

Course Roadmap

Rasterization Pipeline

Core Concepts

- Sampling
- Antialiasing
- Transforms

Intro

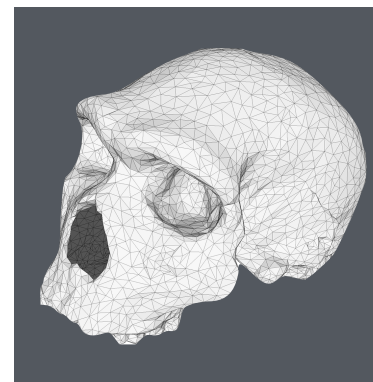
Rasterization

Transforms & Projection

Texture Mapping

Visibility, Shading, Overall Pipeline

Geometric Modeling



← Starting today

Lighting & Materials



Cameras & Imaging



Lecture 7:

Introduction to Geometry, Splines and Bezier Curves

**Computer Graphics and Imaging
UC Berkeley CS184/284A**

Examples of Geometry



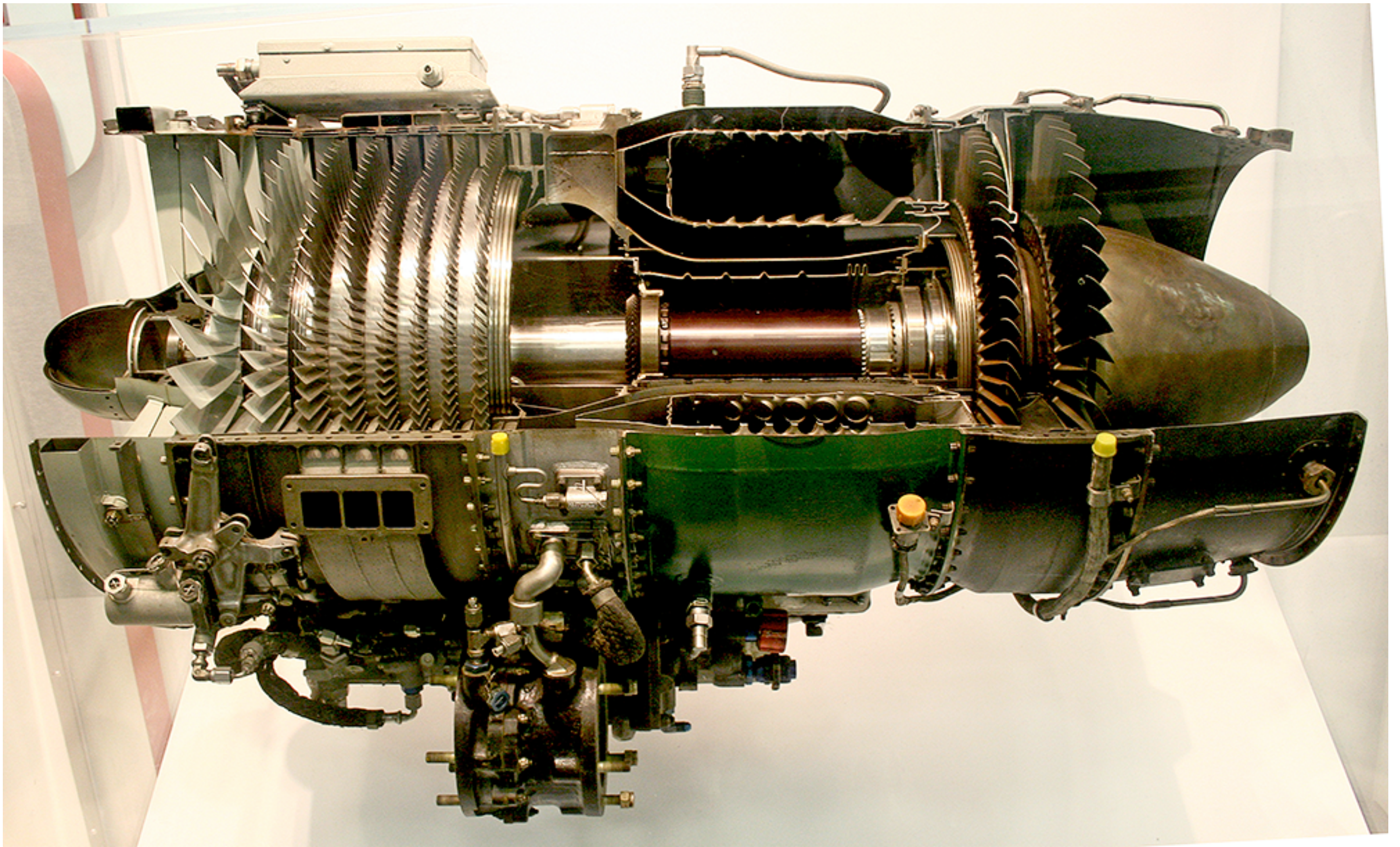
Examples of Geometry



CS184/284A

Ren Ng, James O'Brien

Examples of Geometry



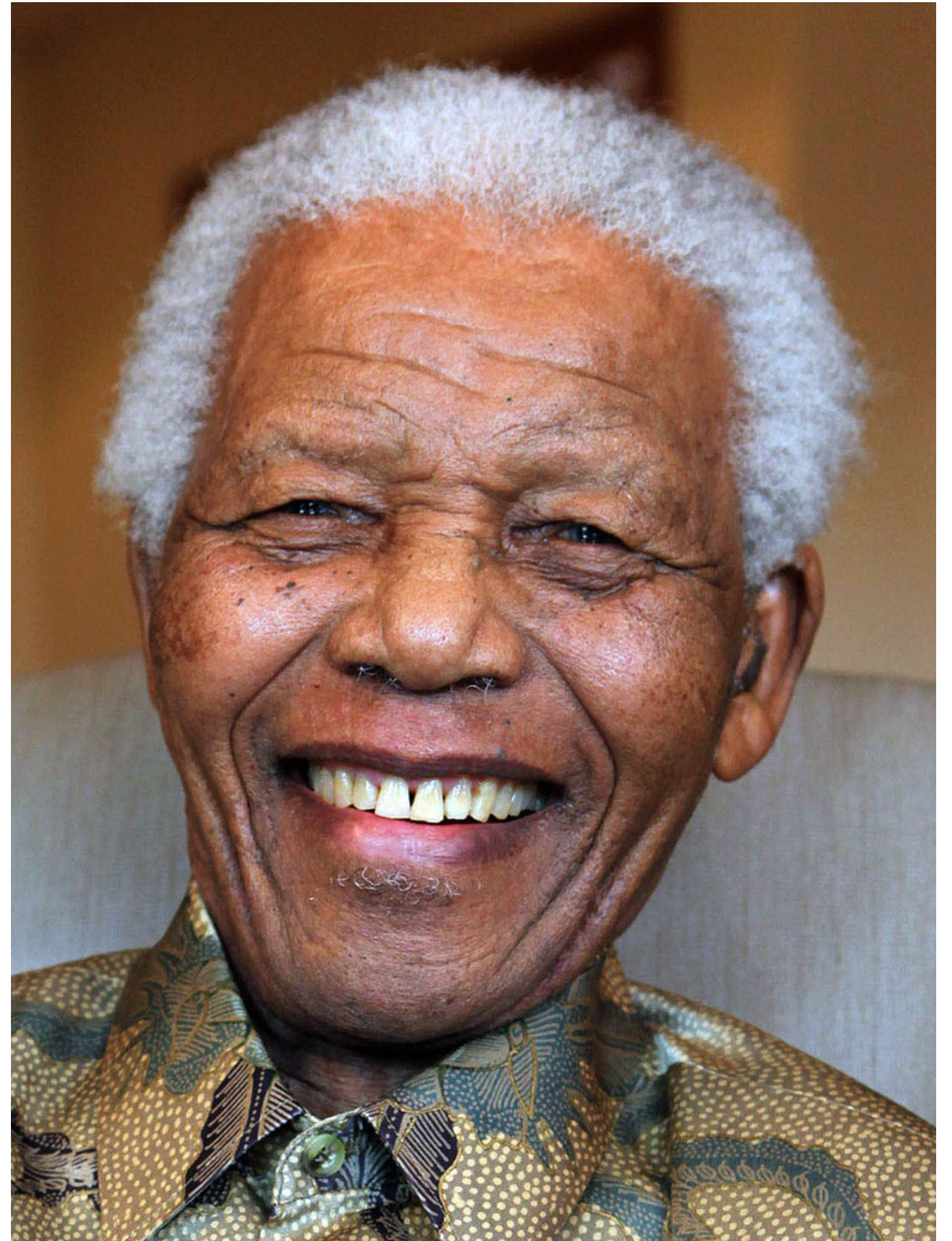
CS184/284A

Ren Ng, James O'Brien

Examples of Geometry



CS184/284A



Ren Ng, James O'Brien

Examples of Geometry



Examples of Geometry



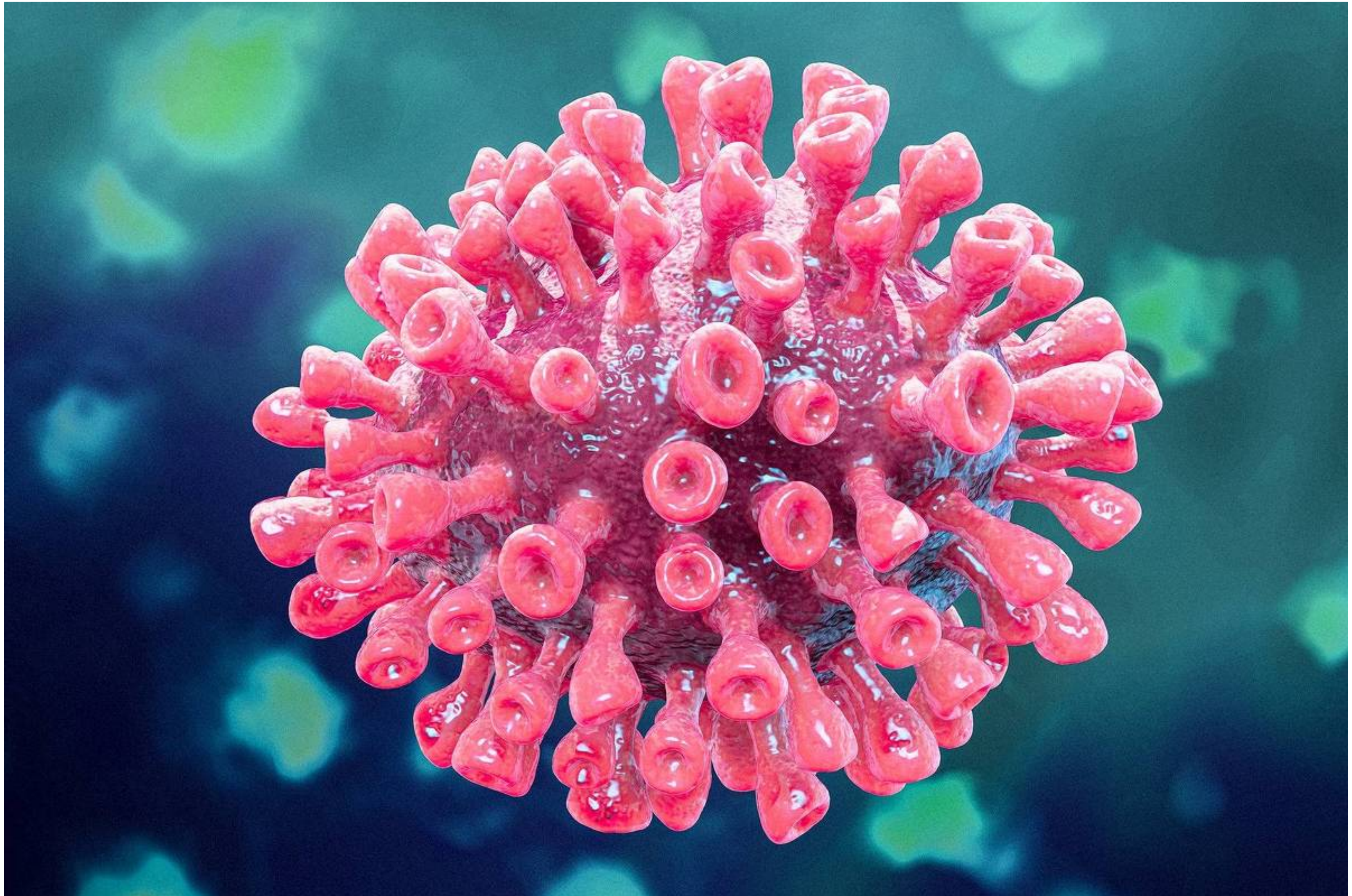
Examples of Geometry



Examples of Geometry



Examples of Geometry

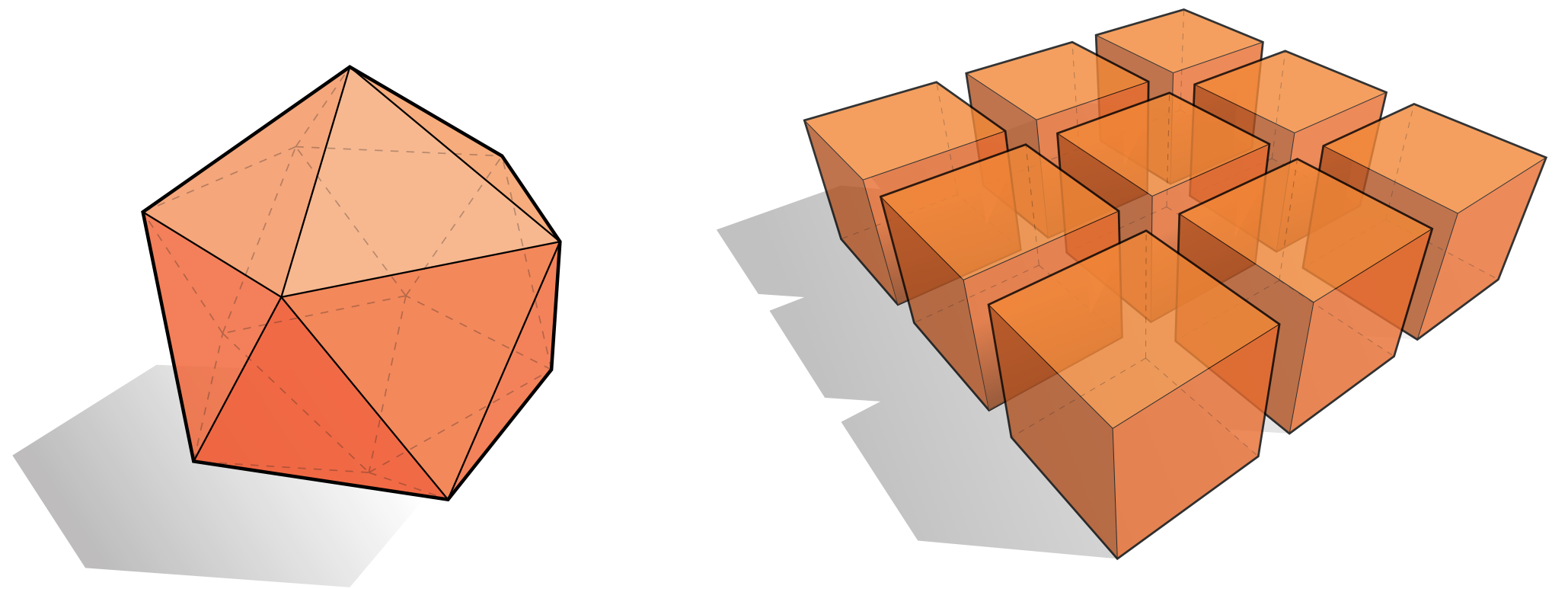


The Lancet

Many Ways to Represent Geometry

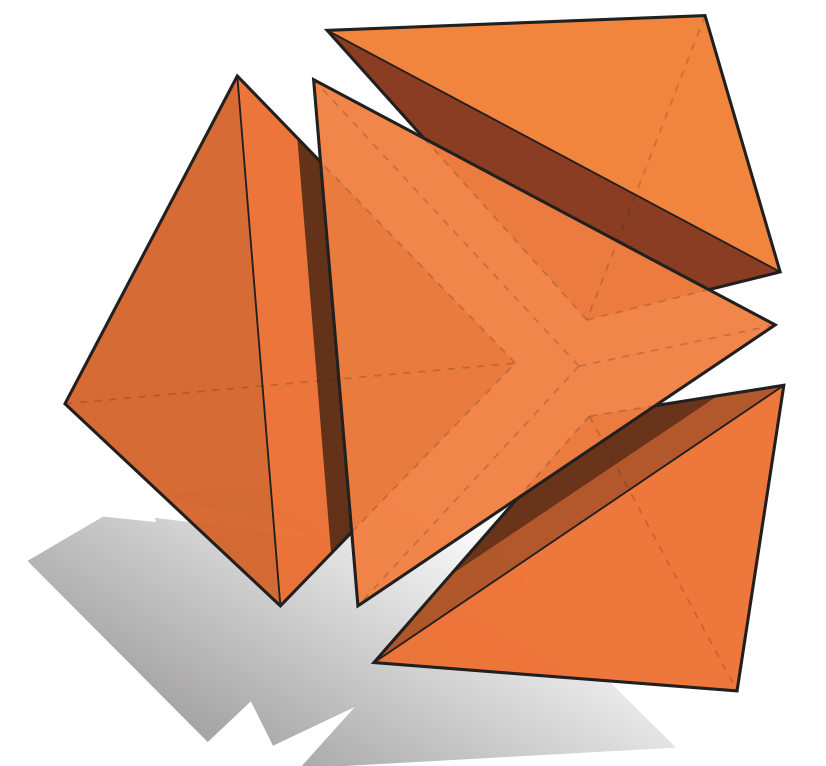
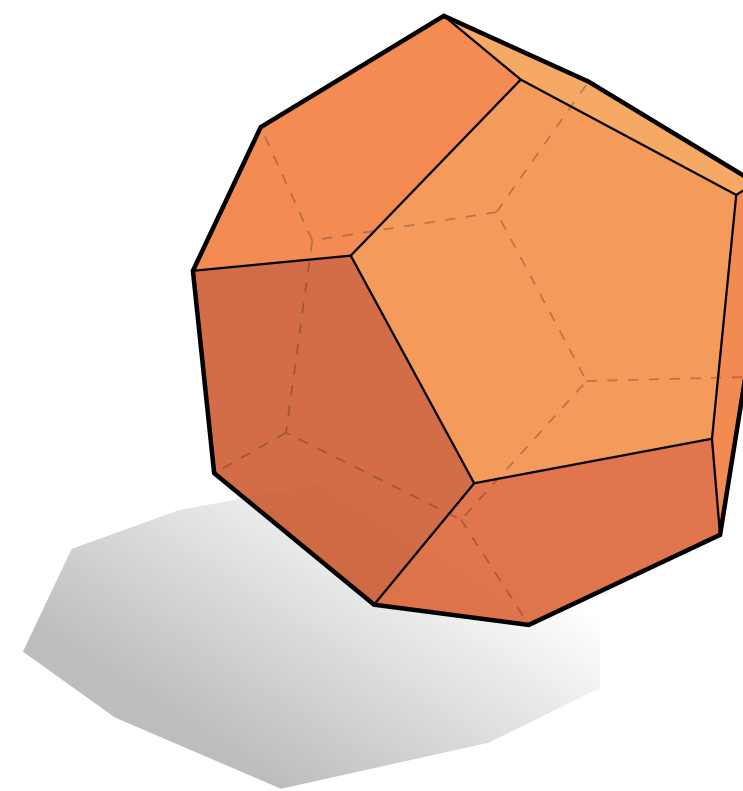
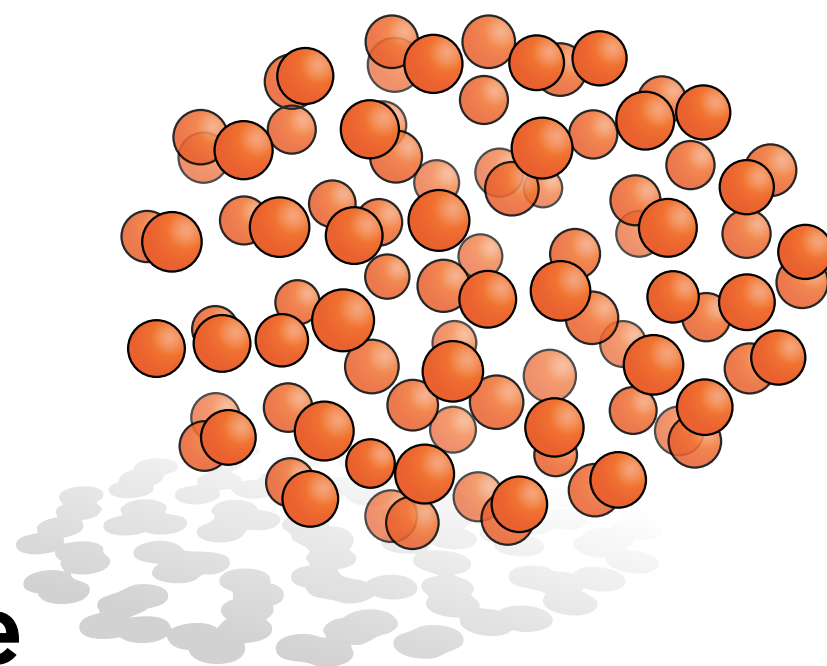
Explicit

- point cloud
- polygon mesh
- subdivision, NURBS
- ...



