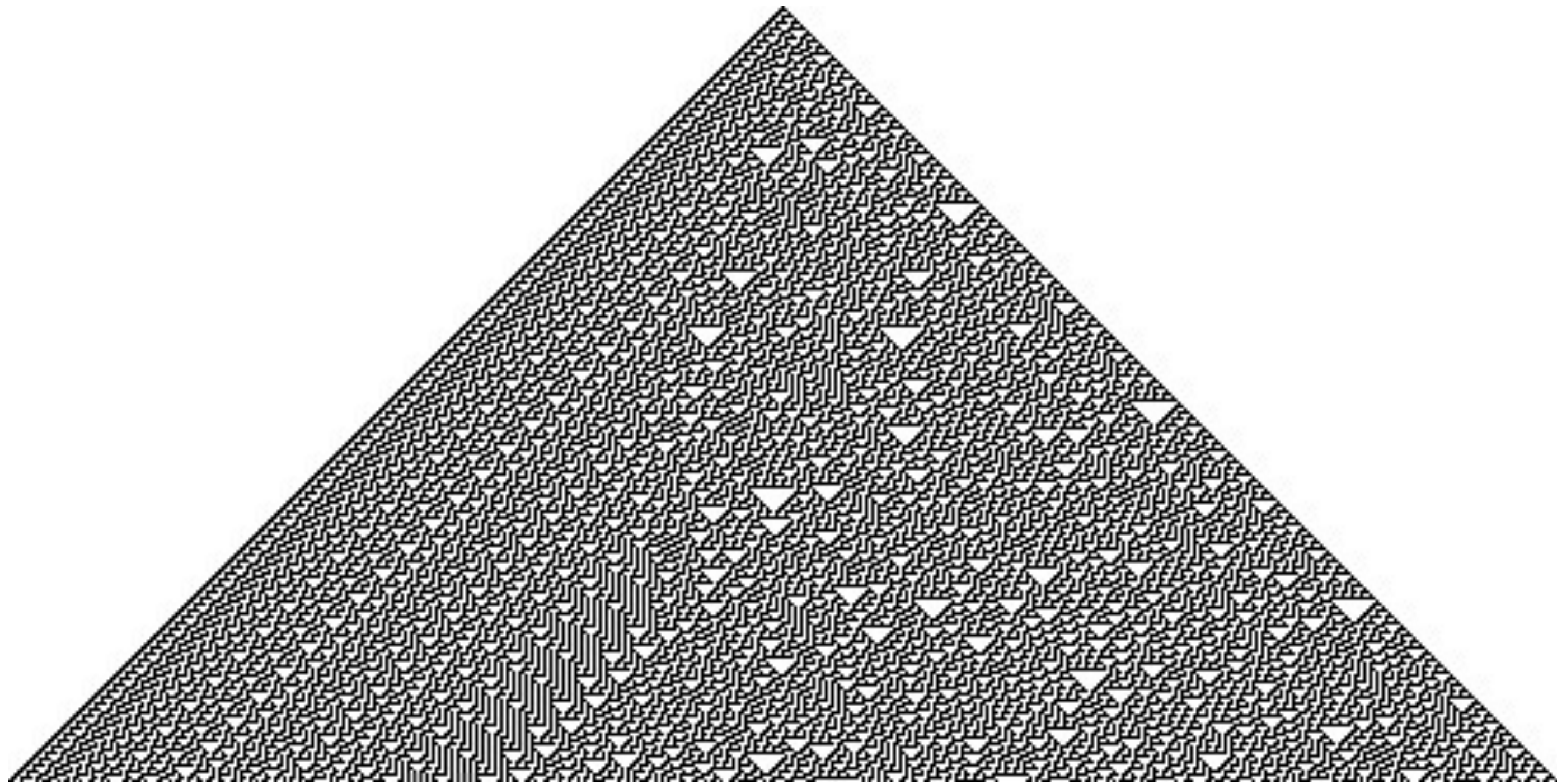


<http://mathworld.wolfram.com/Rule30.html>



# Pseudo-Random Number Generation

Example: cellular automata #30



<http://mathworld.wolfram.com/Rule30.html>

Center line values are a high-quality random bit sequence  
Once used for random number generator in Mathematica



# Pseudo-Random Number Generation



Credit: Richard Ling



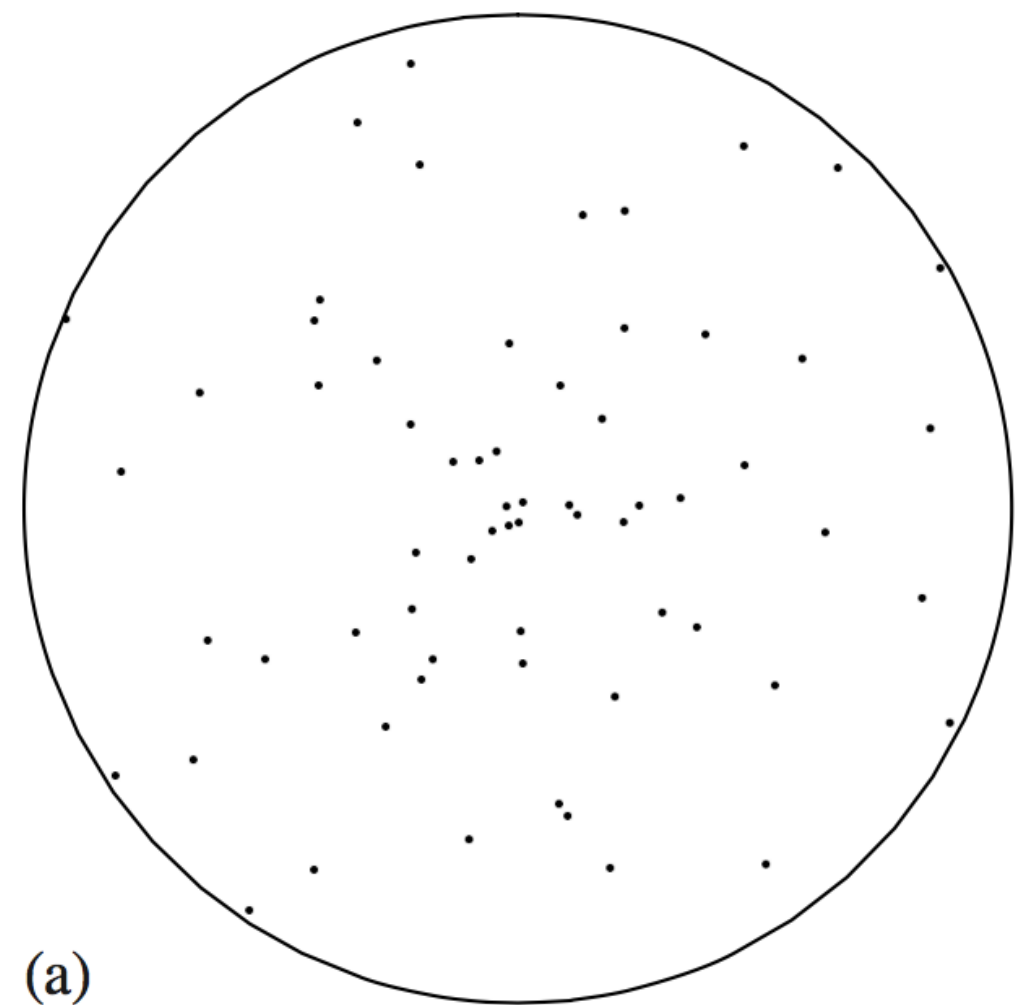
# **Random Sampling of Disks & Hemispheres**