Implicit

- level sets
- algebraic surface
- distance functions
- ...



Each choice best suited to a different task/type of geometry

Smooth Curves

Smooth Curves and Surfaces

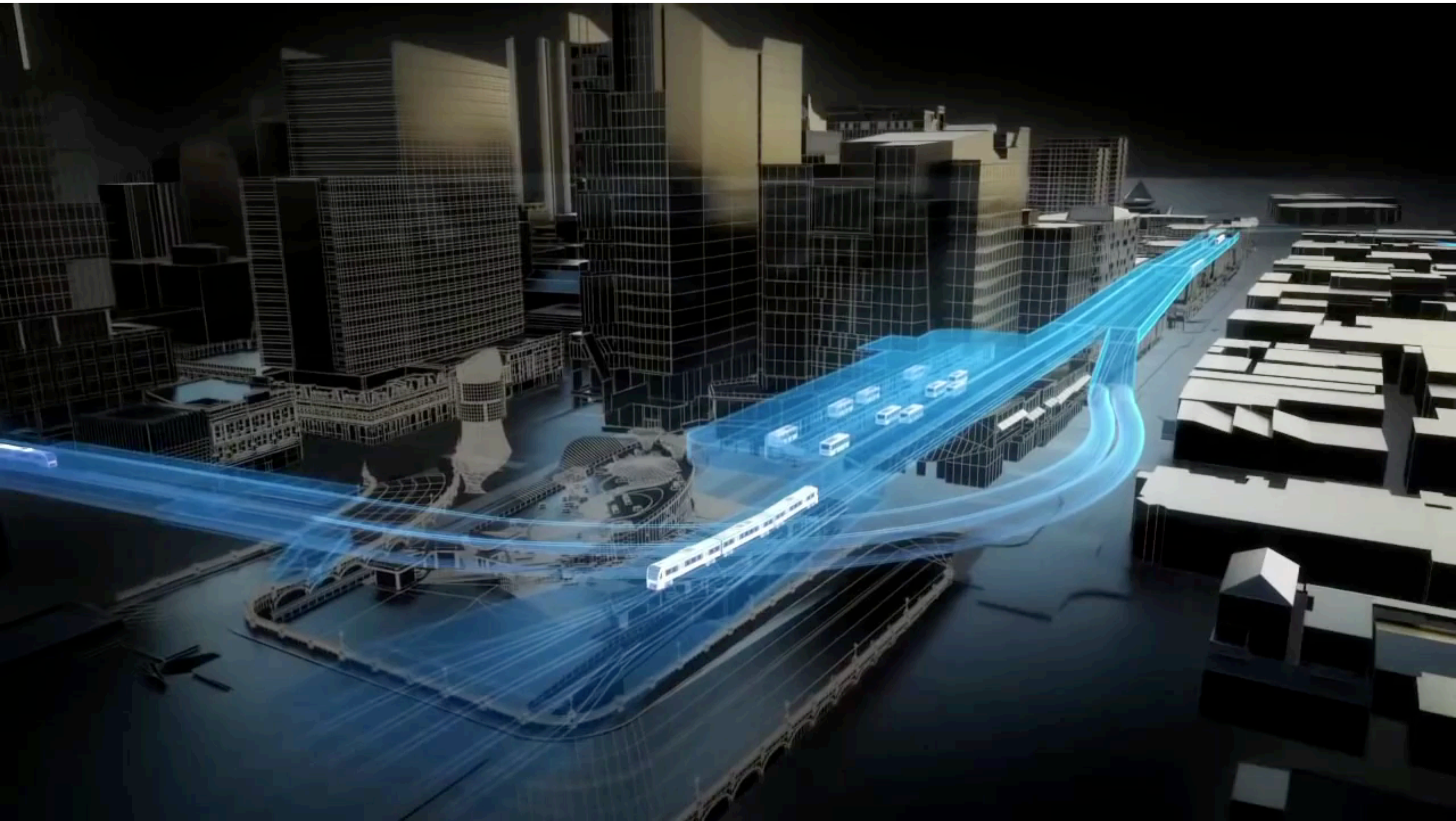
So far we can make:

- Things with corners (lines, triangles, squares, ...)
- Specialty shapes (circles, ellipses, ...)

Many applications require designed, smooth shapes

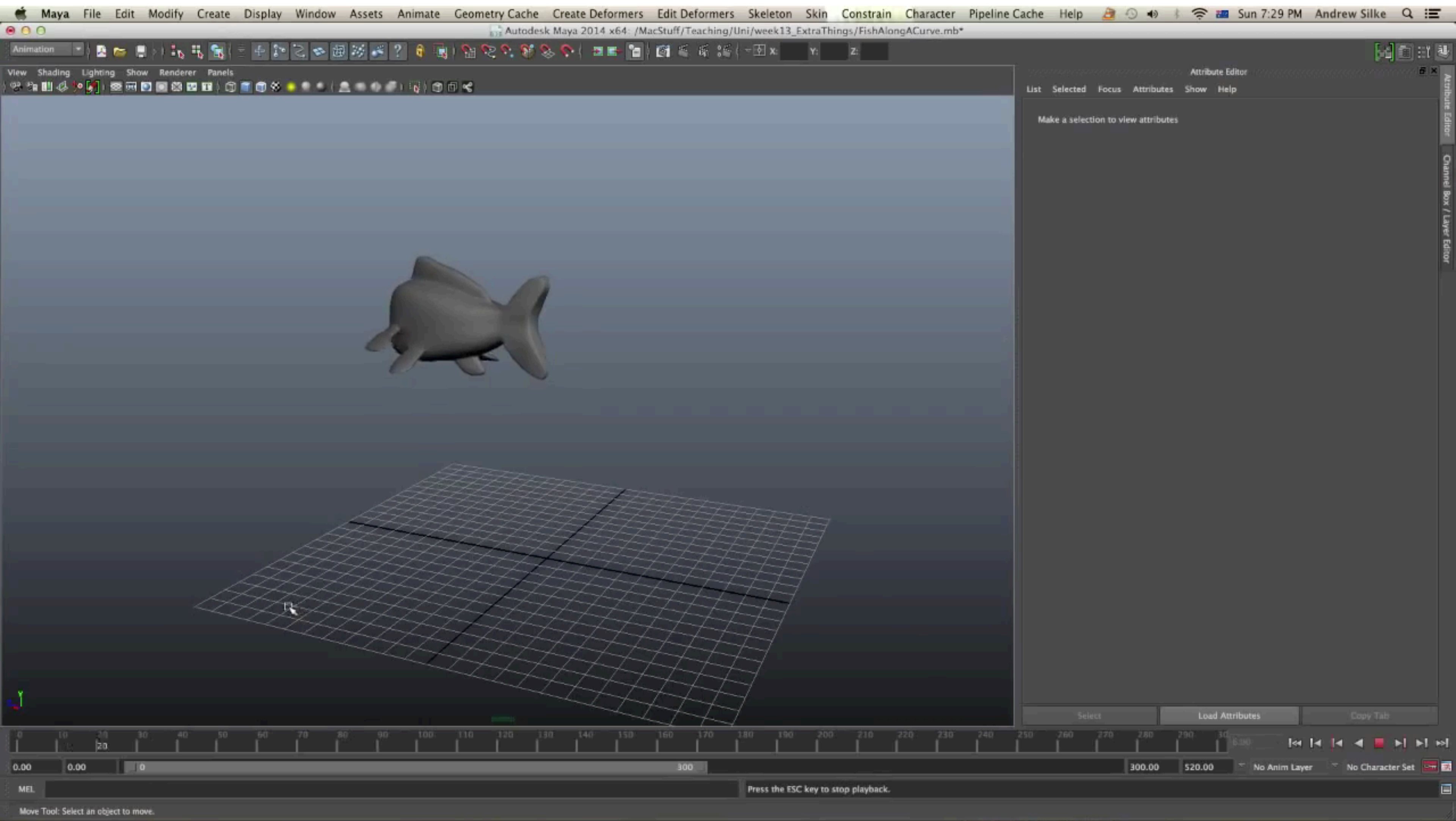
- Camera paths, vector fonts, ...
- Resampling filter functions
- CAD design, object modeling, ...

Camera Paths



Flythrough of proposed Perth Citylink subway, <https://youtu.be/rIJMuQPwr3E>

Animation Curves

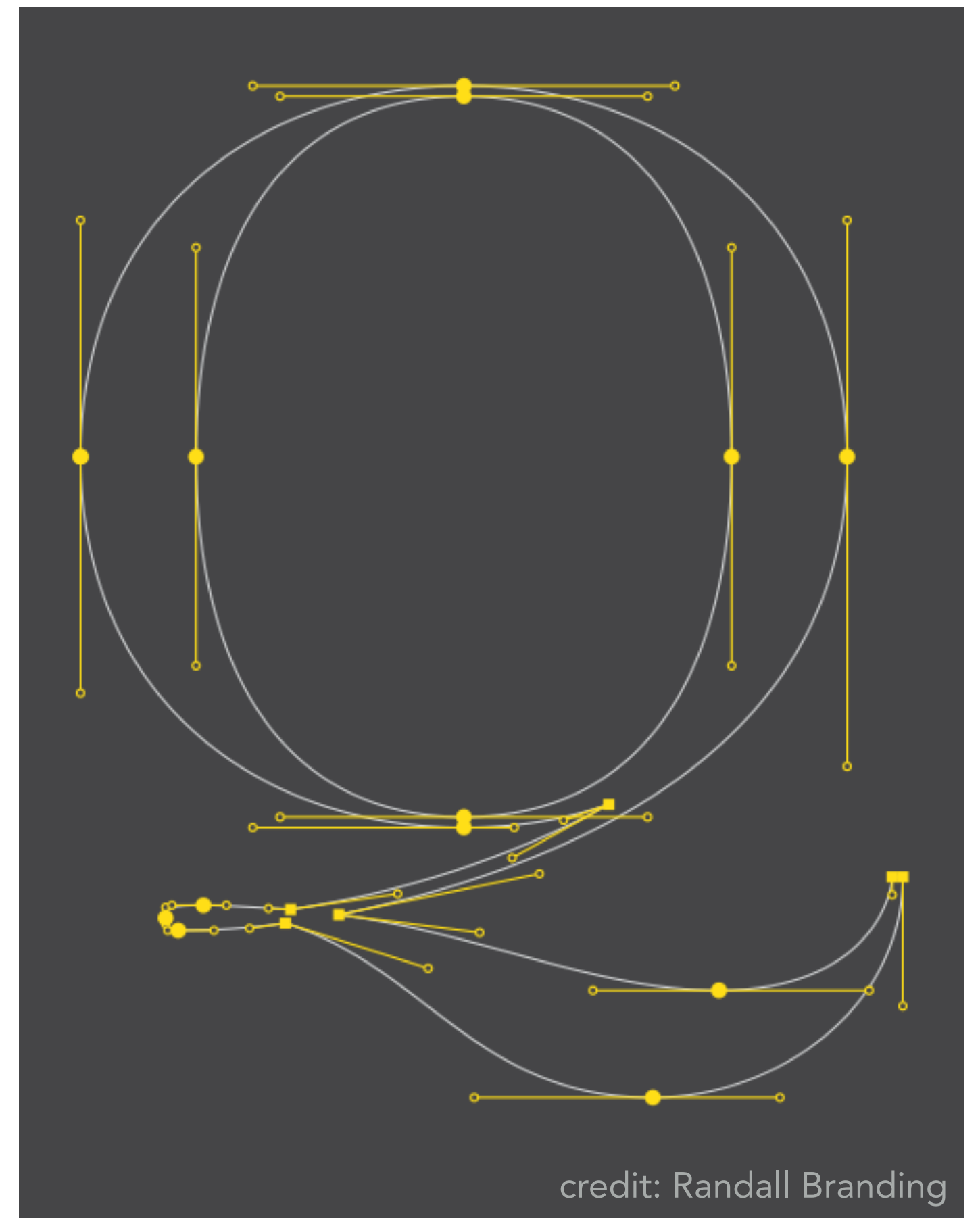


Maya Animation Tutorial: <https://youtu.be/b-o5wtZlJPc>

Vector Fonts

The Quick Brown
Fox Jumps Over
The Lazy Dog

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 0123456789



Baskerville font - represented as cubic Bézier splines

CAD Design



3D Car Modeling with Rhinoceros

Parametric Representations

Curves: $\mathbf{x} = \mathbf{x}(u)$ $\mathbf{x} \in \mathbb{R}^n$ $u \in \mathbb{R}$

Surfaces: $\mathbf{x} = \mathbf{x}(u, v)$ $\mathbf{x} \in \mathbb{R}^n$ $u, v \in \mathbb{R}$
 $\mathbf{x} = \mathbf{x}(\mathbf{u})$ $\mathbf{u} \in \mathbb{R}^2$

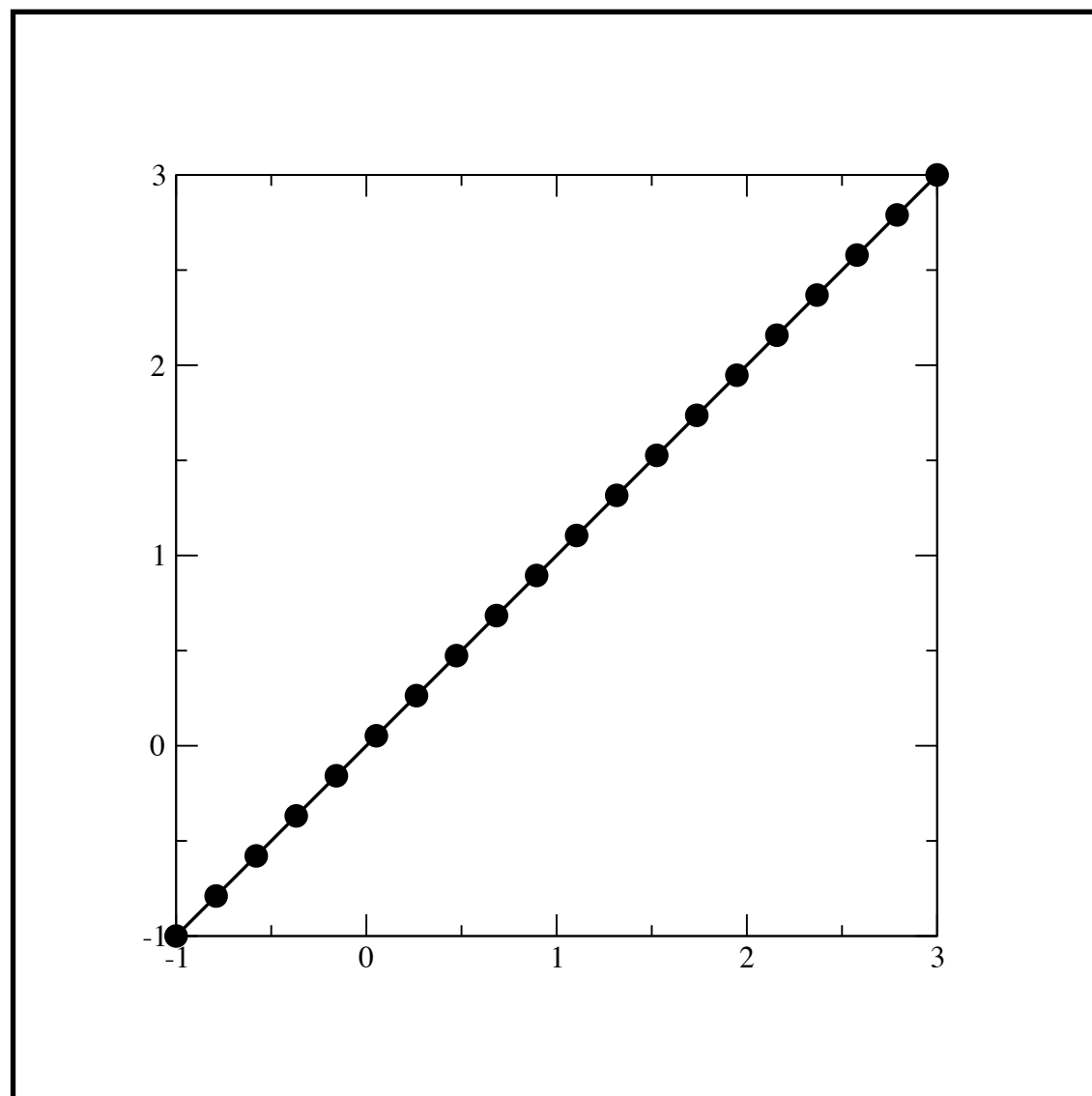
Volumes: $\mathbf{x} = \mathbf{x}(u, v, w)$ $\mathbf{x} \in \mathbb{R}^n$ $u, v, w \in \mathbb{R}$
 $\mathbf{x} = \mathbf{x}(\mathbf{u})$ $\mathbf{u} \in \mathbb{R}^3$

and so on...

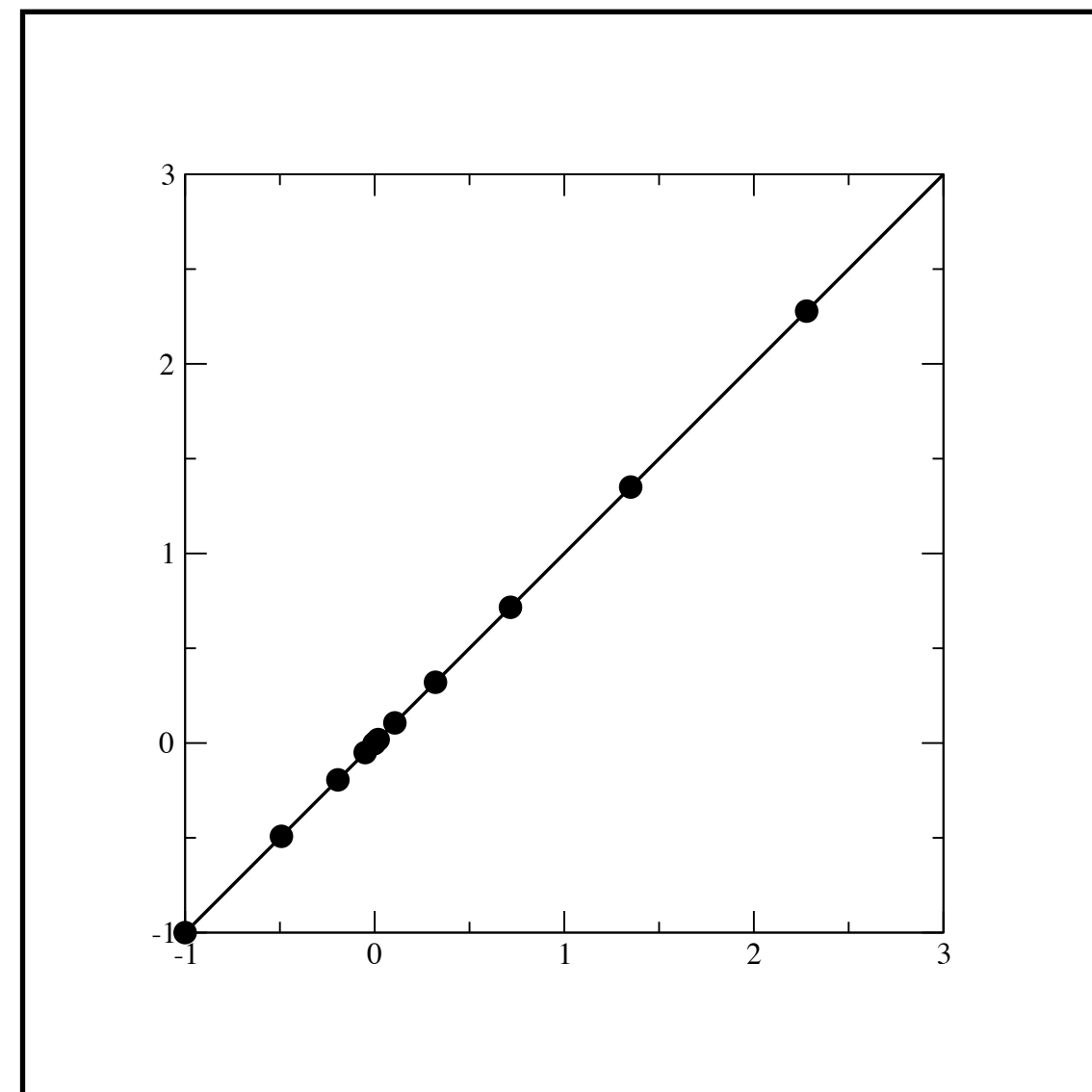
Note: a vector function is equivalently n scalar functions