# Sampling Unit Circle: Simple but Wrong Method

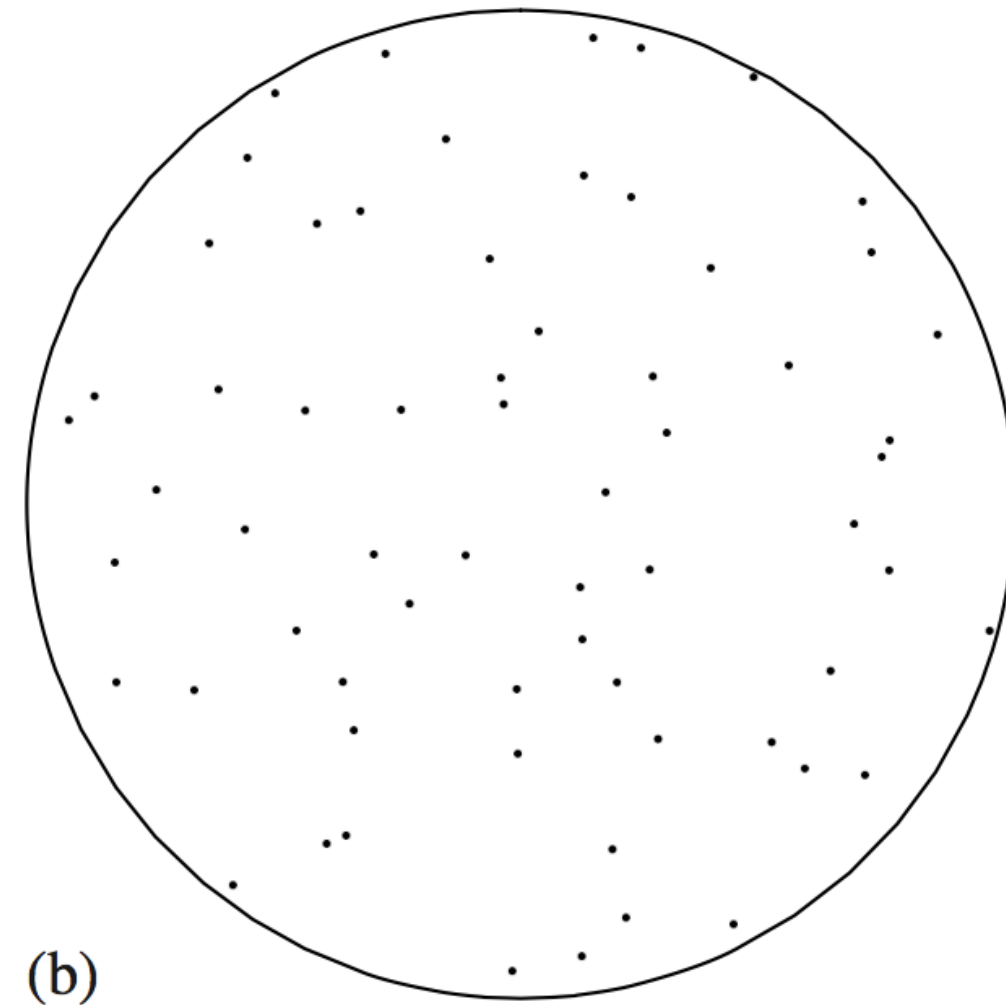
$\theta$  = uniform random angle between 0 and  $2\pi$

$r$  = uniform random radius between 0 and 1

Return point:  $(r \cos \theta, r \sin \theta)$



**Result**

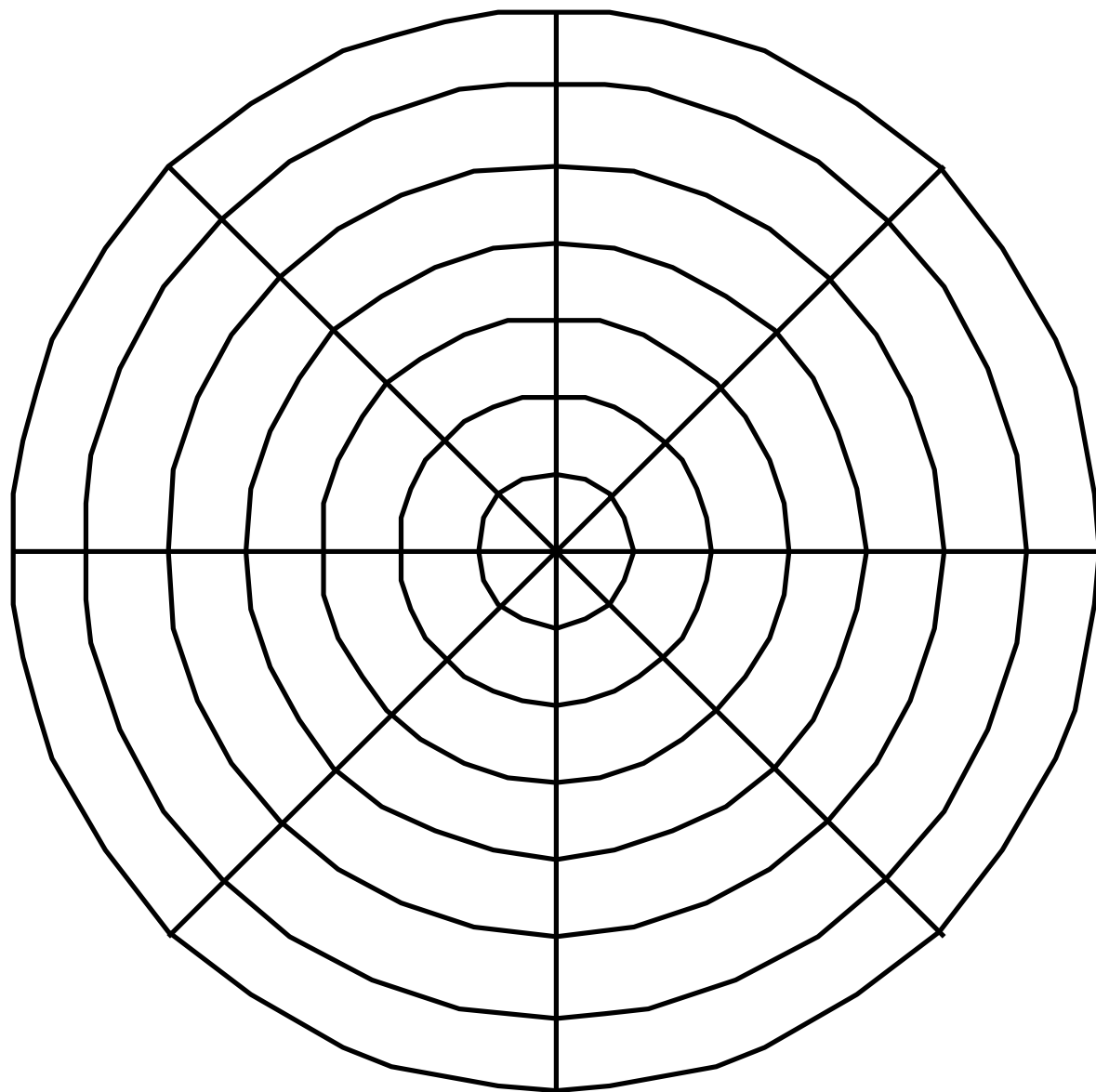


**Reference**

Source: [PBRT]