Parametric Rep. Non-unique

Same curve/surface may have multiple formulae



$$x(u) = [u, u]$$

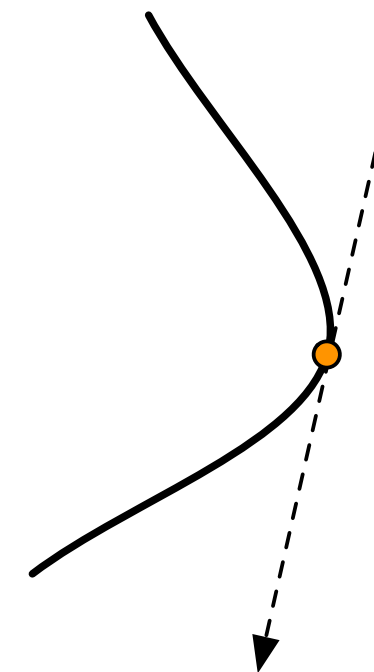


$$x(u) = [u^3, u^3]$$

Some Differential Geometry

Tangent to curve

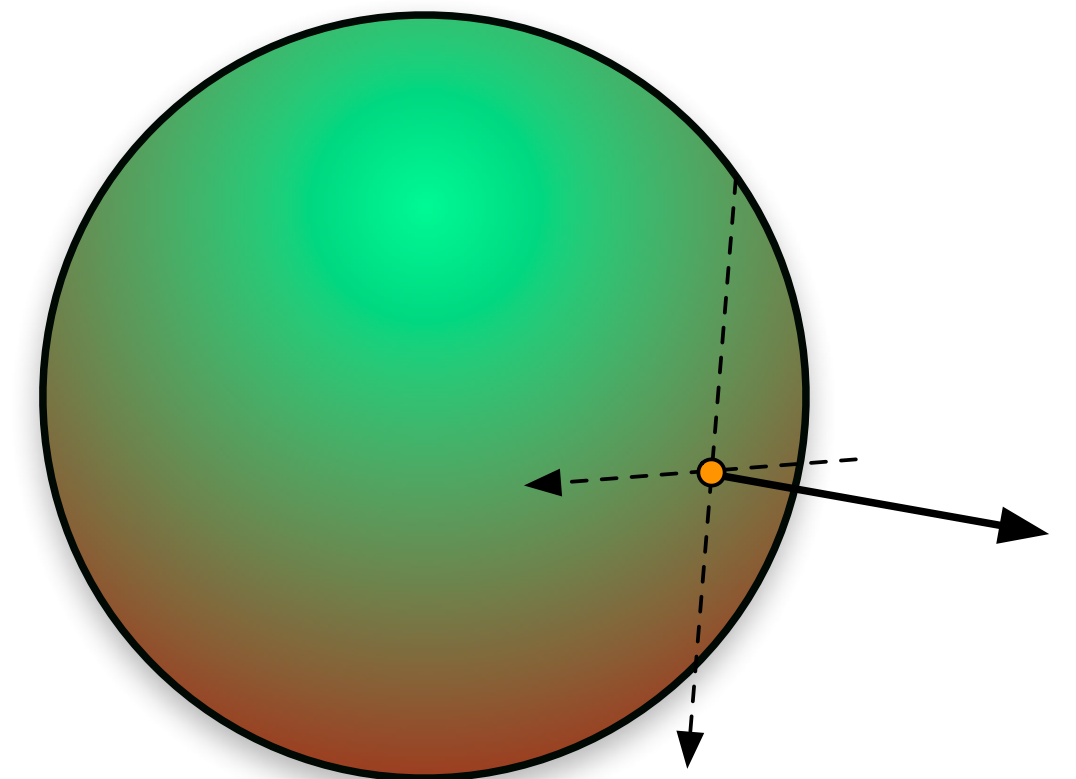
$$\mathbf{t}(u) = \left. \frac{\partial \mathbf{x}}{\partial u} \right|_u$$



Tangents to surface

$$\mathbf{t}_u(u, v) = \left. \frac{\partial \mathbf{x}}{\partial u} \right|_{u, v}$$

$$\mathbf{t}_v(u, v) = \left. \frac{\partial \mathbf{x}}{\partial v} \right|_{u, v}$$



Normal of surface

$$\hat{\mathbf{n}} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|}$$

Degeneracies: $\partial \mathbf{x} / \partial u = 0$ **or** $\mathbf{t}_u \times \mathbf{t}_v = 0$

Tangent Space

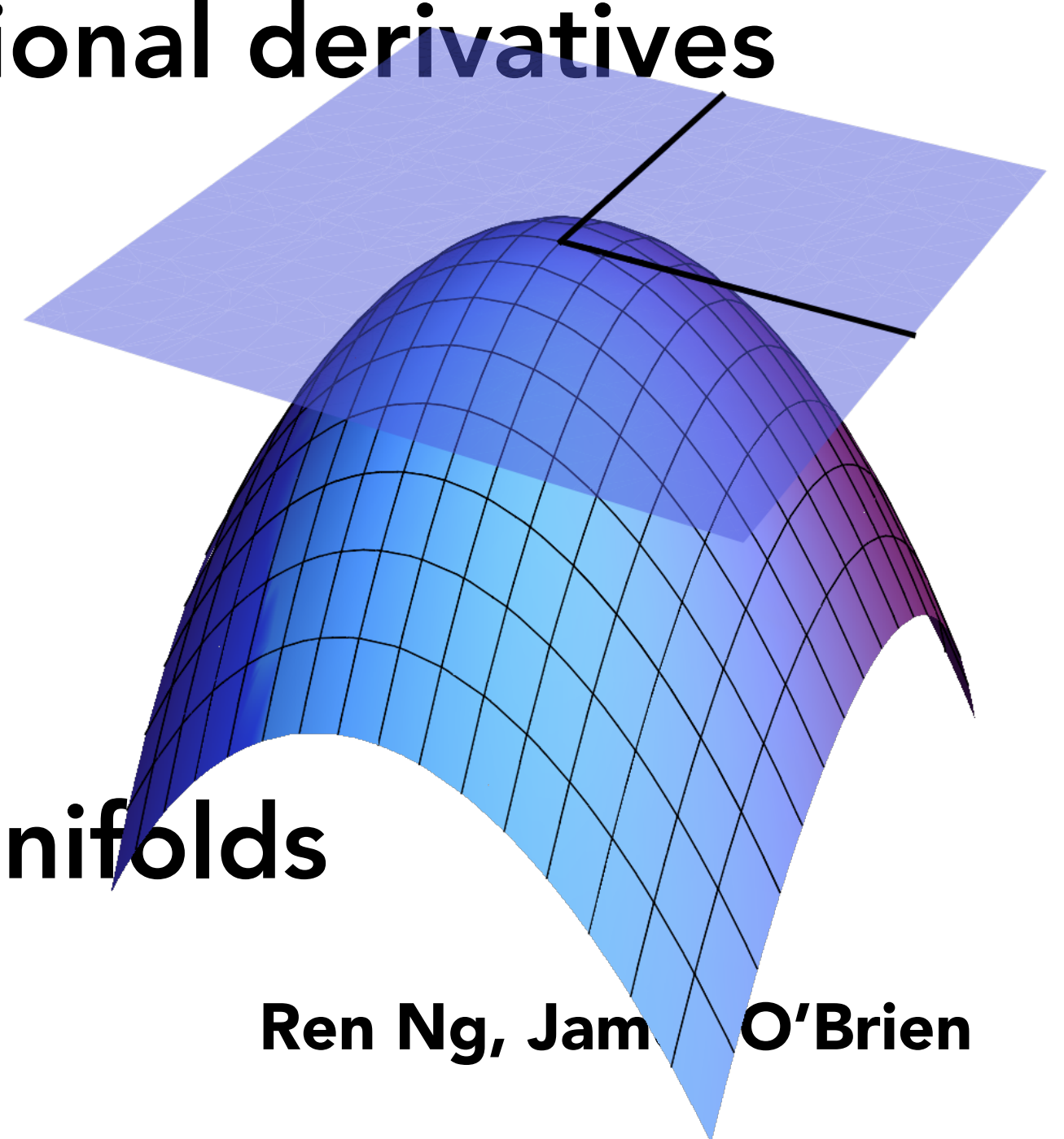
The *tangent space* at a point on a surface is the vector space spanned by

$$\frac{\partial \mathbf{x}(\mathbf{u})}{\partial u} \quad \frac{\partial \mathbf{x}(\mathbf{u})}{\partial v}$$

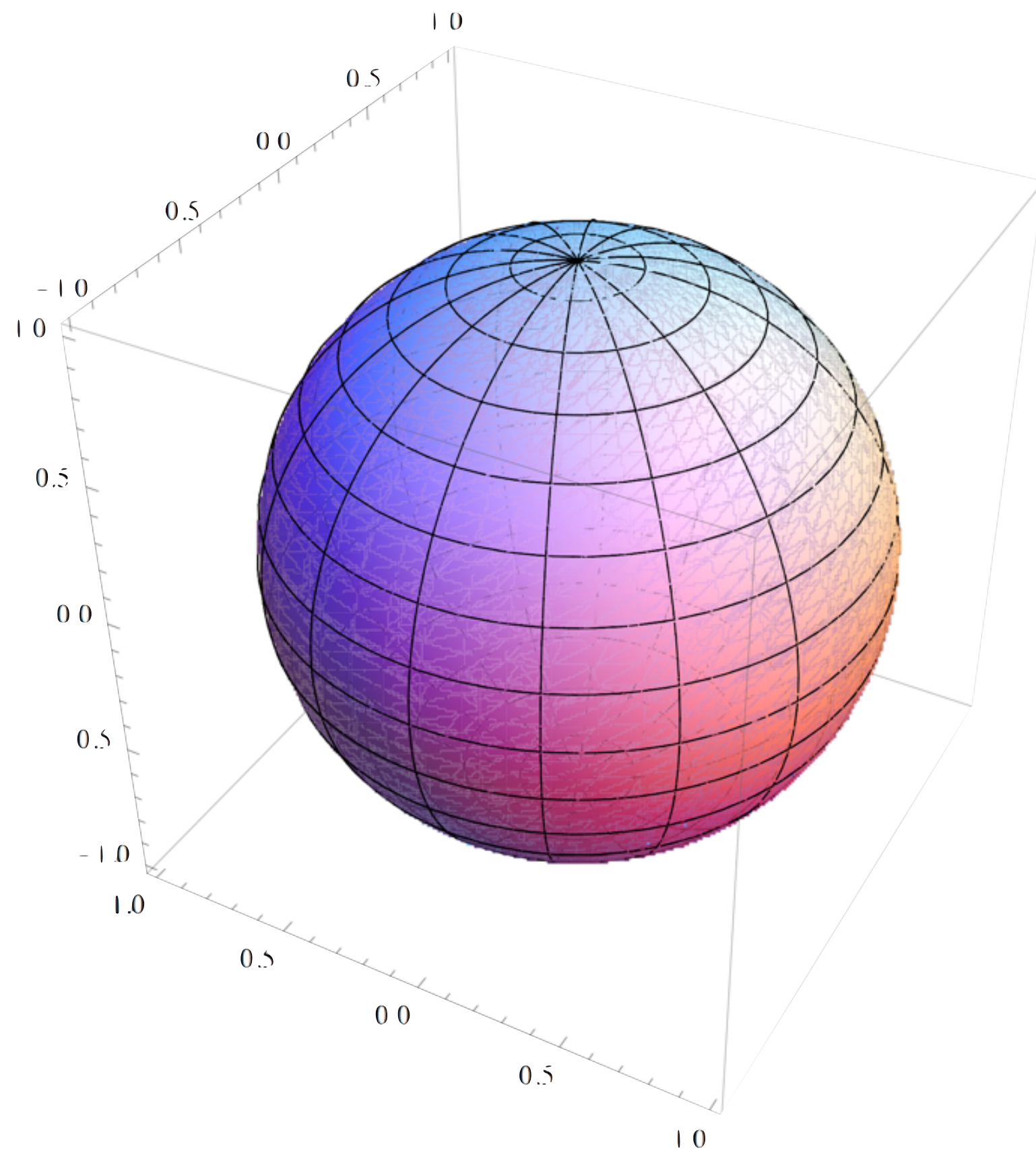
- Definition assumes that these directional derivatives are linearly independent.
- Tangent space of surface may exist even if the parameterization is bad

For surface the space is a plane

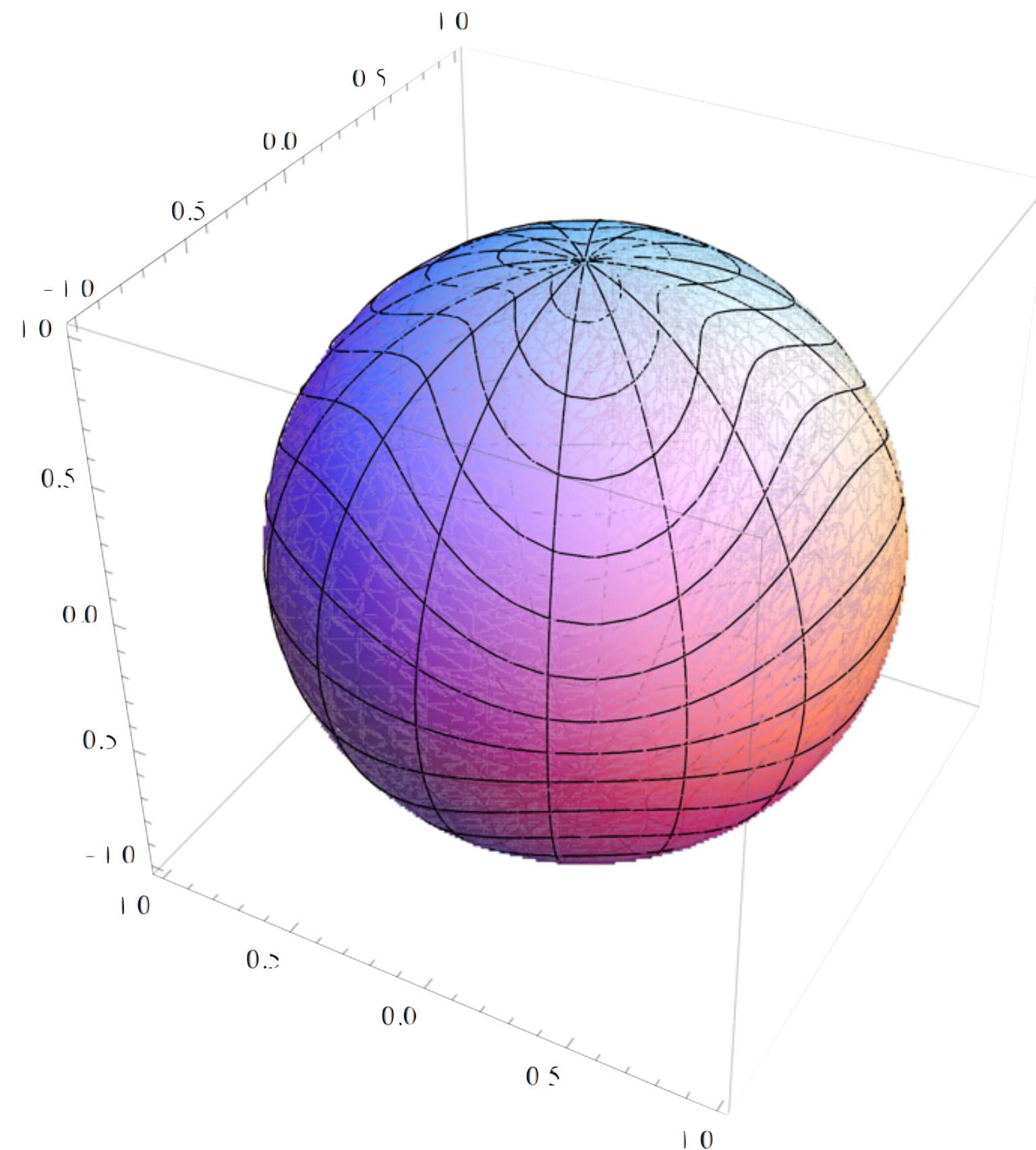
- Generalized to higher dimension manifolds



Non Orthogonal Tangents



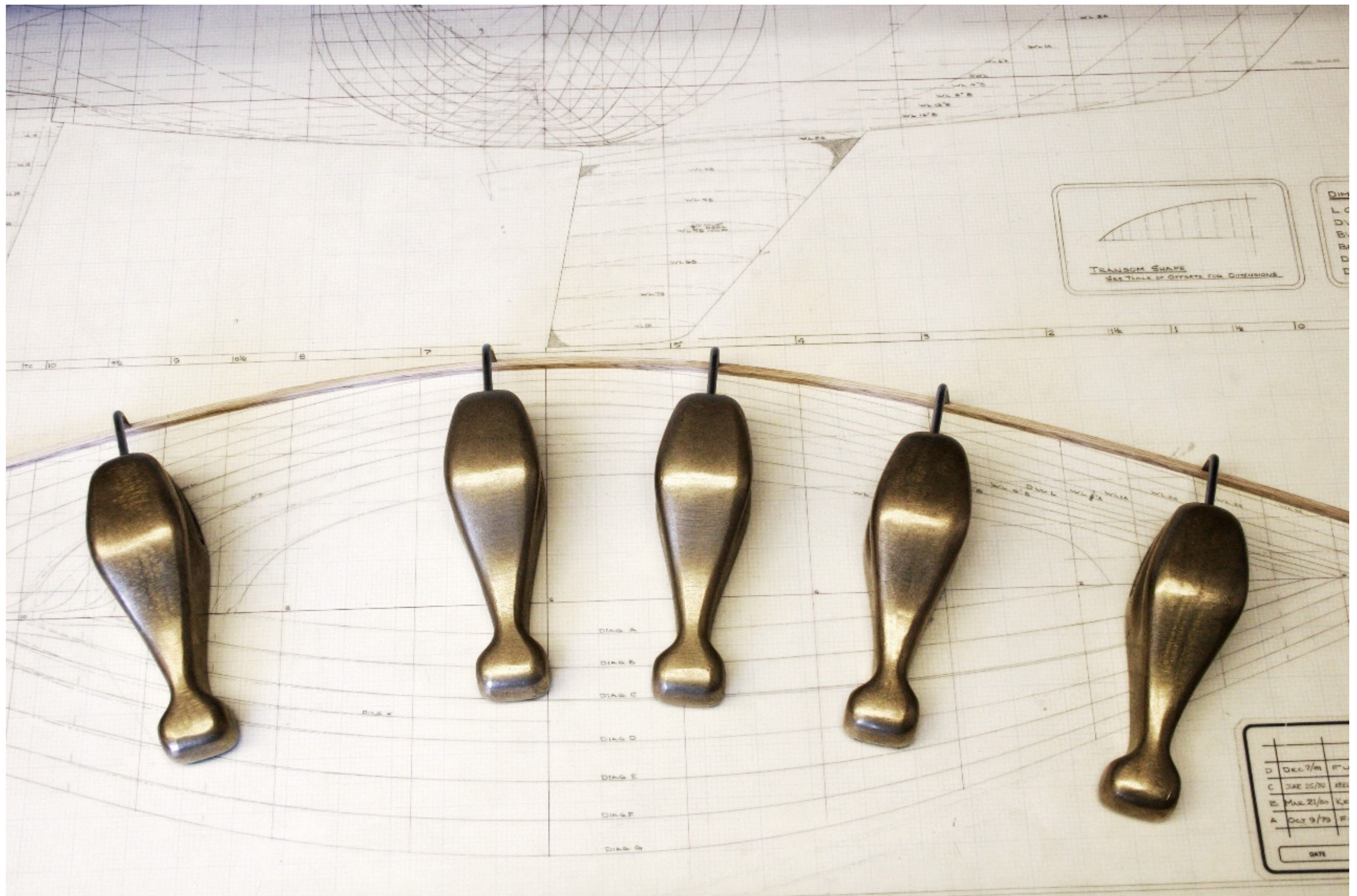
$$\begin{bmatrix} \cos(\theta 2\pi) \cos(\phi \pi / 2) \\ \sin(\theta 2\pi) \cos(\phi \pi / 2) \\ \sin(\phi \pi / 2) \end{bmatrix}$$



$$\begin{bmatrix} \cos(2\pi\theta) \cos\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi + \phi\right)\right) \\ \cos\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi + \phi\right)\right) \sin(2\pi\theta) \\ \sin\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi + \phi\right)\right) \end{bmatrix}$$