# Need to Sample Uniformly in Area

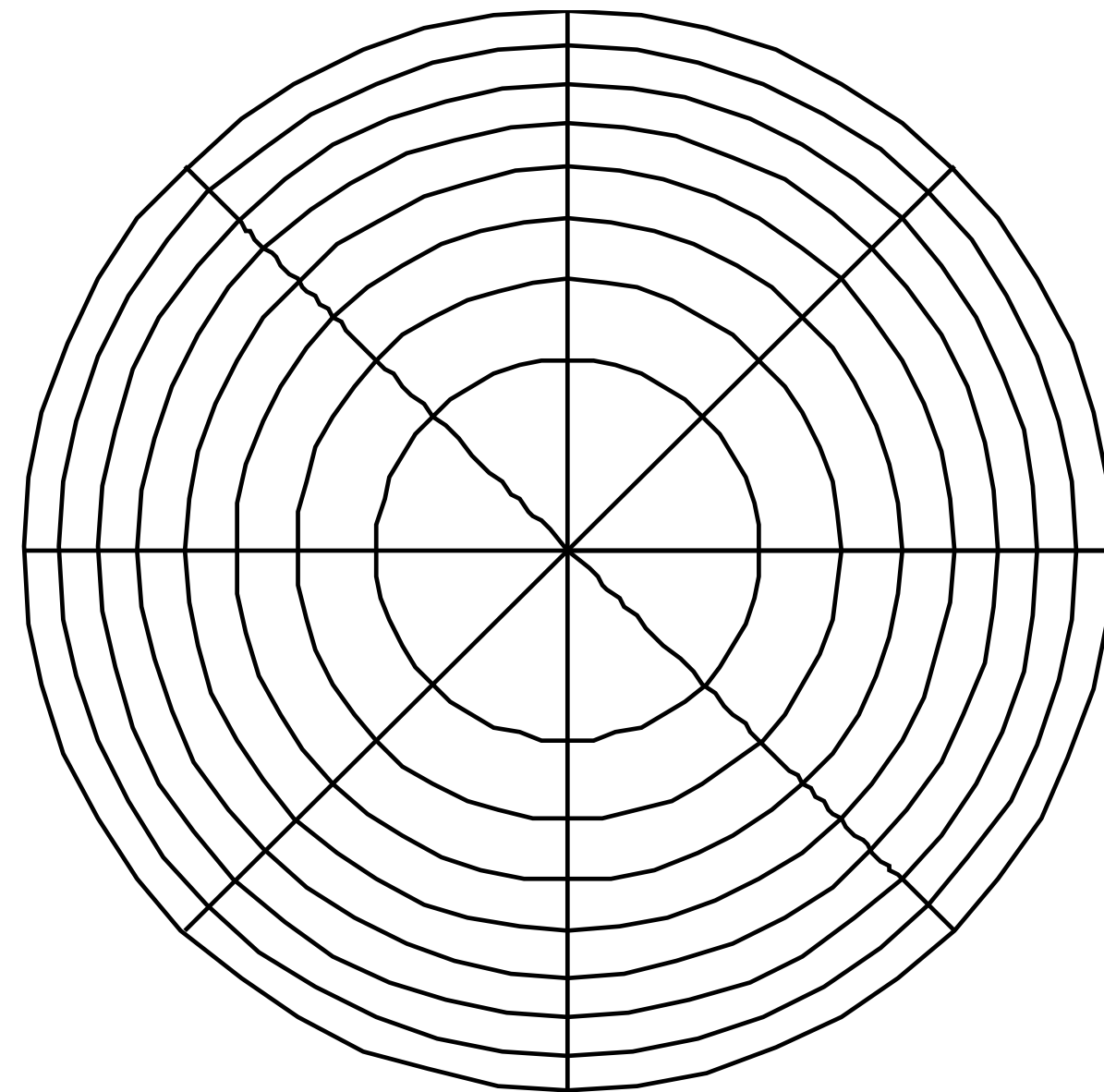
**Incorrect  
Not Equi-area**



$$\theta = 2\pi\xi_1$$

$$r = \xi_2$$

**Correct  
Equi-area**



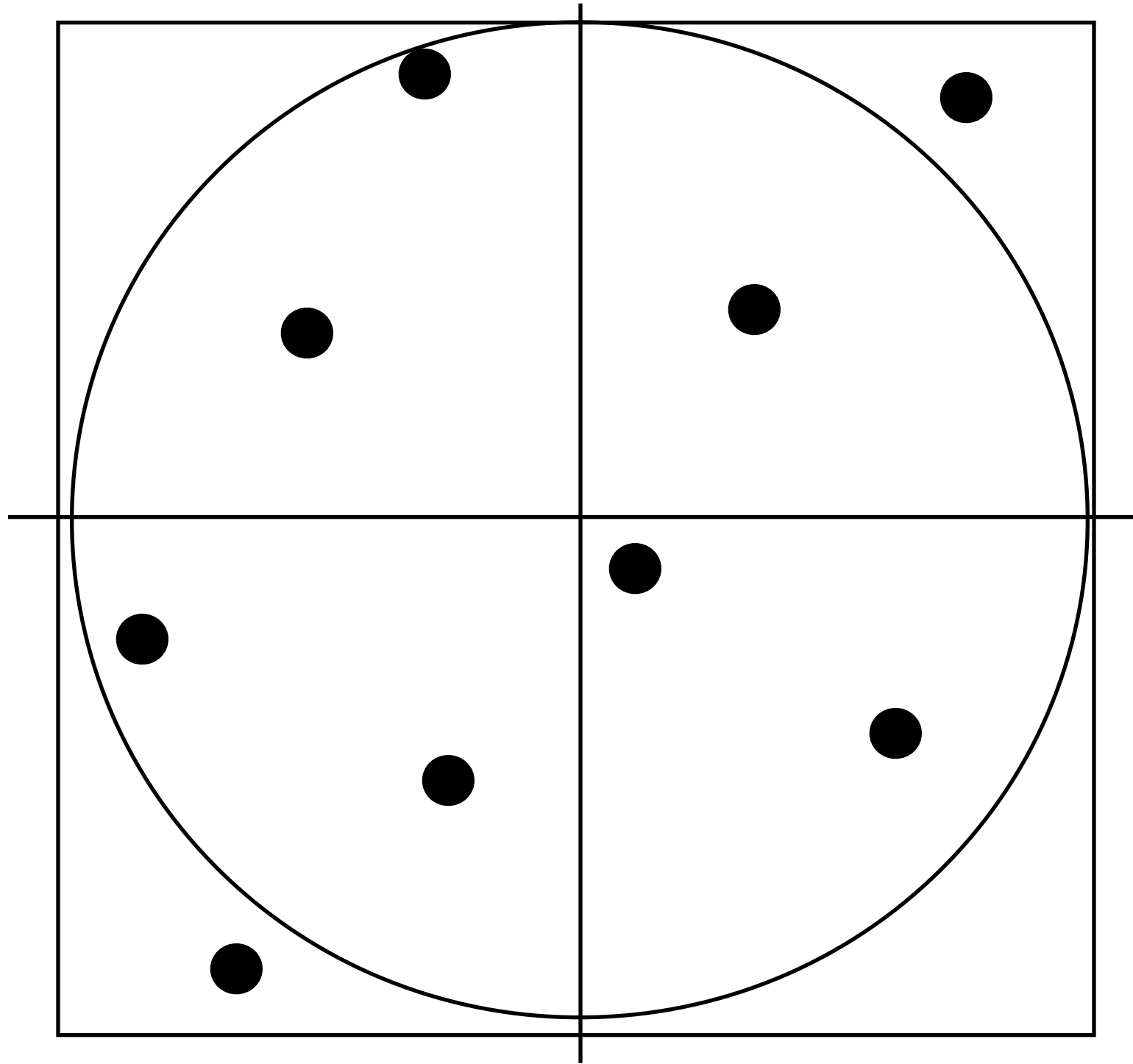
$$\theta = 2\pi\xi_1$$

$$r = \sqrt{\xi_2}$$

**\* See Shirley et al. p.331 for full explanation using inversion method**



# Rejection Sampling Circle's Area



```
do {  
    x = 1 - 2 * rand01();  
    y = 1 - 2 * rand01();  
} while (x*x + y*y > 1.);
```