Splines

A Real Draftsperson's Spline



Spline Topics

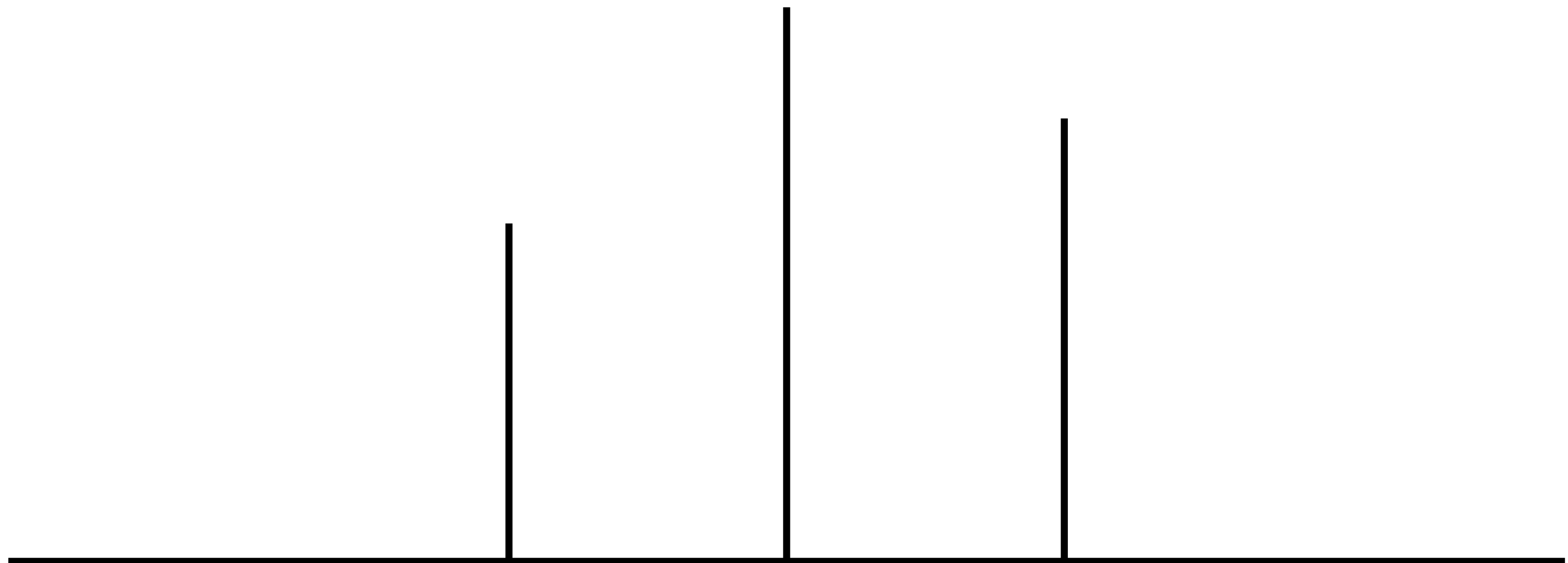
Interpolation

- Cubic Hermite interpolation
- Catmull-Rom interpolation

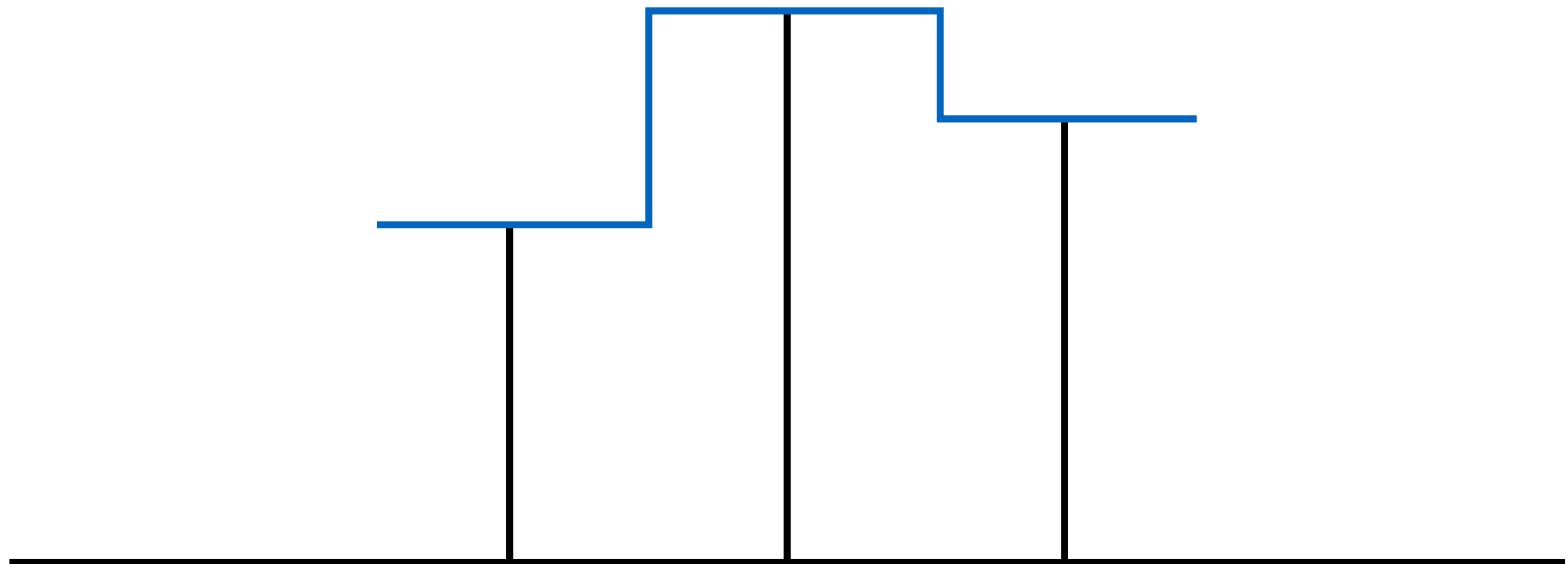
Bezier curves

Bezier surfaces

Goal: Interpolate Values

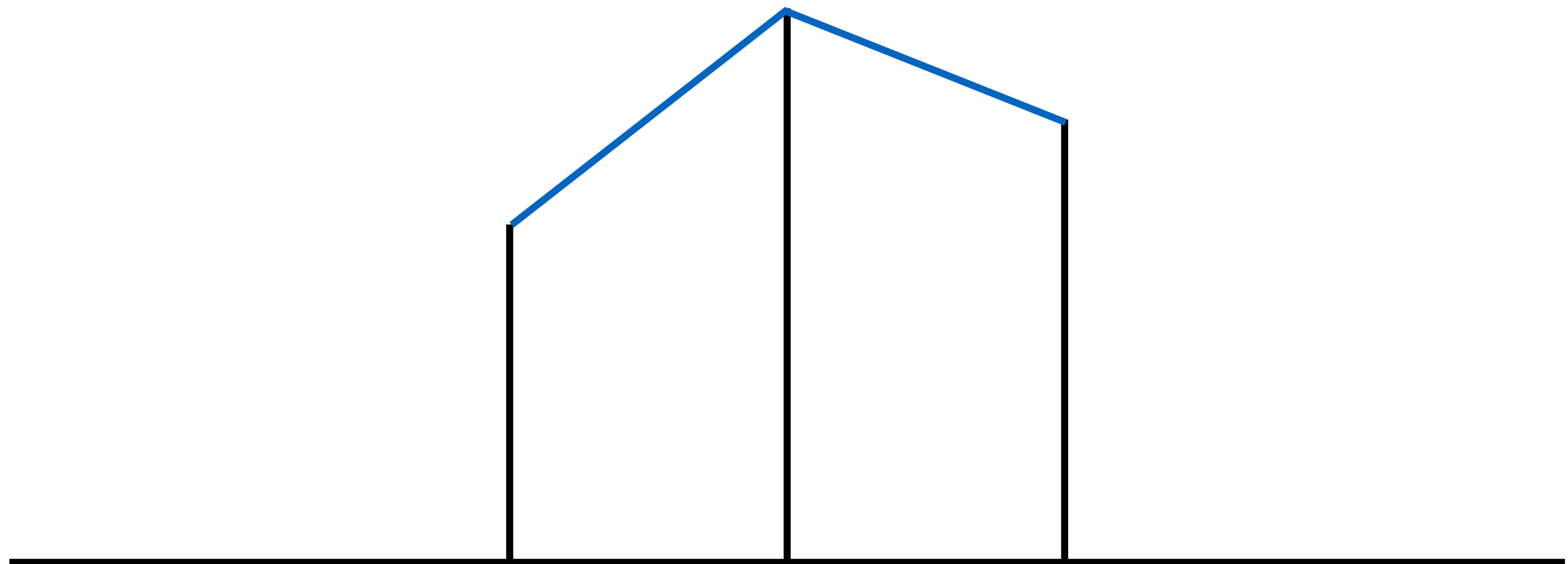


Nearest Neighbor Interpolation



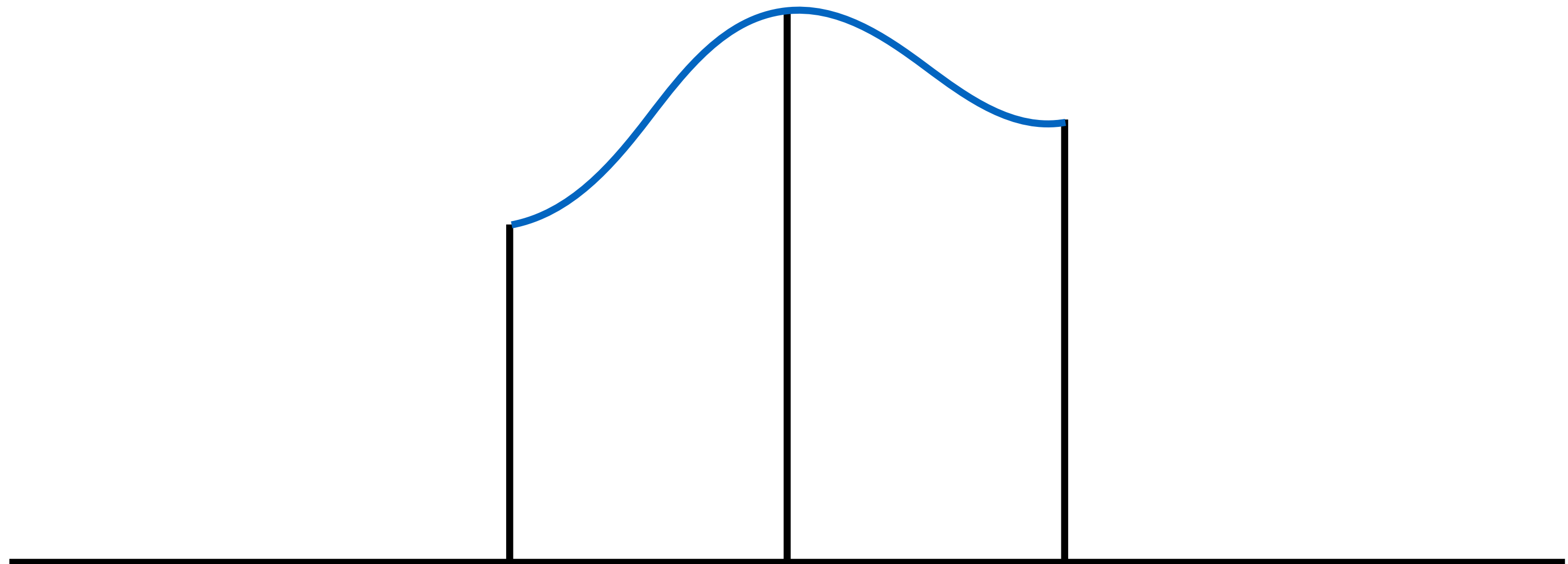
Problem: values not continuous

Linear Interpolation



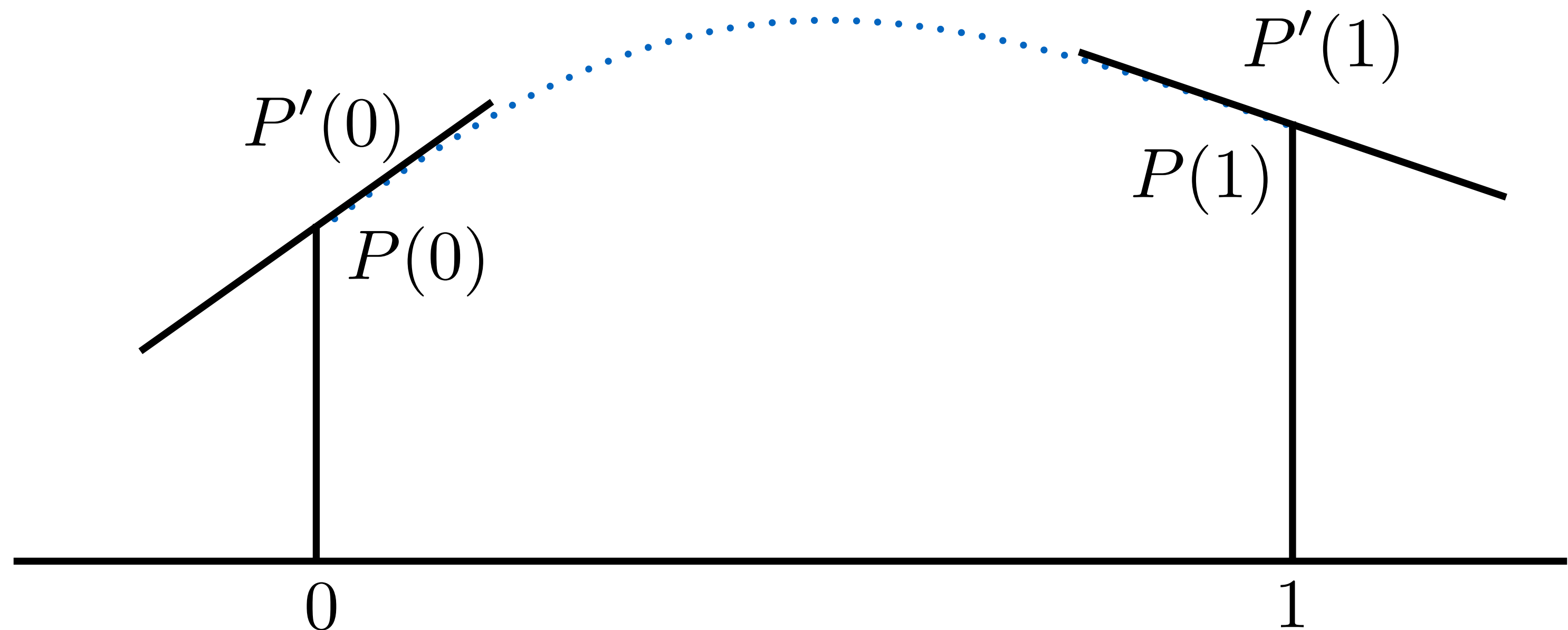
Problem: derivatives not continuous

Smooth Interpolation?



Cubic Hermite Interpolation

Cubic Hermite Interpolation



Inputs: values and derivatives at endpoints

Polynomial Basis Functions

Power Basis

$$x(u) = \sum_{i=0}^d c_i u^i$$

e.g. a cubic polynomial

$$x(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$$

$$x(u) = \mathbf{C} \cdot \mathcal{P}^d$$
$$\mathbf{C} = [c_0, c_1, c_2, \dots, c_d]$$
$$\mathcal{P}^d = [1, u, u^2, \dots, u^d]$$

The elements of $\mathcal{P}^d$ are *linearly independent*
i.e. no good approximation

$$u^k \not\approx \sum_{i \neq k} c_i u^i$$

Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$x(u) = c_0 + c_1u + c_2u^2 + c_3u^3$$

$$x(u_0) = x_0$$

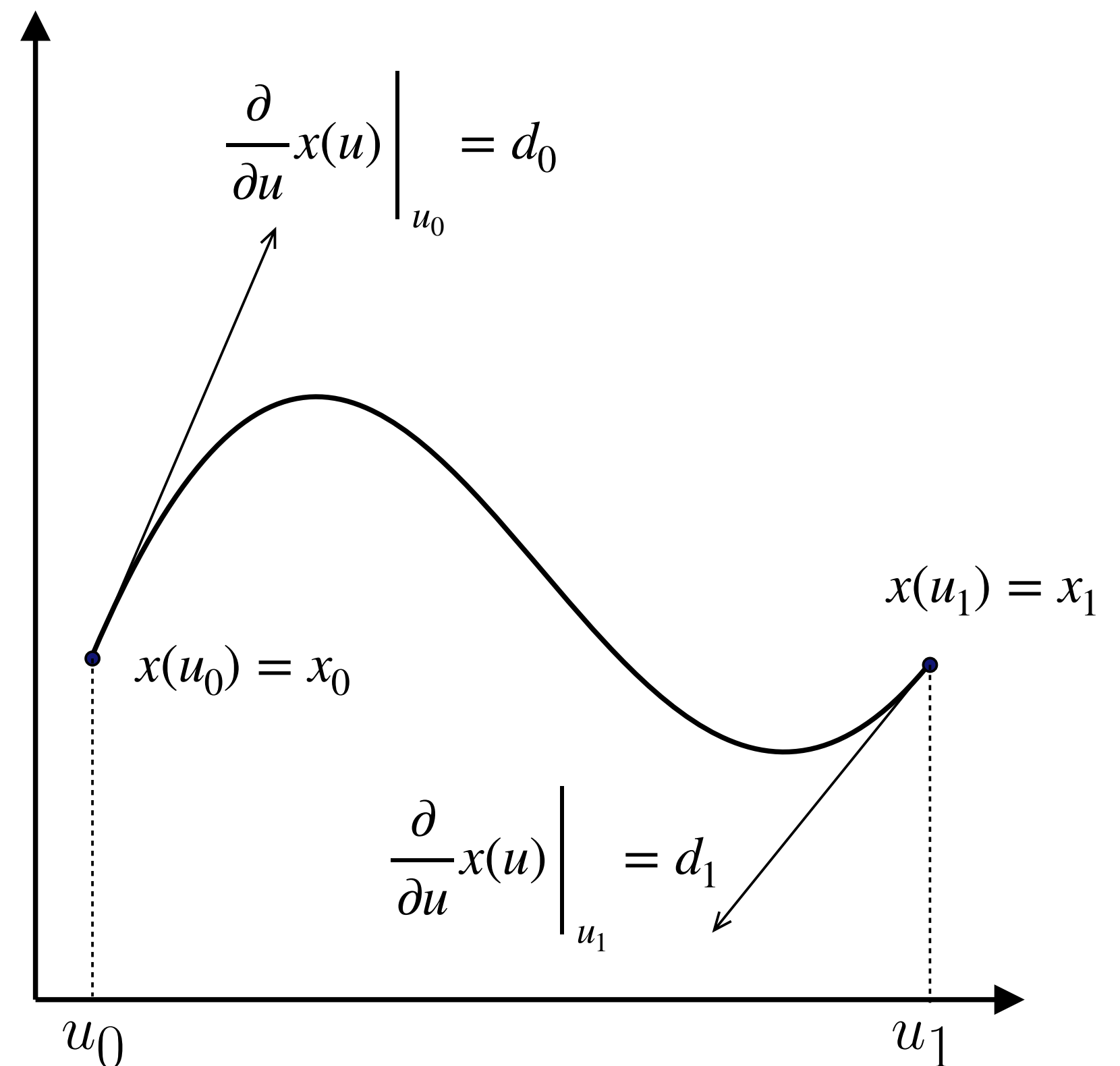
$$x(u_1) = x_1$$

$$\left. \frac{\partial}{\partial u} x(u) \right|_{u_0} = d_0$$

$$\left. \frac{\partial}{\partial u} x(u) \right|_{u_1} = d_1$$

For now, assume

$$u_0 = 0 \quad u_1 = 1$$



Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$x(u) = c_0 + c_1u + c_2u^2 + c_3u^3$$

$$x'(u) = c_1 + 2c_2u + 3c_3u^2$$

$$x(u_0) = x_0$$

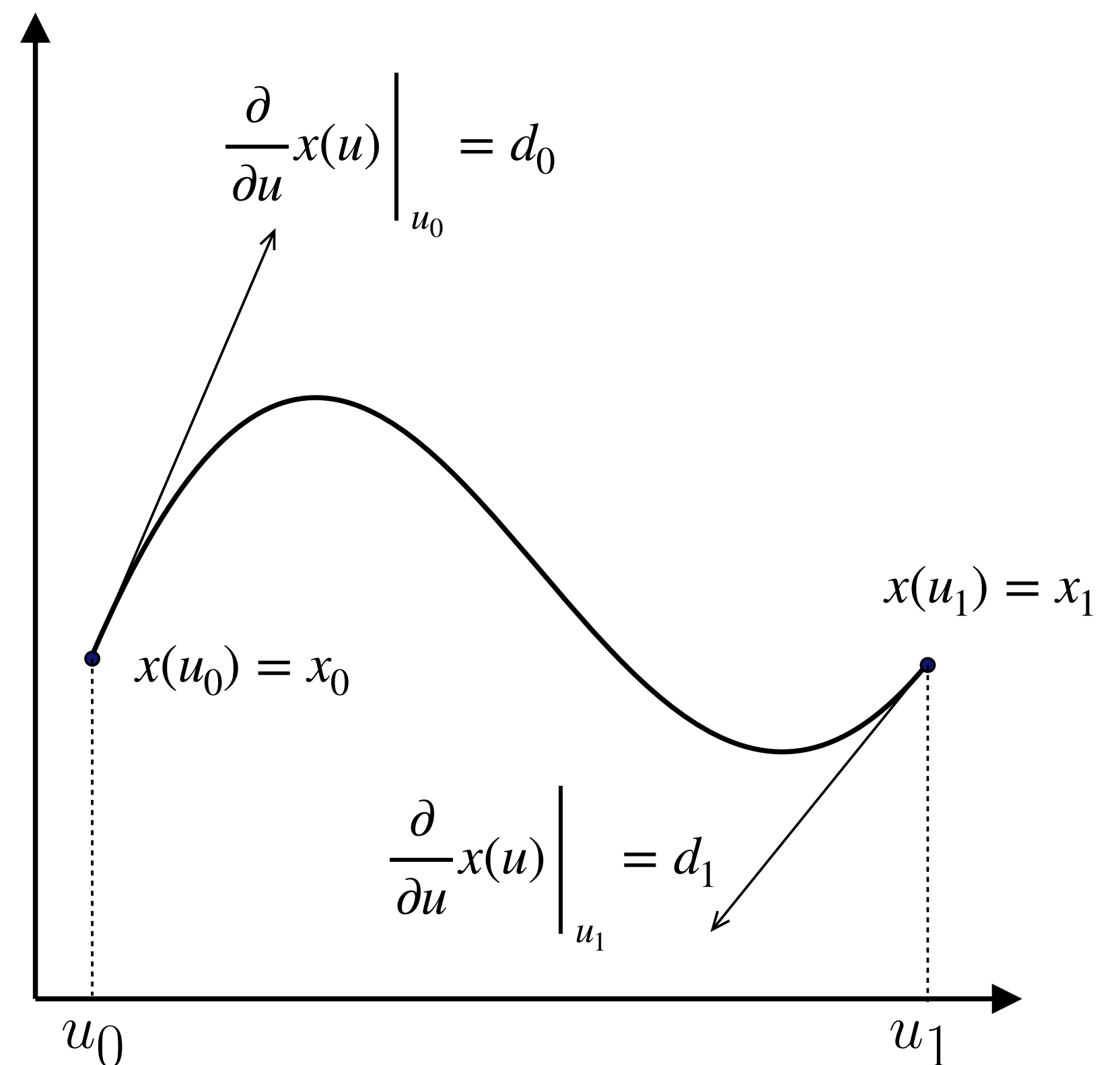
$$x(u_1) = x_1$$

$$x'(u_0) = d_0$$

$$x'(u_1) = d_1$$

For now, assume

$$u_0 = 0 \quad u_1 = 1$$



Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$x(u) = c_0 + c_1u + c_2u^2 + c_3u^3$$

$$x'(u) = c_1 + 2c_2u + 3c_3u^2$$

$$c_0 + c_1u_0 + c_2u_0^2 + c_3u_0^3 = x_0$$

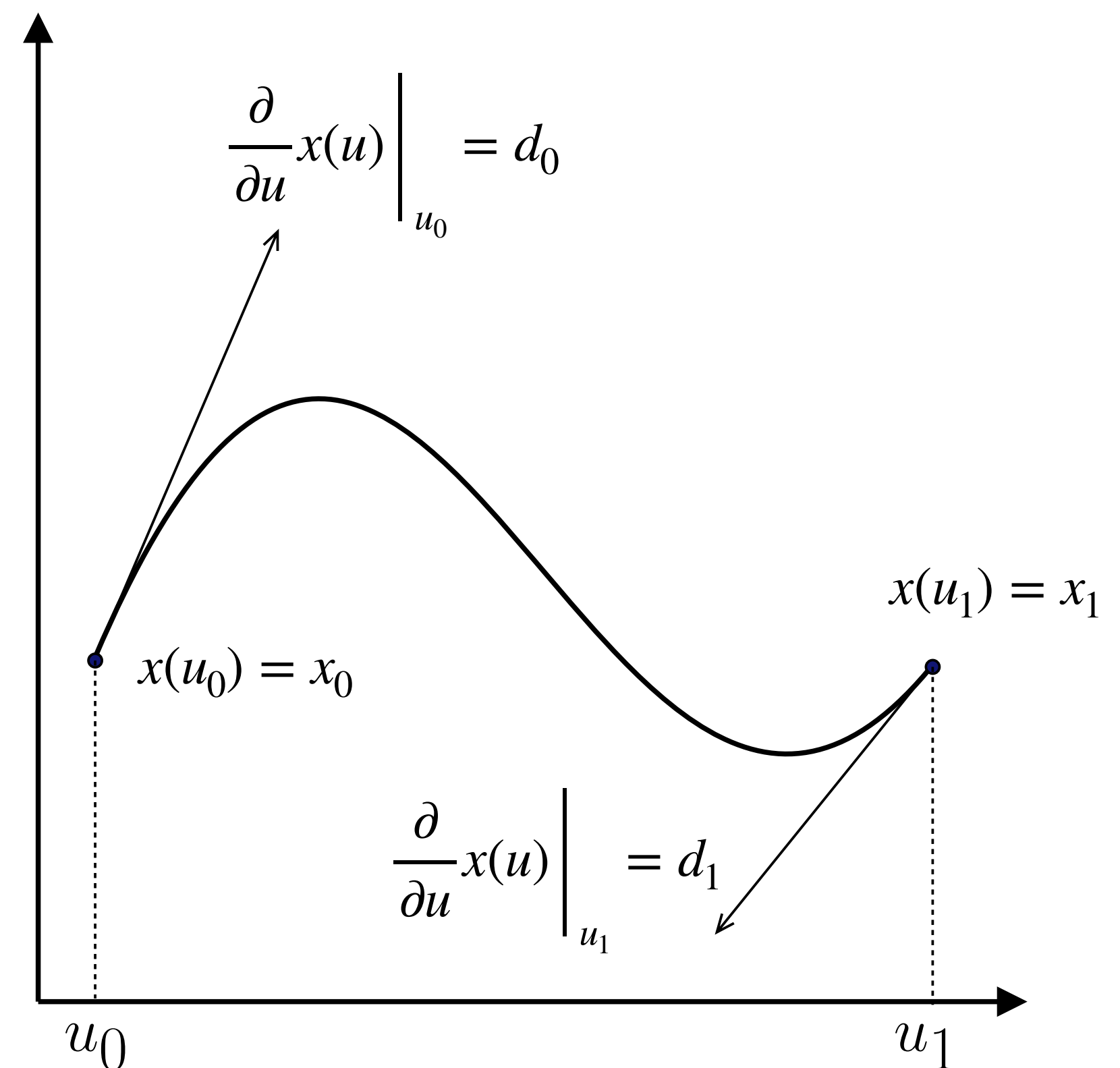
$$c_0 + c_1u_1 + c_2u_1^2 + c_3u_1^3 = x_1$$

$$c_1 + 2c_2u_0 + 3c_3u_0^2 = d_0$$

$$c_1 + 2c_2u_1 + 3c_3u_1^2 = d_1$$

For now, assume

$$u_0 = 0 \quad u_1 = 1$$



Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$x(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$$

$$x'(u) = c_1 + 2c_2 u + 3c_3 u^2$$

$$c_0 + c_1 0 + c_2 0 + c_3 0 = x_0$$

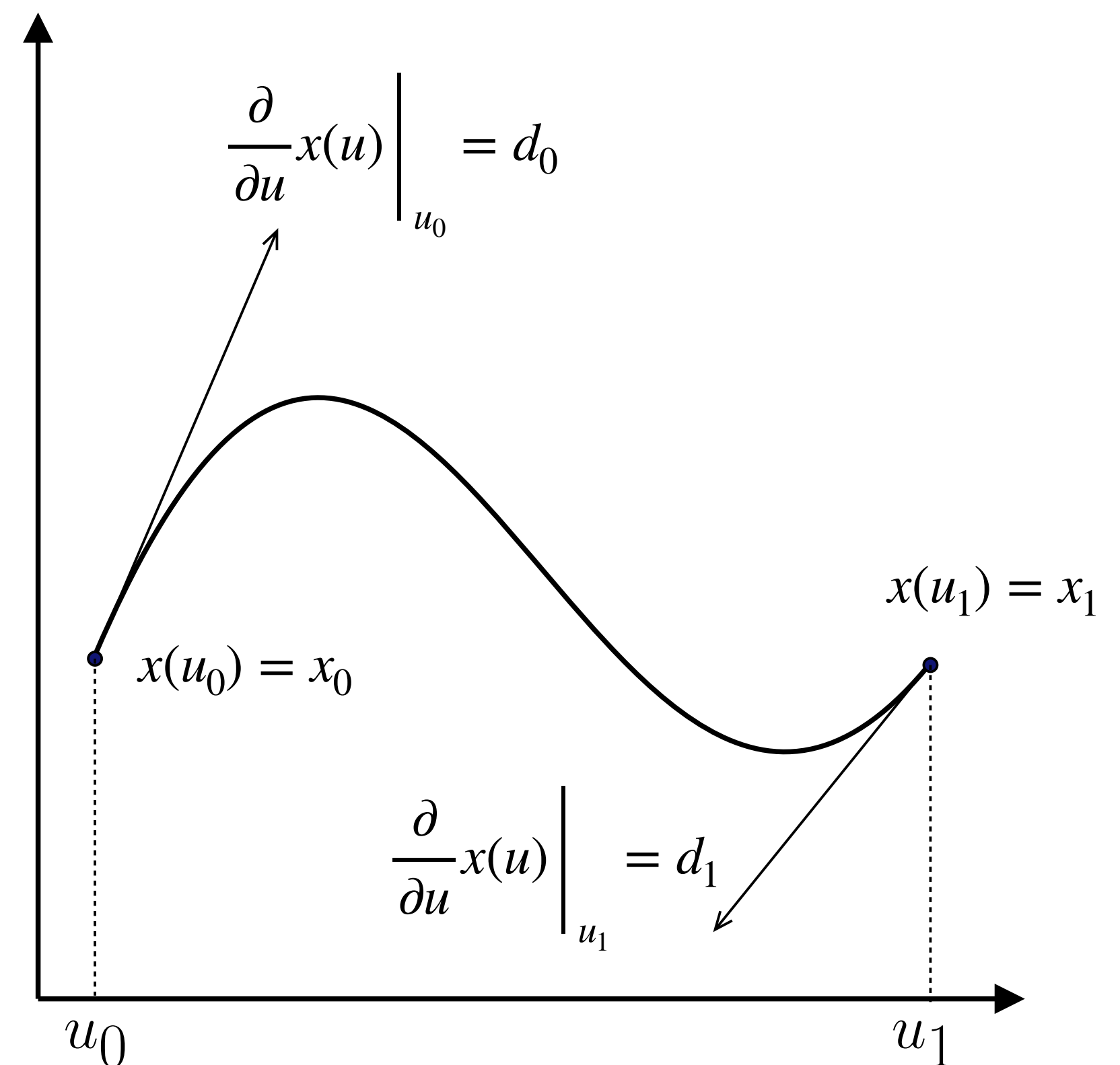
$$c_0 + c_1 1 + c_2 1 + c_3 1 = x_1$$

$$c_1 + 2c_2 0 + 3c_3 0 = d_0$$

$$c_1 + 2c_2 1 + 3c_3 1 = d_1$$

For now, assume

$$u_0 = 0 \quad u_1 = 1$$



Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

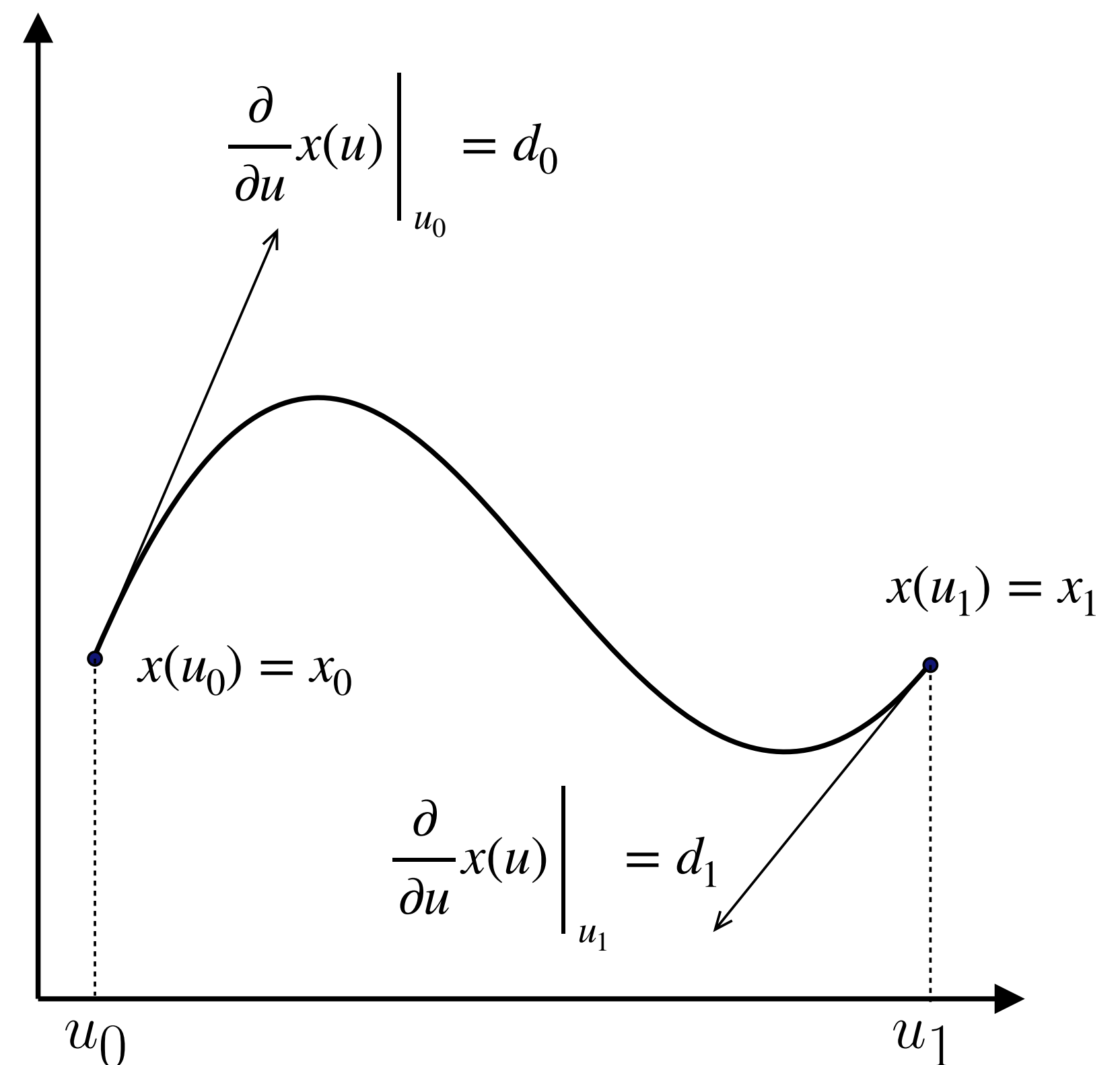
$$x(u) = c_0 + c_1u + c_2u^2 + c_3u^3$$

$$x'(u) = c_1 + 2c_2u + 3c_3u^2$$

For now, assume

$$u_0 = 0 \quad u_1 = 1$$

c_0	$=$	x_0
$c_0 + c_1 + c_2 + c_3$	$=$	x_1
c_1	$=$	d_0
$c_1 + 2c_2 + 3c_3$	$=$	d_1



Specifying a Curve

(From previous slide for reference.)

$$\begin{aligned} c_0 &= x_0 \\ c_0 + c_1 + c_2 + c_3 &= x_1 \\ c_1 &= d_0 \\ c_1 + 2c_2 + 3c_3 &= d_1 \end{aligned}$$

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$x(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$$

$$x'(u) = c_1 + 2c_2 u + 3c_3 u^2$$

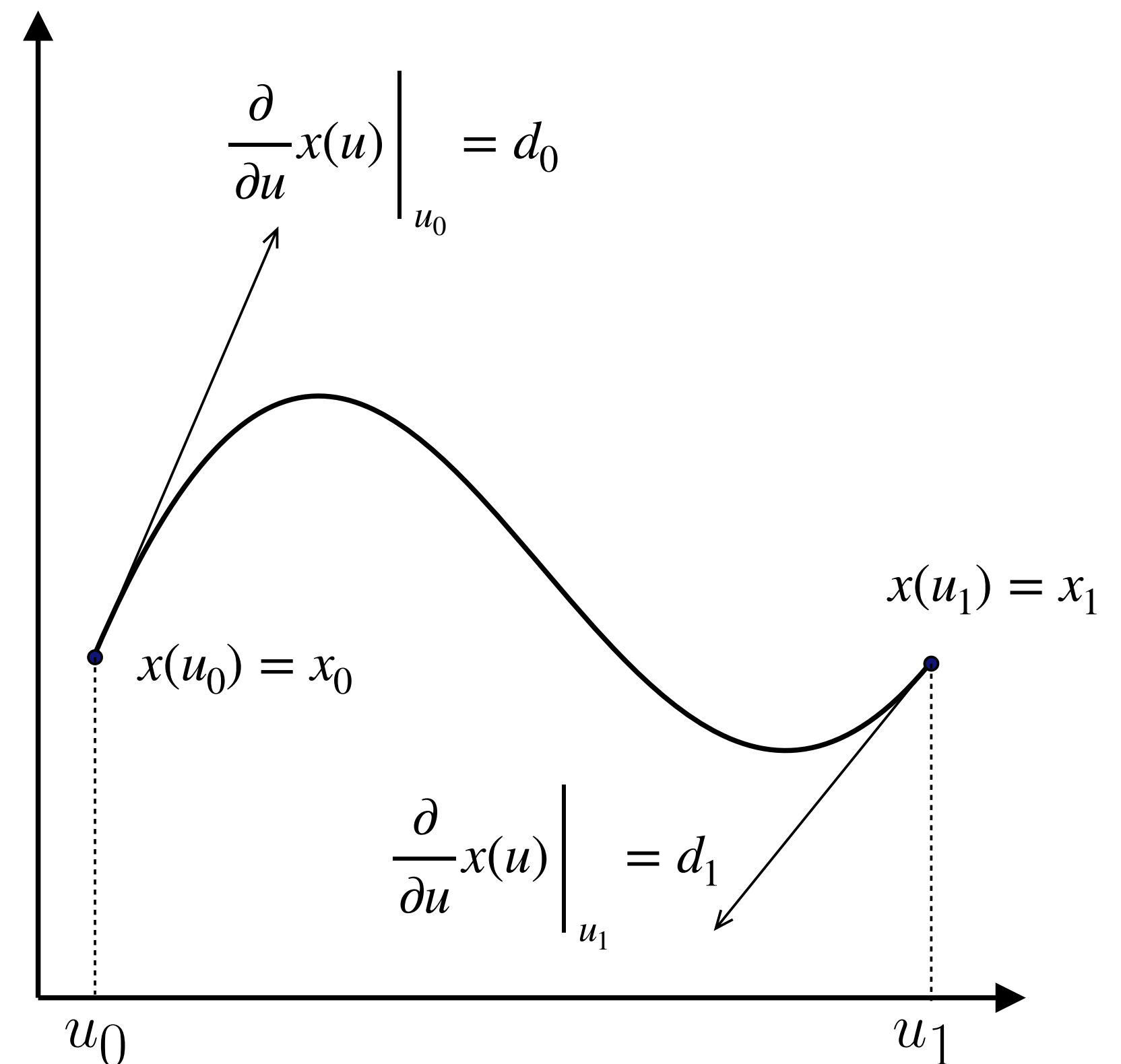
For now, assume
 $u_0 = 0 \quad u_1 = 1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ d_0 \\ d_1 \end{bmatrix}$$

Matrix

Coefficients (unknown)

Constraint Values



Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$x(u) = c_0 + c_1u + c_2u^2 + c_3u^3$$

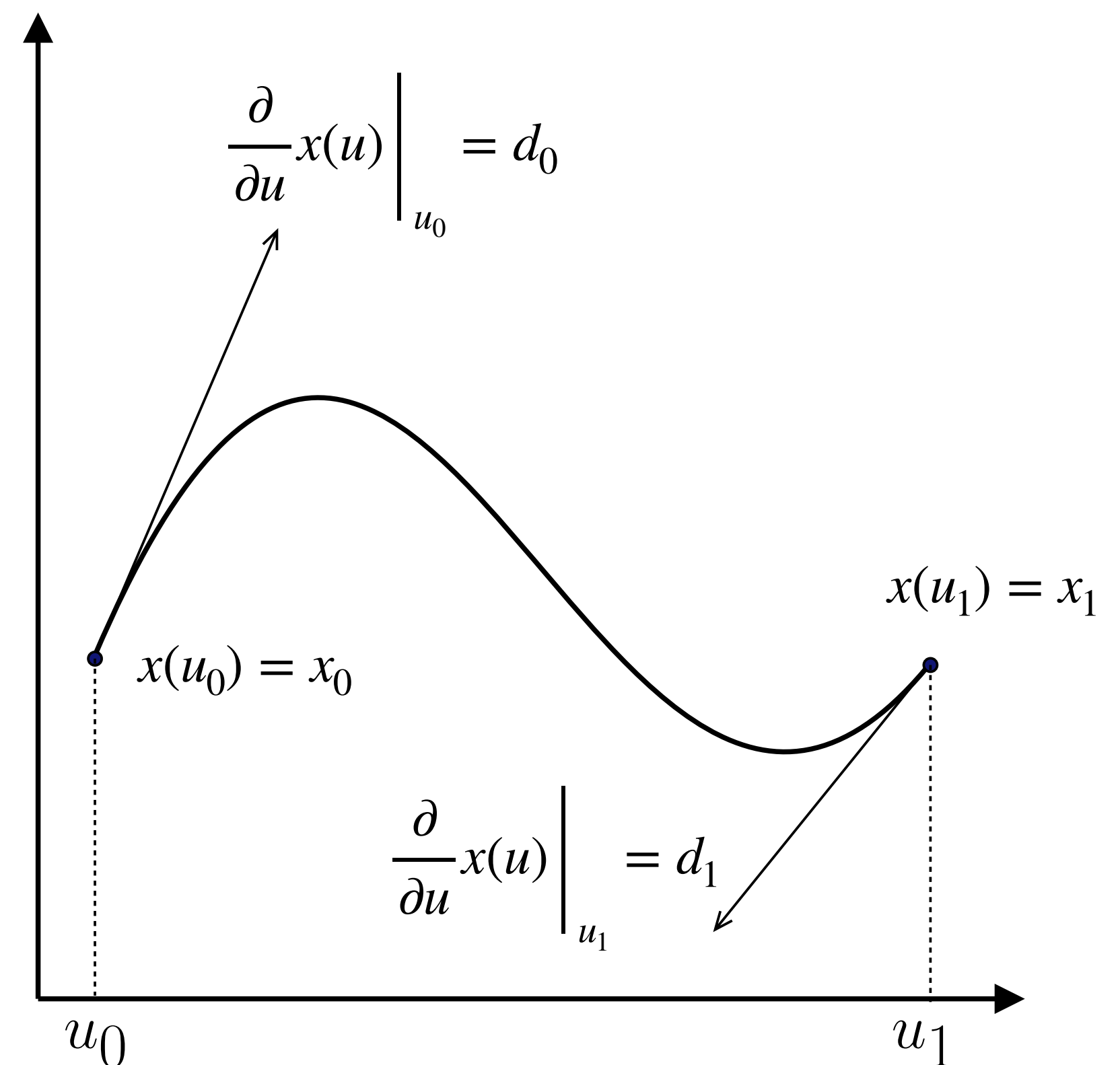
$$x'(u) = c_1 + 2c_2u + 3c_3u^2$$

For now, assume
 $u_0 = 0 \quad u_1 = 1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ d_0 \\ d_1 \end{bmatrix}$$

$$\mathbf{B} \cdot \mathbf{c} = \mathbf{h}$$

Often called
"control points", but note
the d in this case are
directions, not points.