Efficiency of technique: area of circle / area of square

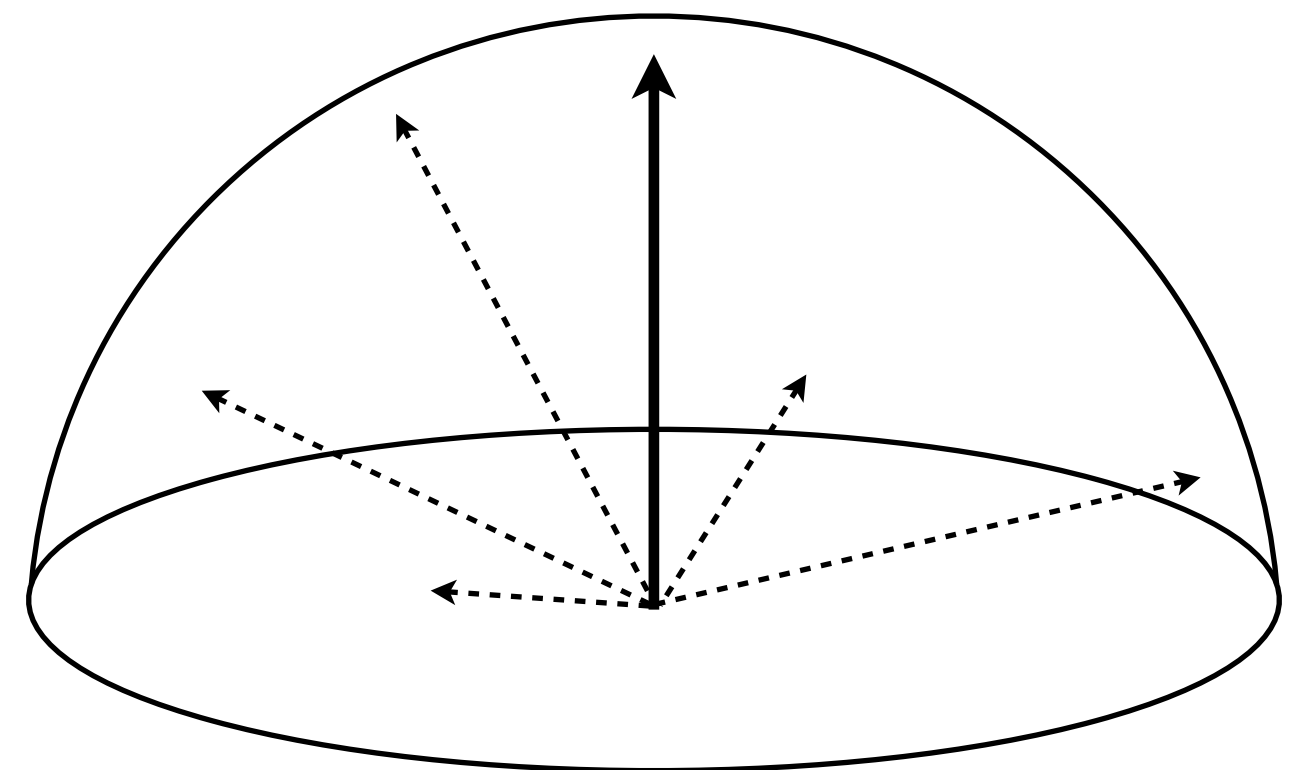
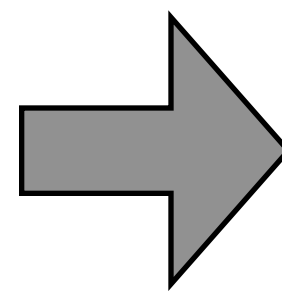
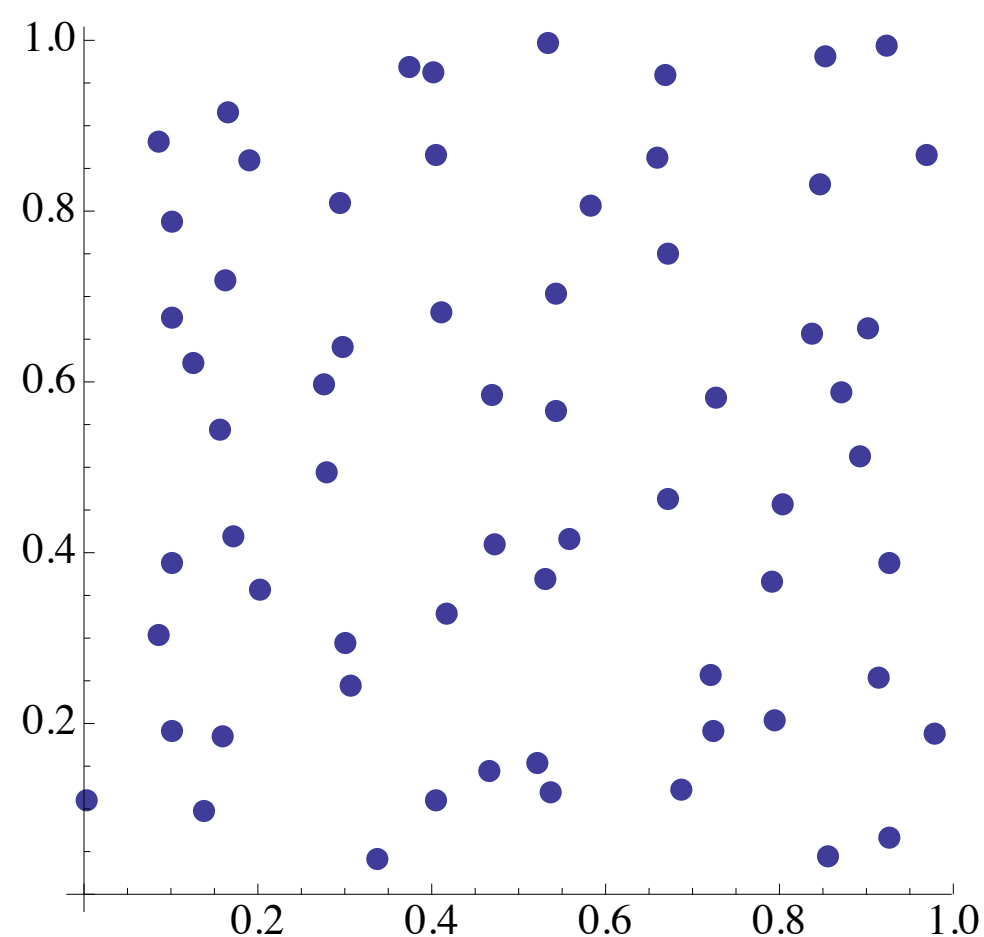
# Uniform Sampling of Hemisphere

Generate random direction on hemisphere (all dirs. equally likely)

$$p(\omega) = \frac{1}{2\pi}$$

Direction computed from uniformly distributed point on 2D square:

$$(\xi_1, \xi_2) \rightarrow \left( \sqrt{1 - \xi_1^2} \cos(2\pi\xi_2), \sqrt{1 - \xi_1^2} \sin(2\pi\xi_2), \xi_1 \right)$$



Full derivation: see PBRT 3rd Ed. 13.6.1