Specifying a Curve

$$x(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$$

$$x'(u) = c_1 + 2c_2 u + 3c_3 u^2$$

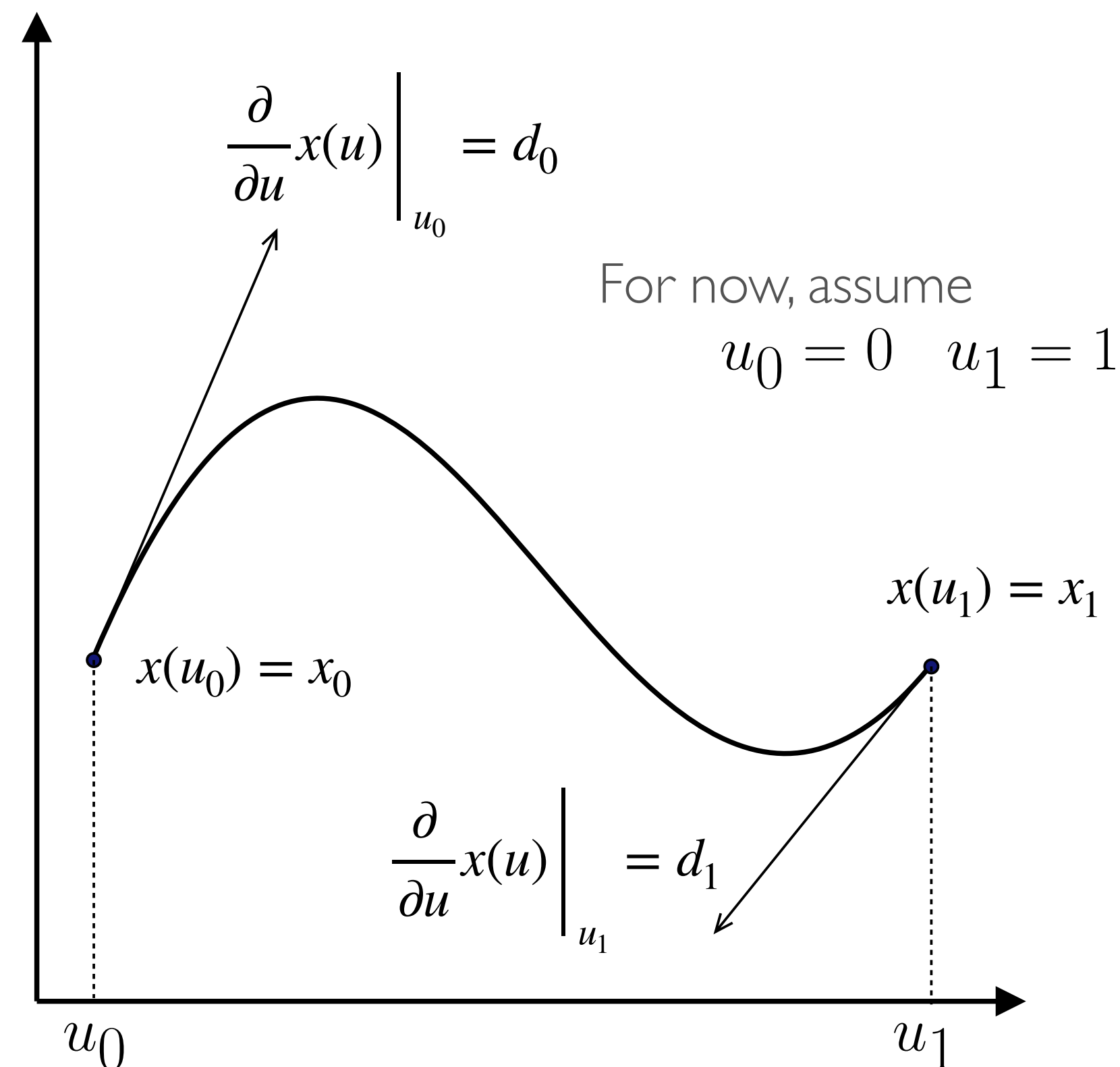
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ d_0 \\ d_1 \end{bmatrix}$$

$$\mathbf{B} \cdot \mathbf{c} = \mathbf{h}$$

$$\mathbf{c} = \mathbf{B}^{-1} \cdot \mathbf{h}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ d_0 \\ d_1 \end{bmatrix}$$

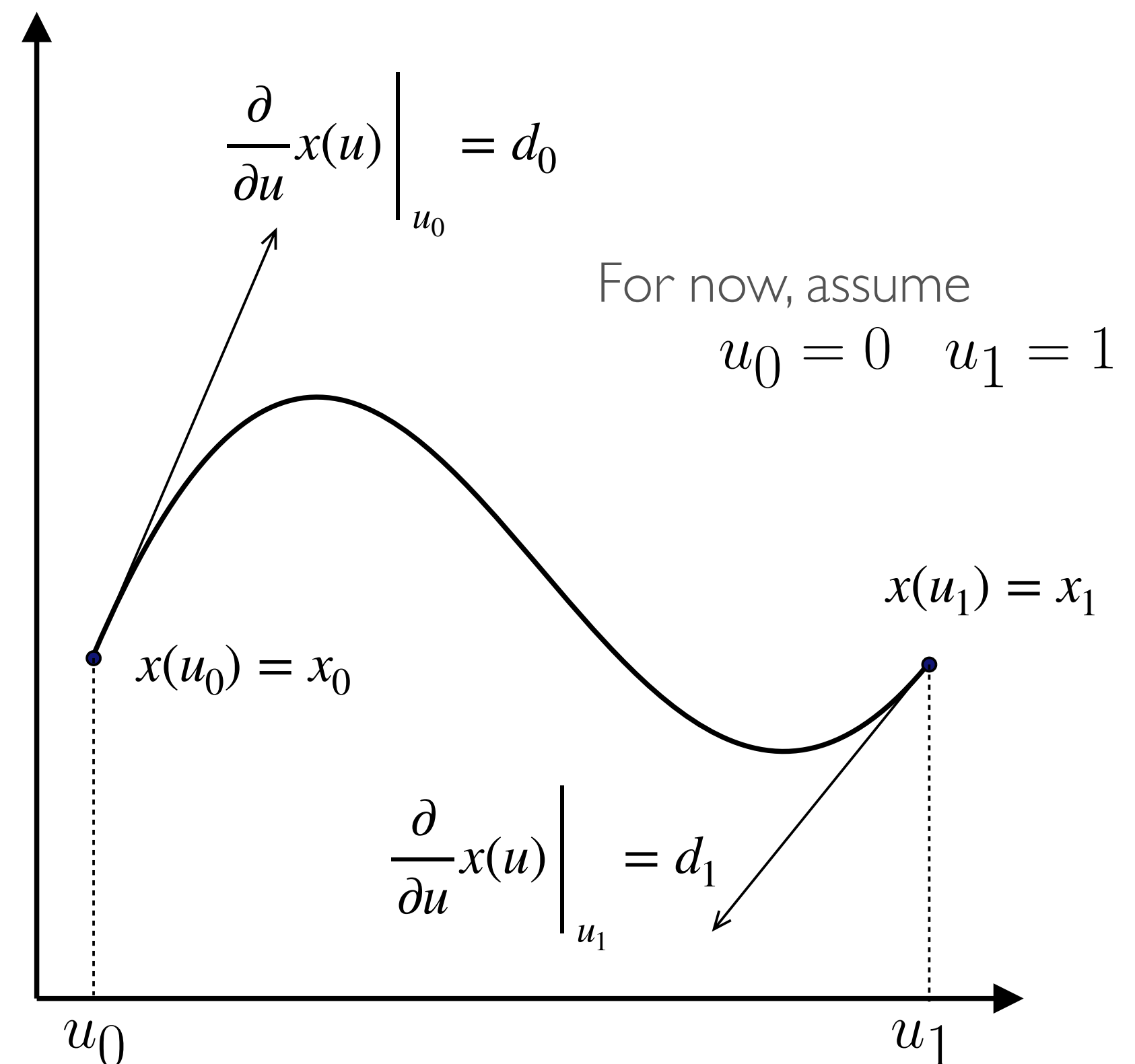


Specifying a Curve

$$x(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$$

$$x'(u) = c_1 + 2c_2 u + 3c_3 u^2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ d_0 \\ d_1 \end{bmatrix}$$



$$\mathbf{B} \cdot \mathbf{c} = \mathbf{h}$$

$$\mathbf{c} = \mathbf{B}^{-1} \cdot \mathbf{h}$$

$$\mathbf{c} = \boldsymbol{\beta}_H \cdot \mathbf{h}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ d_0 \\ d_1 \end{bmatrix}$$

Cubic Hermite Basis

$$\mathbf{c} = \boldsymbol{\beta}_H \cdot \mathbf{h}$$

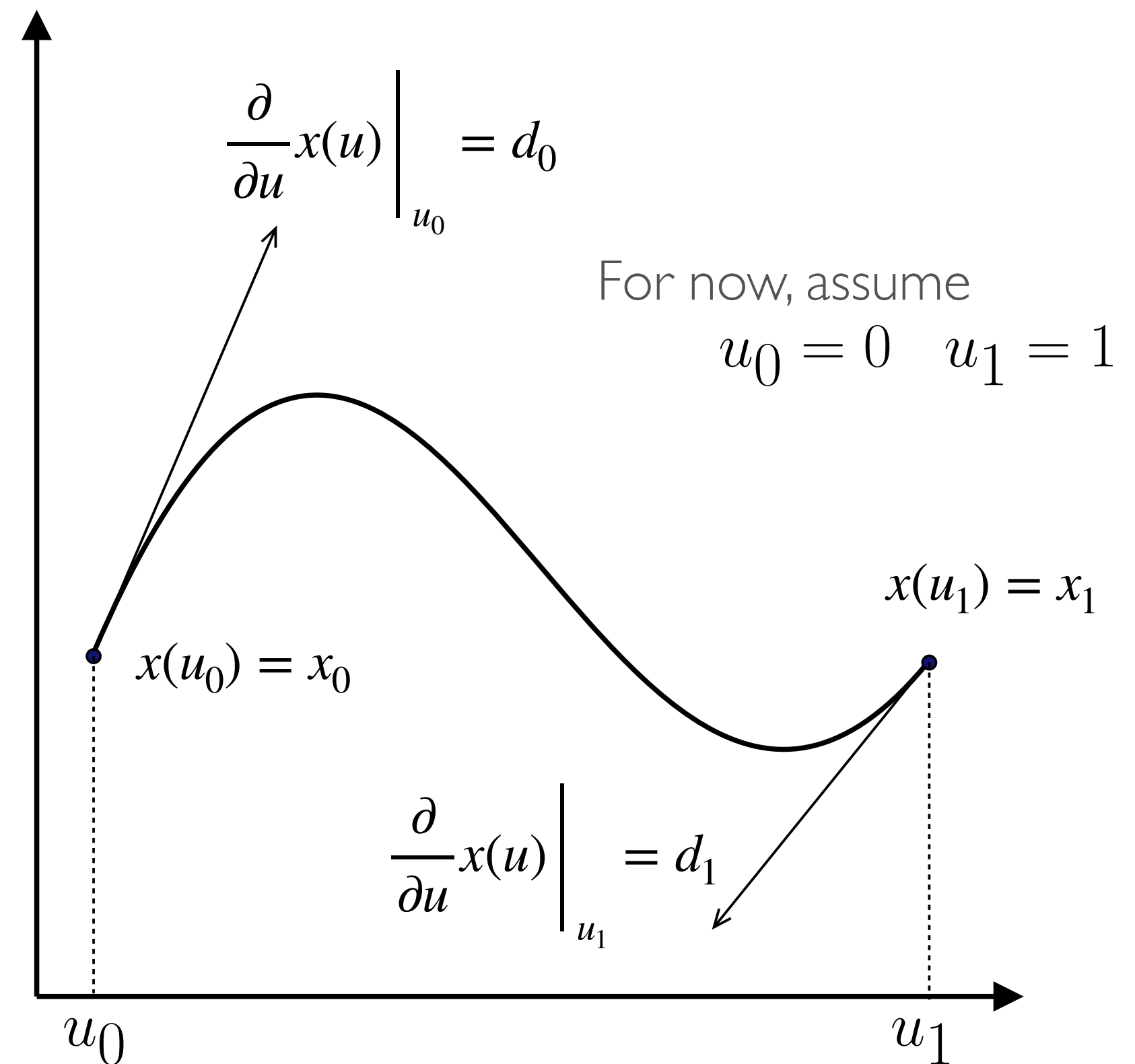
$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ d_0 \\ d_1 \end{bmatrix}$$

$$x(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$$

$$= \mathcal{P}^3 \cdot \mathbf{c}$$

$$= \mathcal{P}^3 \cdot \boldsymbol{\beta}_H \cdot \mathbf{h}$$

$$x(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_H \cdot \mathbf{h}$$



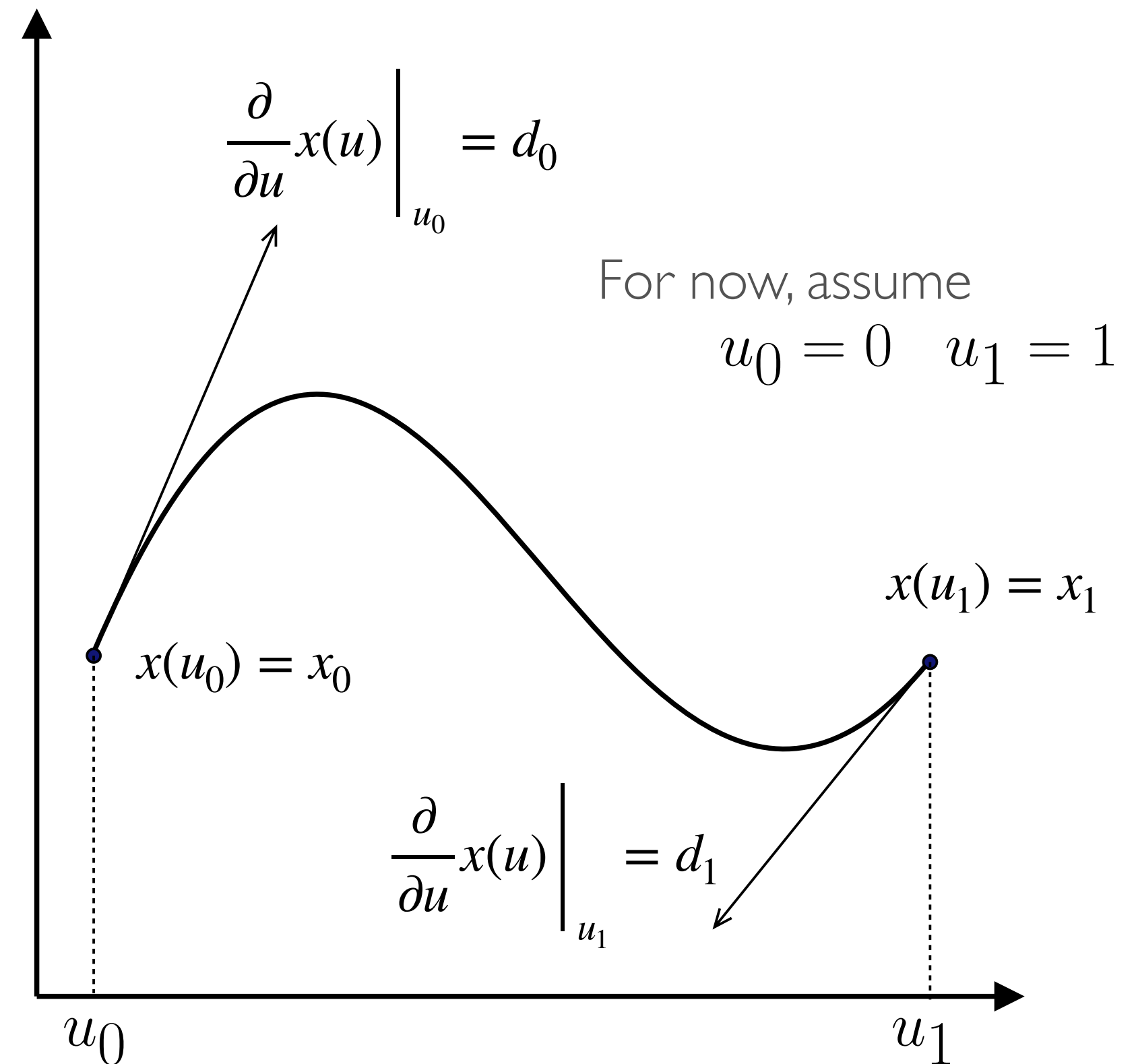
$\boldsymbol{\beta}_H \leftarrow$ Hermite Basis Matrix

$$\mathcal{P}^3(u) = [1 \ u \ u^2 \ u^3]$$

Cubic Hermite Basis

$$\mathbf{c} = \boldsymbol{\beta}_H \cdot \mathbf{h}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ d_0 \\ d_1 \end{bmatrix}$$



$$x(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_H \cdot \mathbf{h}$$

$$x(u) = \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix}^T \cdot \mathbf{h}$$

Cubic Hermite Basis

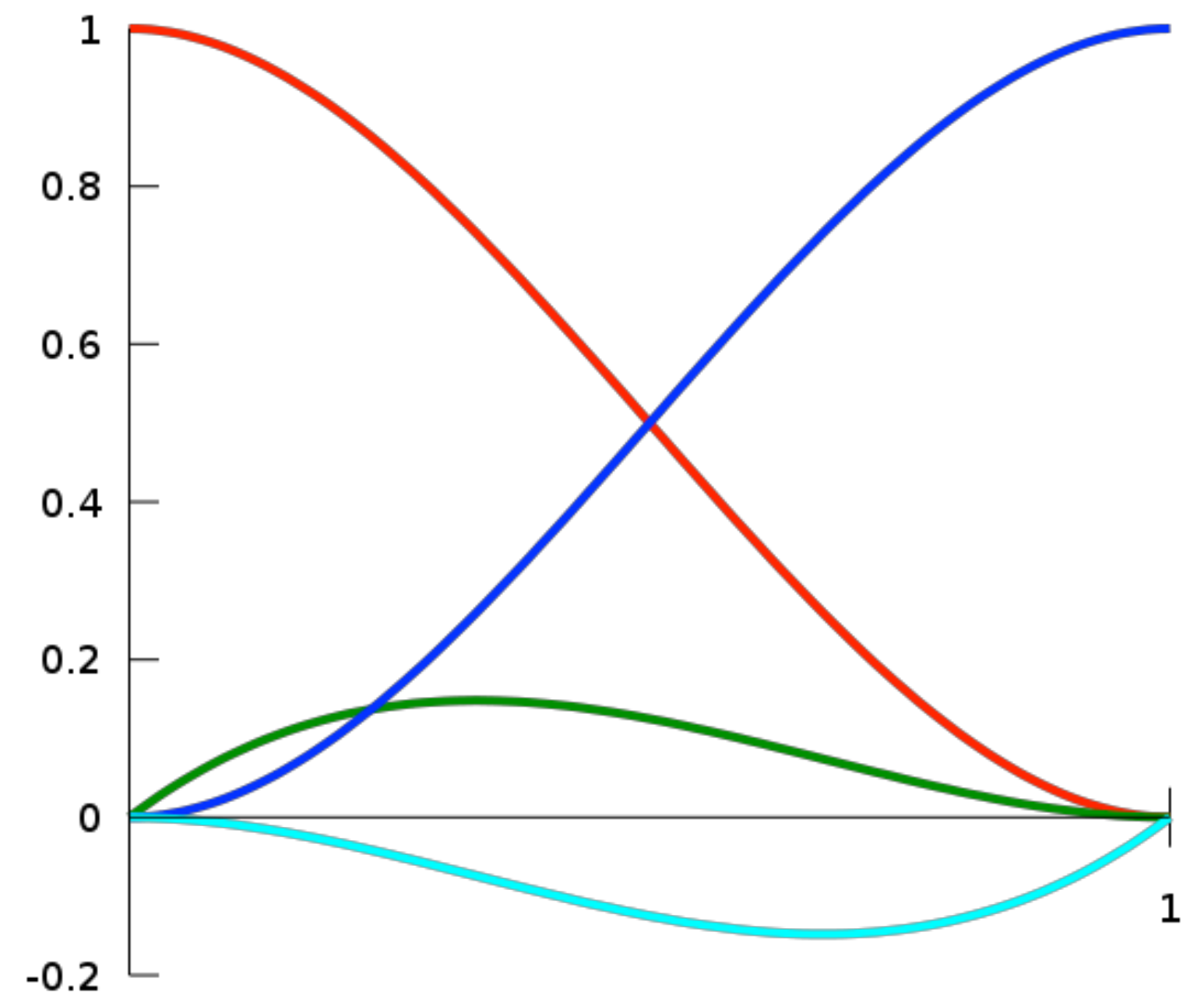
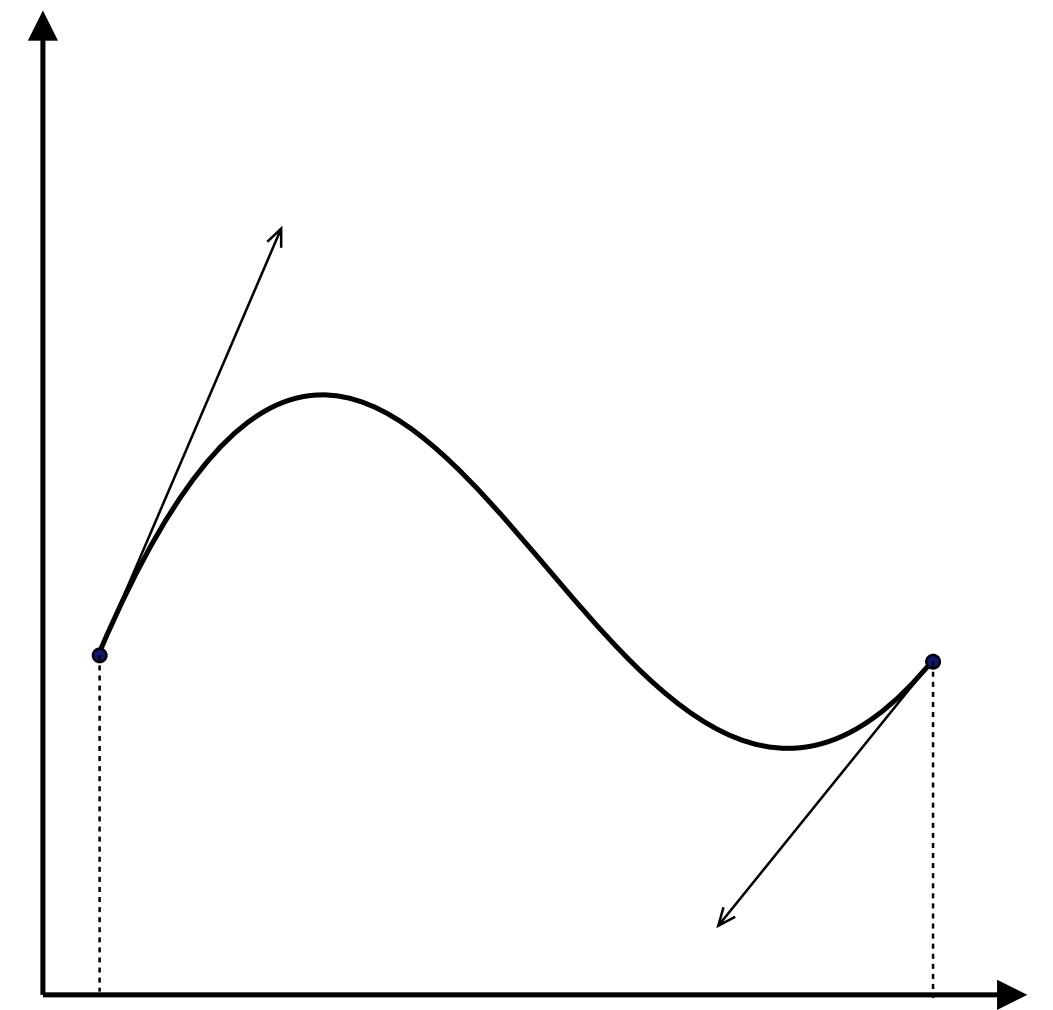
$$x(u) = \mathcal{P}^3(u) \cdot \beta_H \cdot \mathbf{h}$$

$$= \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix}^T \cdot \mathbf{h}$$

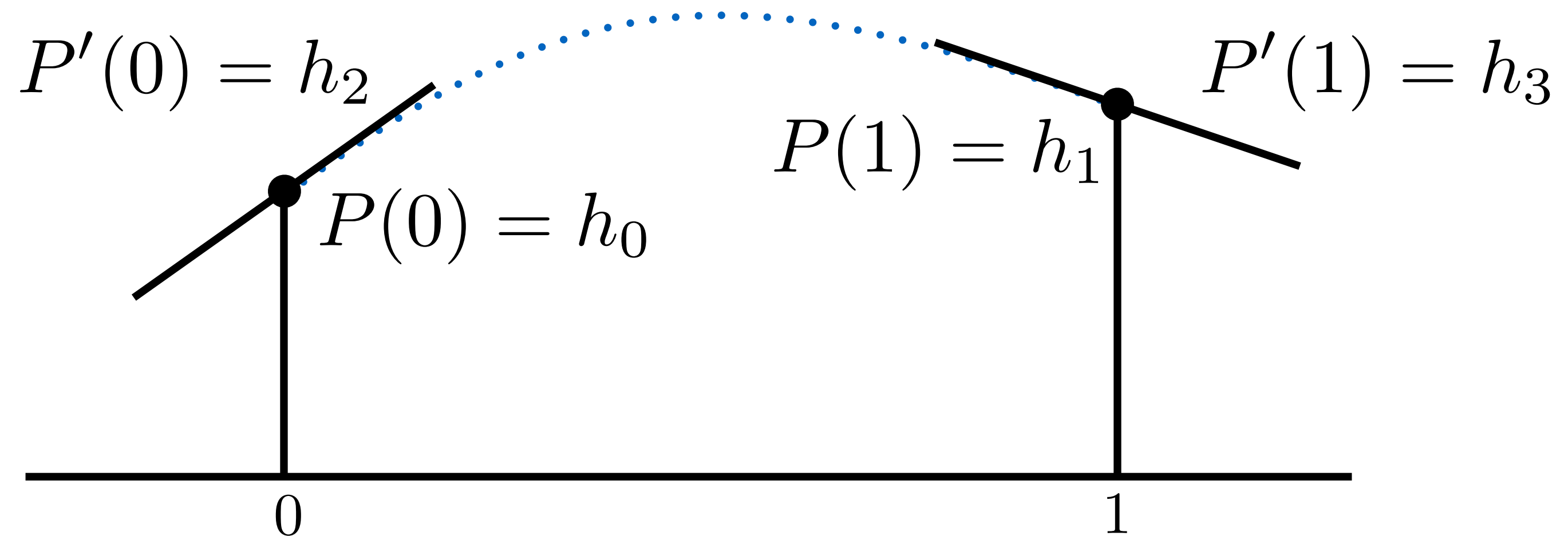
Hermite Basis Functions

Basis matrix depends on u_0 and u_1

We could use this same approach for higher order than cubic.



Recap: Cubic Hermite Interpolation



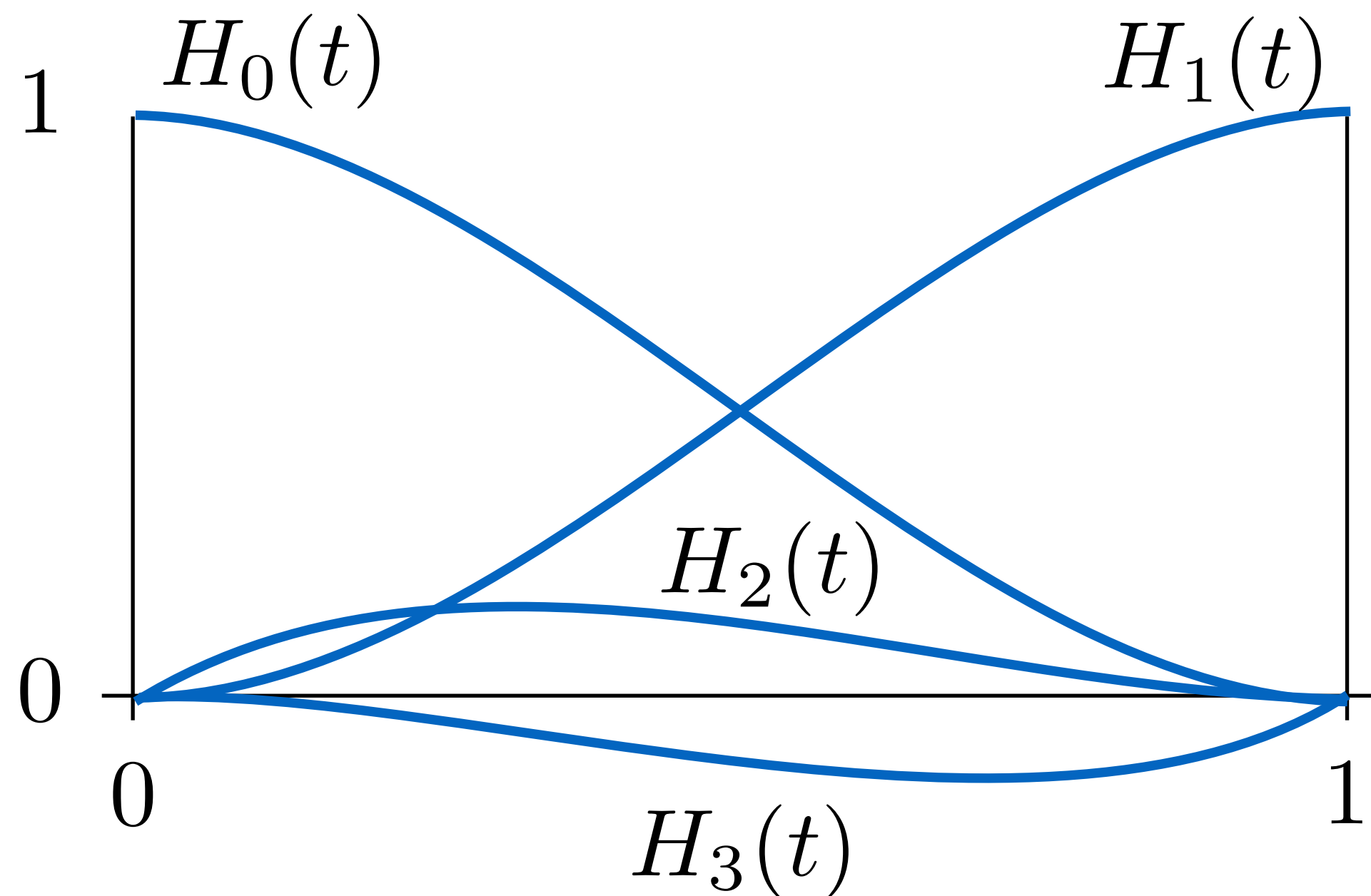
Inputs: values and derivatives at endpoints

Output: cubic polynomial that interpolates

Solution: weighted sum of Hermite basis functions

$$P(t) = h_0 H_0(t) + h_1 H_1(t) + h_2 H_2(t) + h_3 H_3(t)$$

Recap: Cubic Hermite Interpolation



$$H_0(t) = 2t^3 - 3t^2 + 1$$

$$H_1(t) = -2t^3 + 3t^2$$

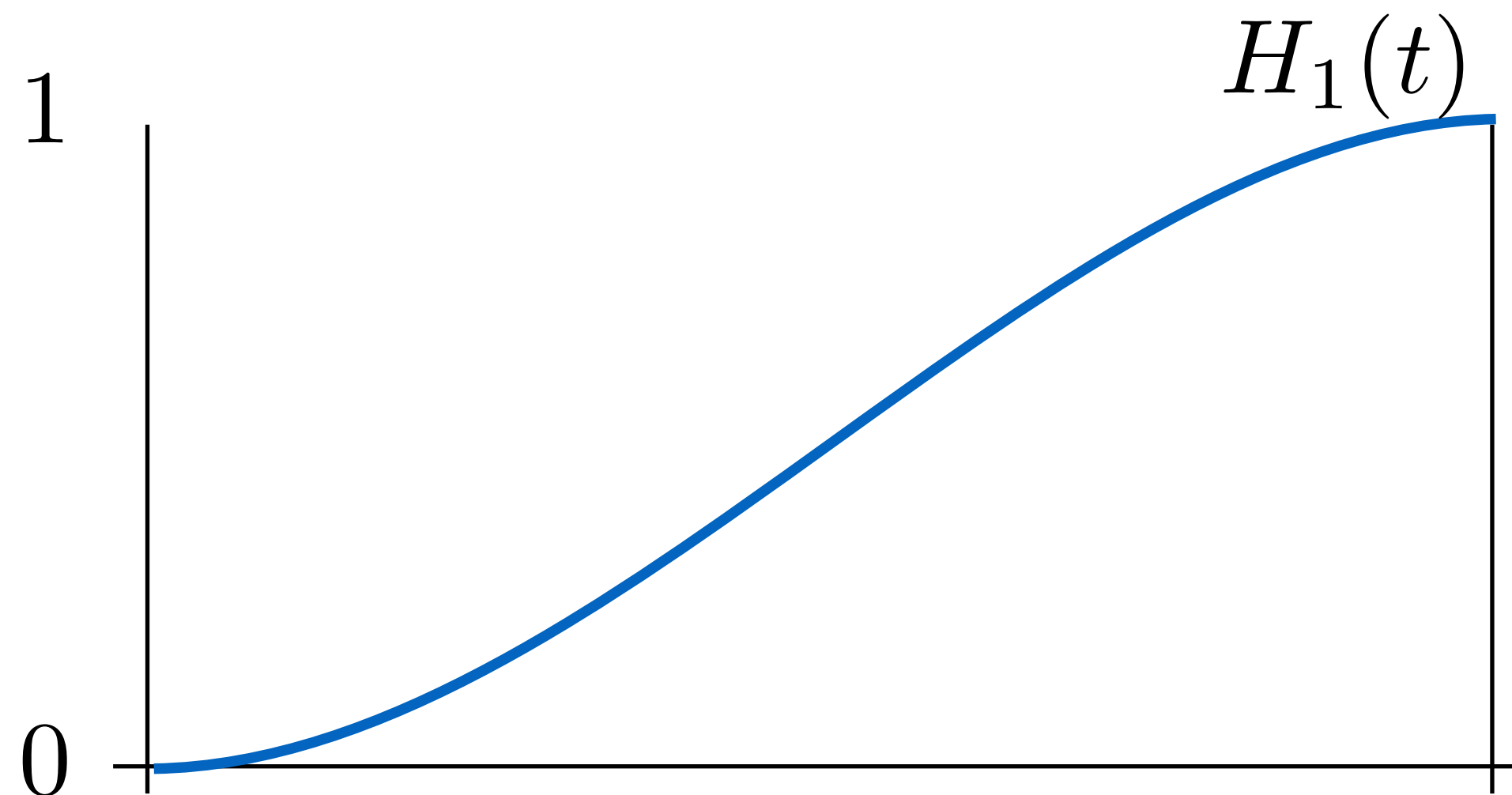
$$H_2(t) = t^3 - 2t^2 + t$$

$$H_3(t) = t^3 - t^2$$

Ease Function

A very useful function

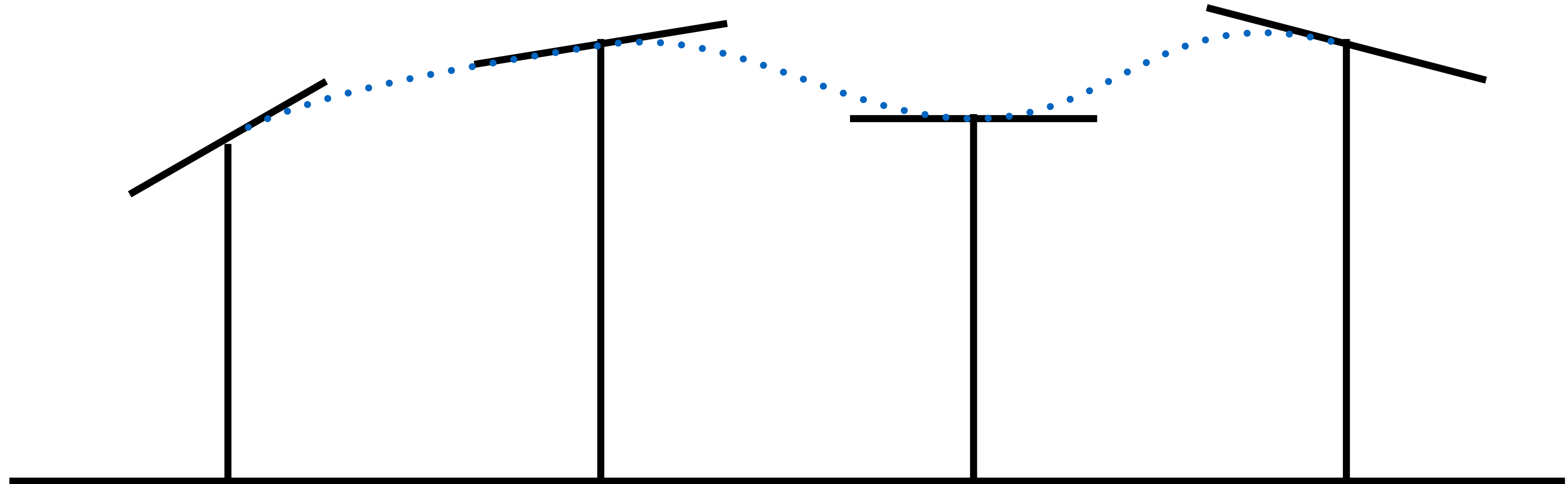
In animation, start and stop gently (zero velocity)



$$H_1(t) = -2t^3 + 3t^2 = t^2(3 - 2t)$$

Catmull-Rom Interpolation

Hermite Spline Interpolation



Inputs: sequence of values and derivatives

Question: Why not use higher-order polynomial?

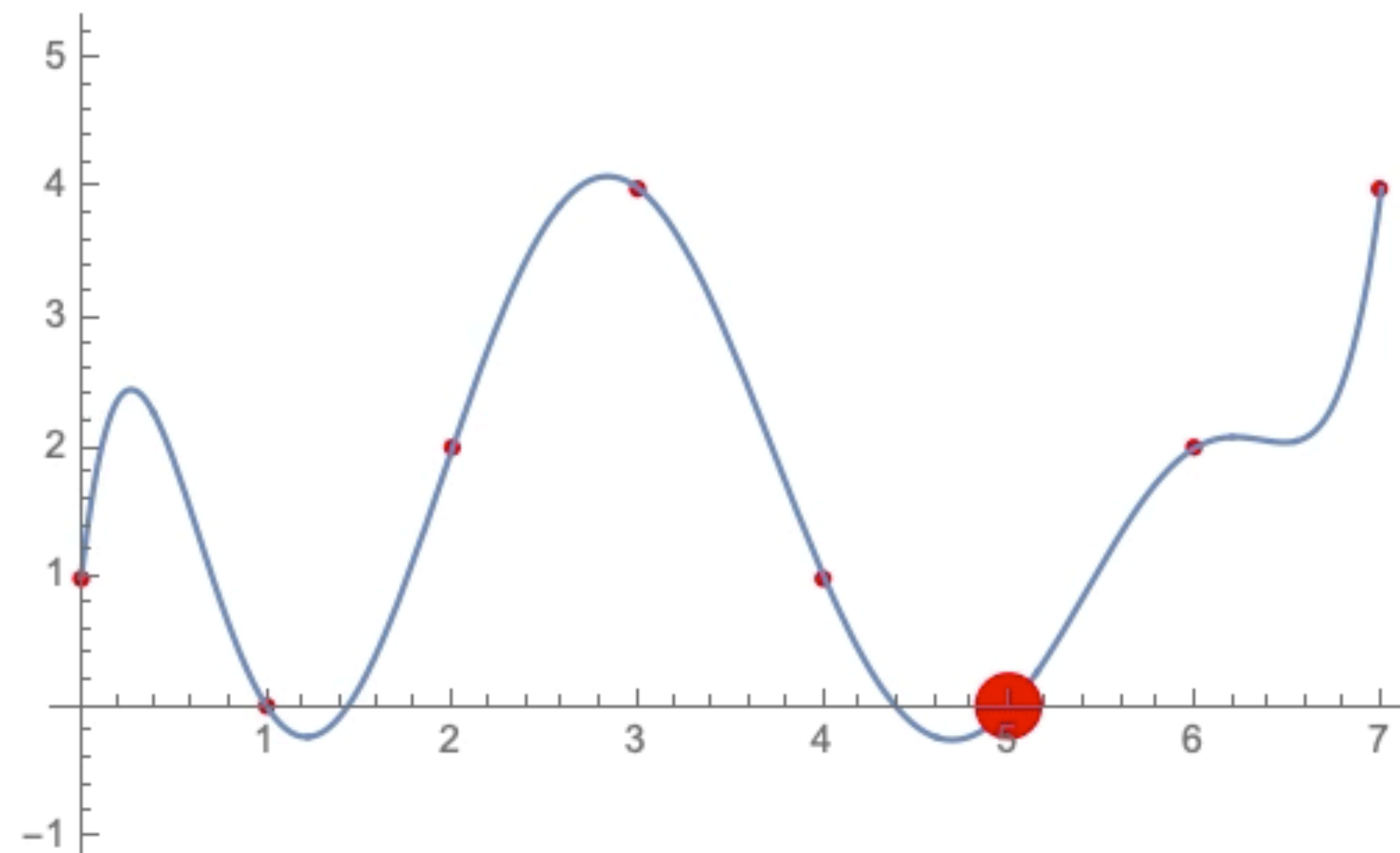
Higher order polynomials tend to wiggle excessively.

Would like to keep effect of edits "local".

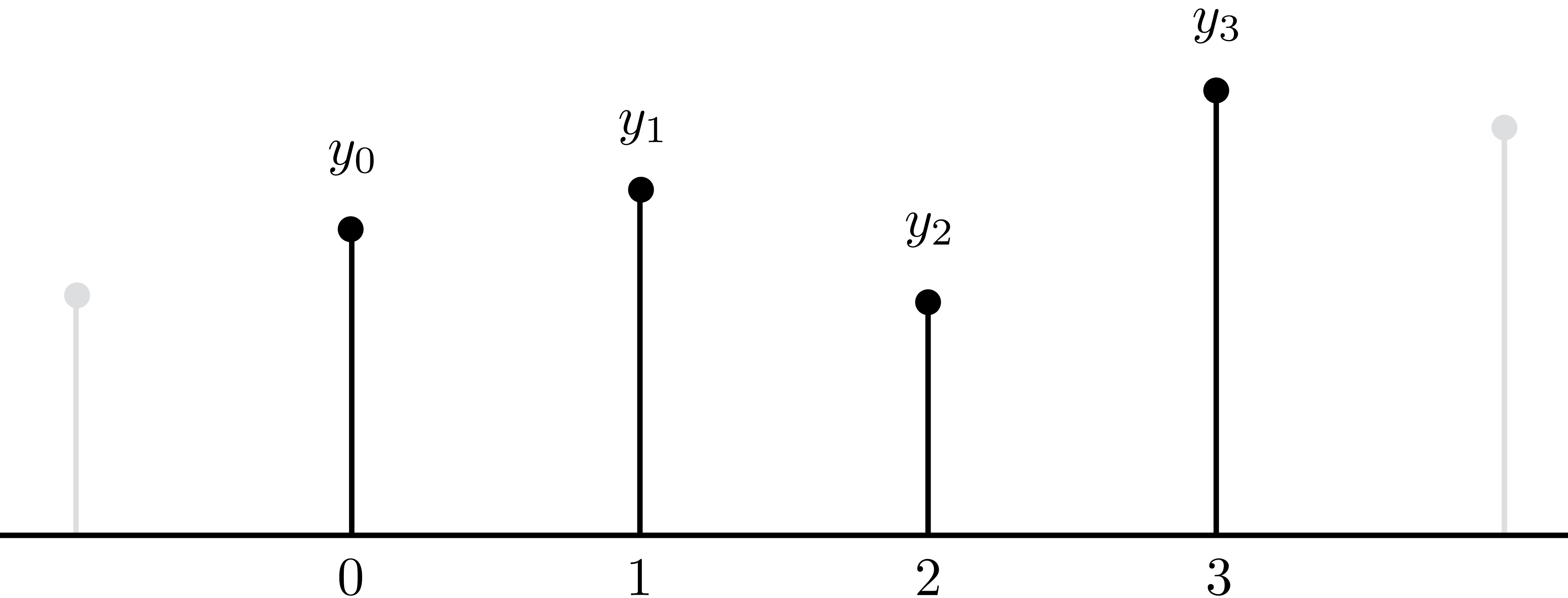
Higher-Order Polynomial?

High-degree polynomials don't behave well

- Unstable
- No local edits
- Expensive

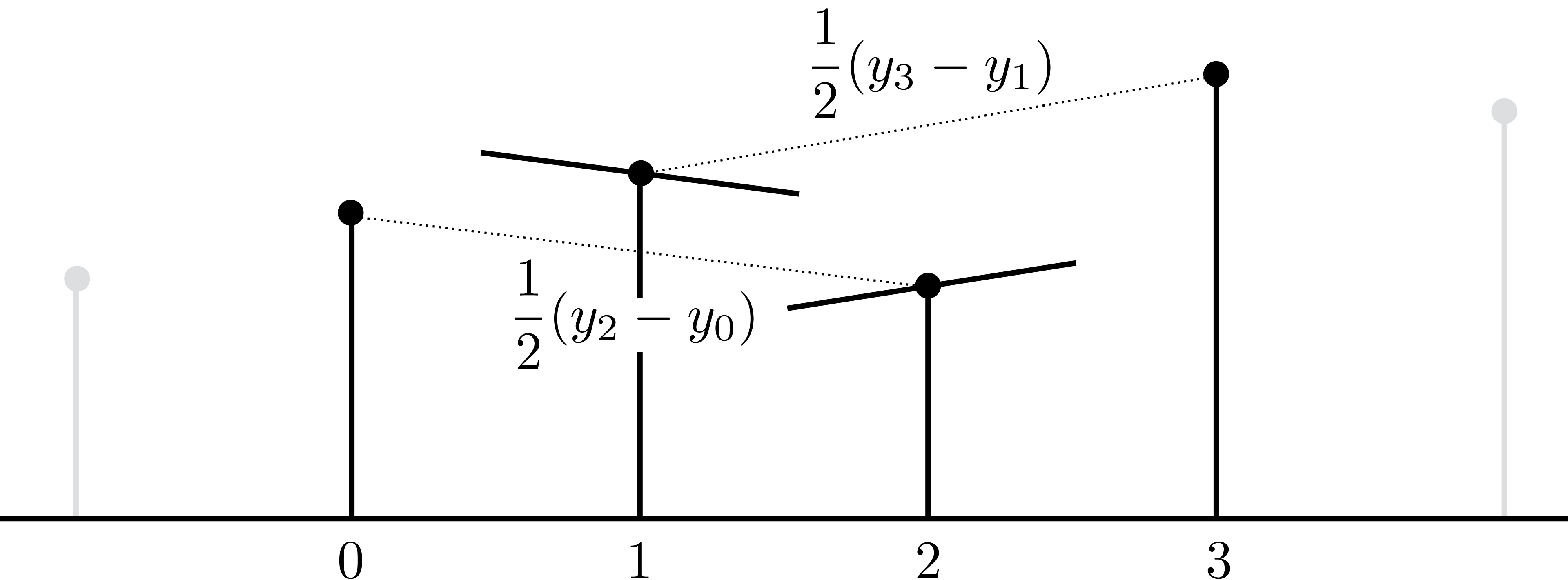


Catmull-Rom Interpolation



Inputs: sequence of values

Catmull-Rom Interpolation

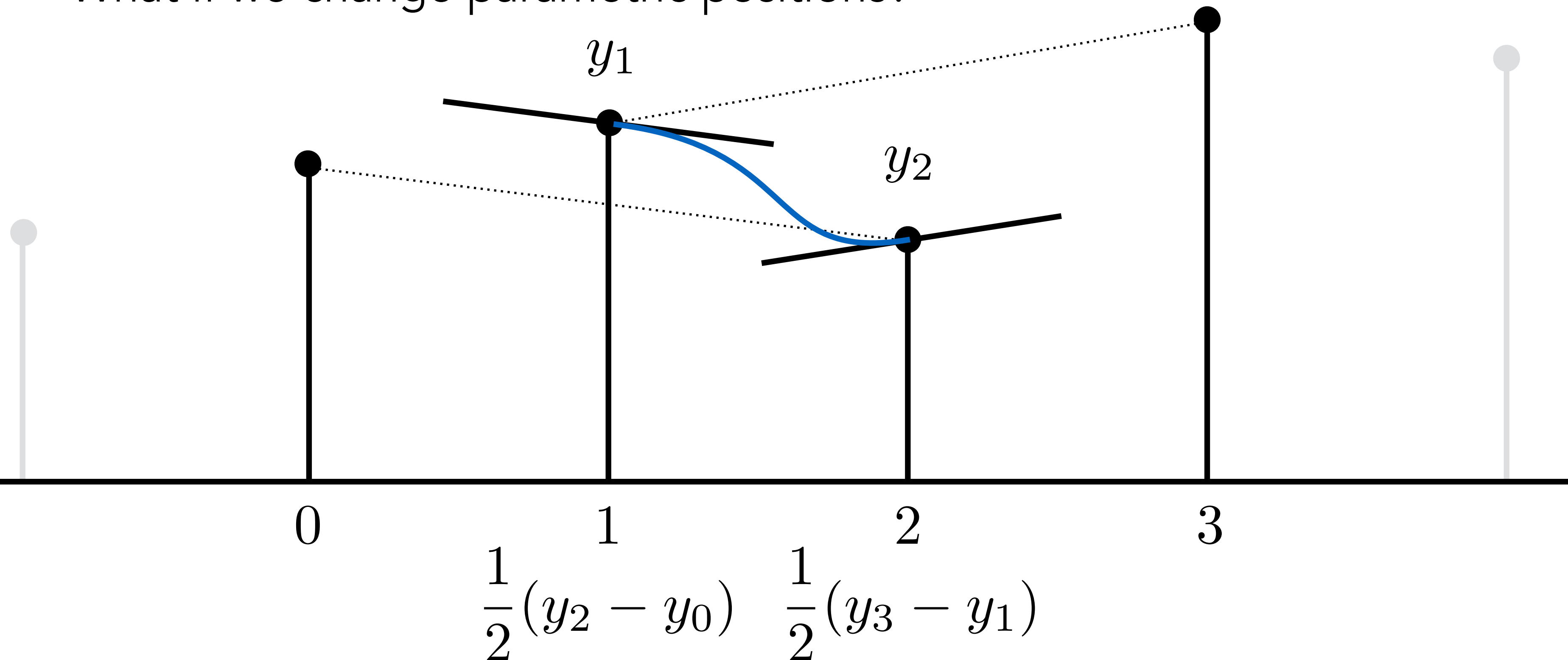


Rule for derivatives:

Match slope between previous and next values

Catmull-Rom Interpolation

Note the domain where curve is defined!
What if we change parametric positions?

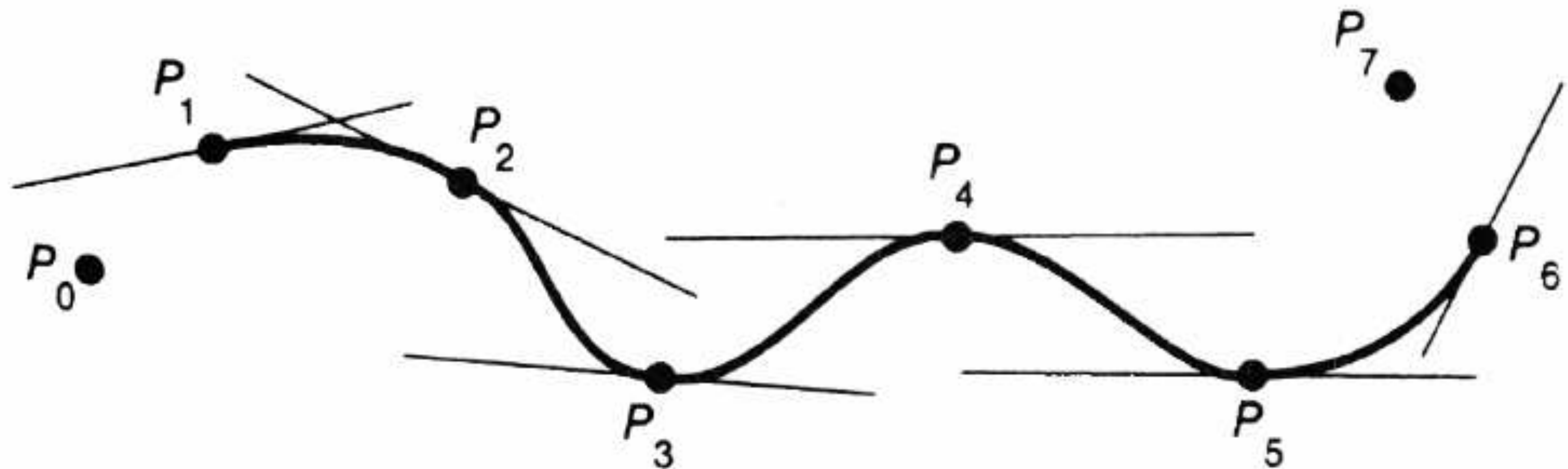


Then use Hermite interpolation

Catmull-Rom Spline

Input: sequence of points

Output: spline that interpolates all points with C1 continuity

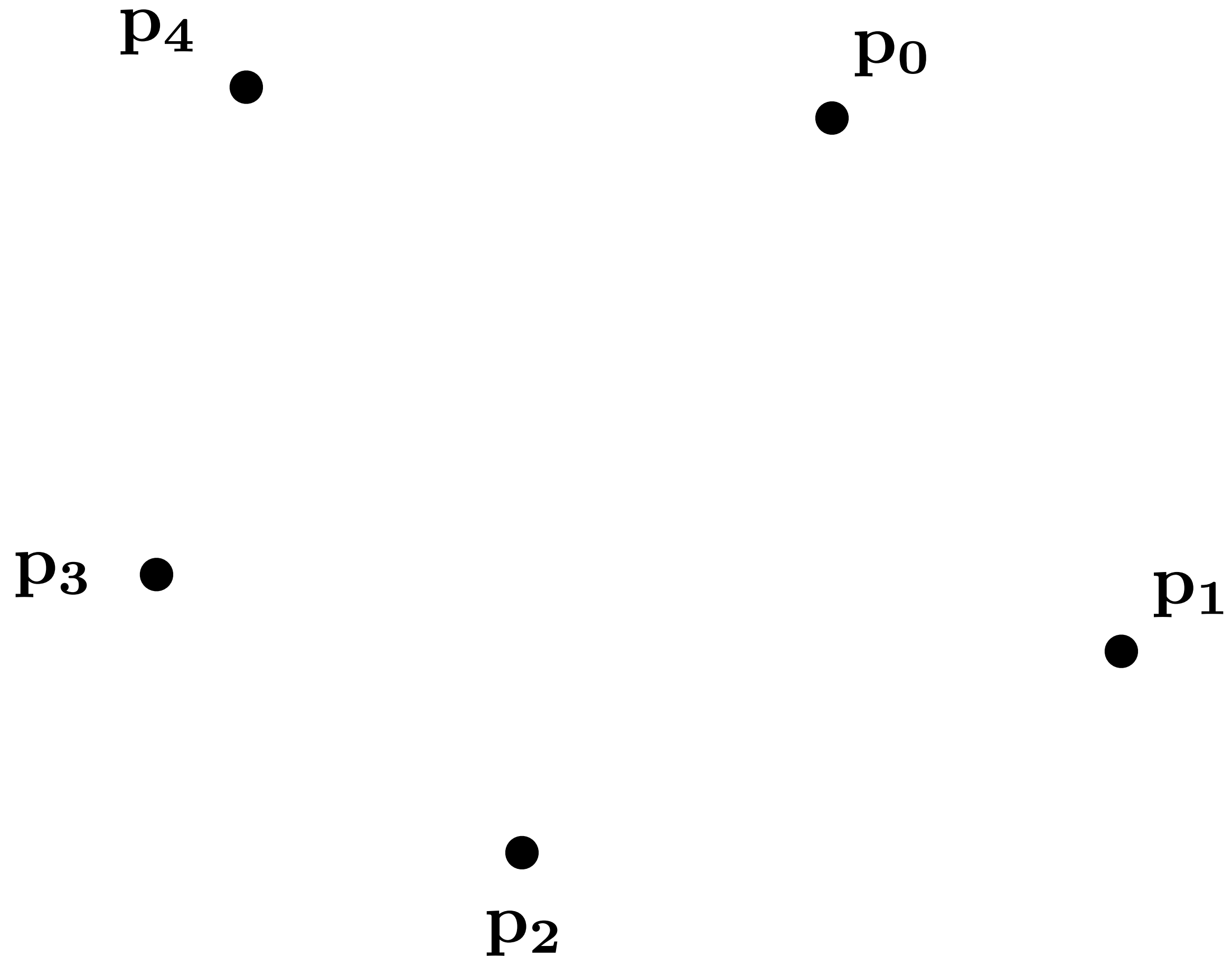


- Piecewise Cubic Curve
- C1 continuous!

Interpolating Points & Vectors

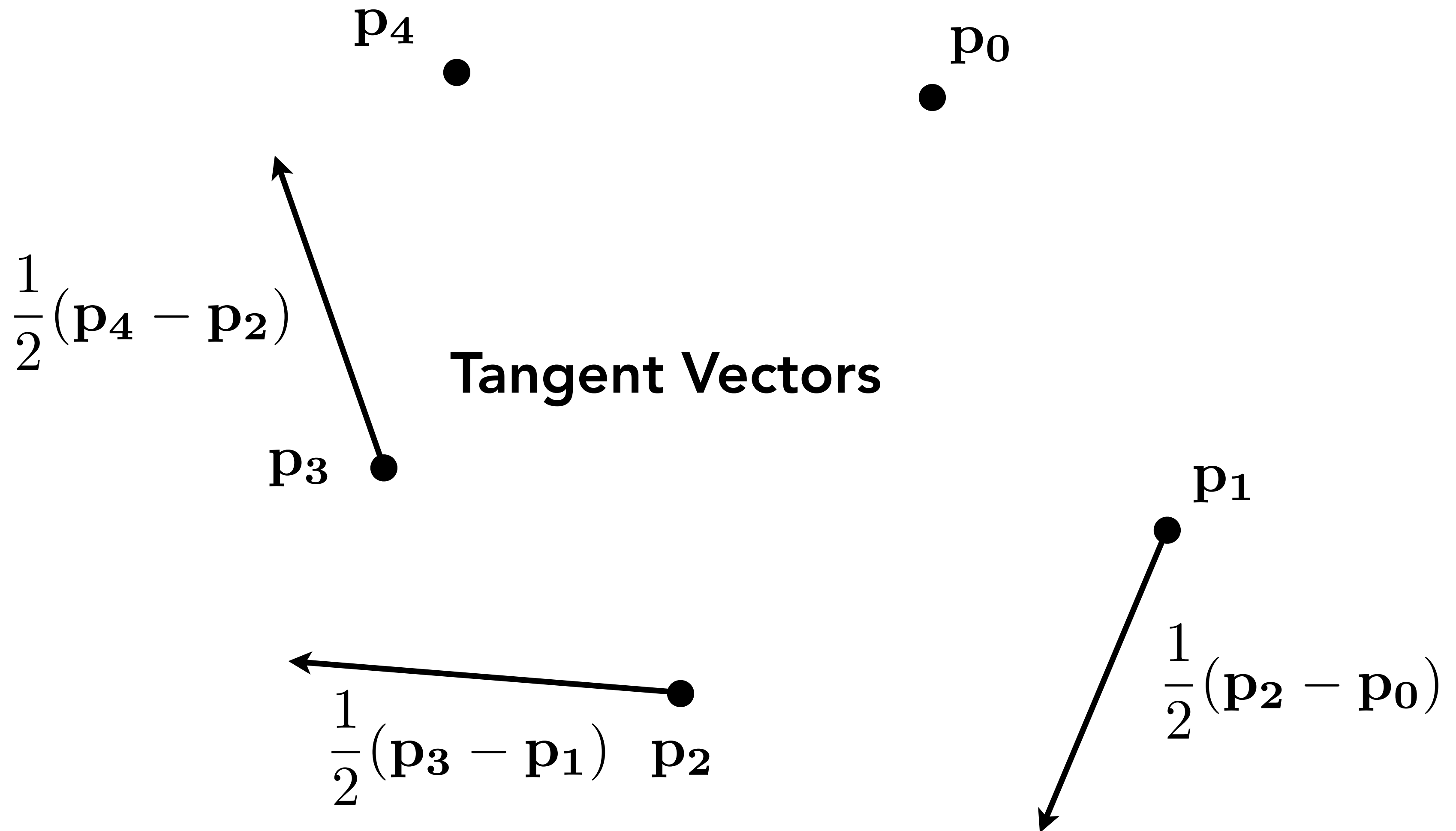
Can Interpolate Points As Easily As Values

E.g. point (0,1,3)
in 3D space, or
even a general
vector in N
dimensions



Catmull-Rom 3D spline control points

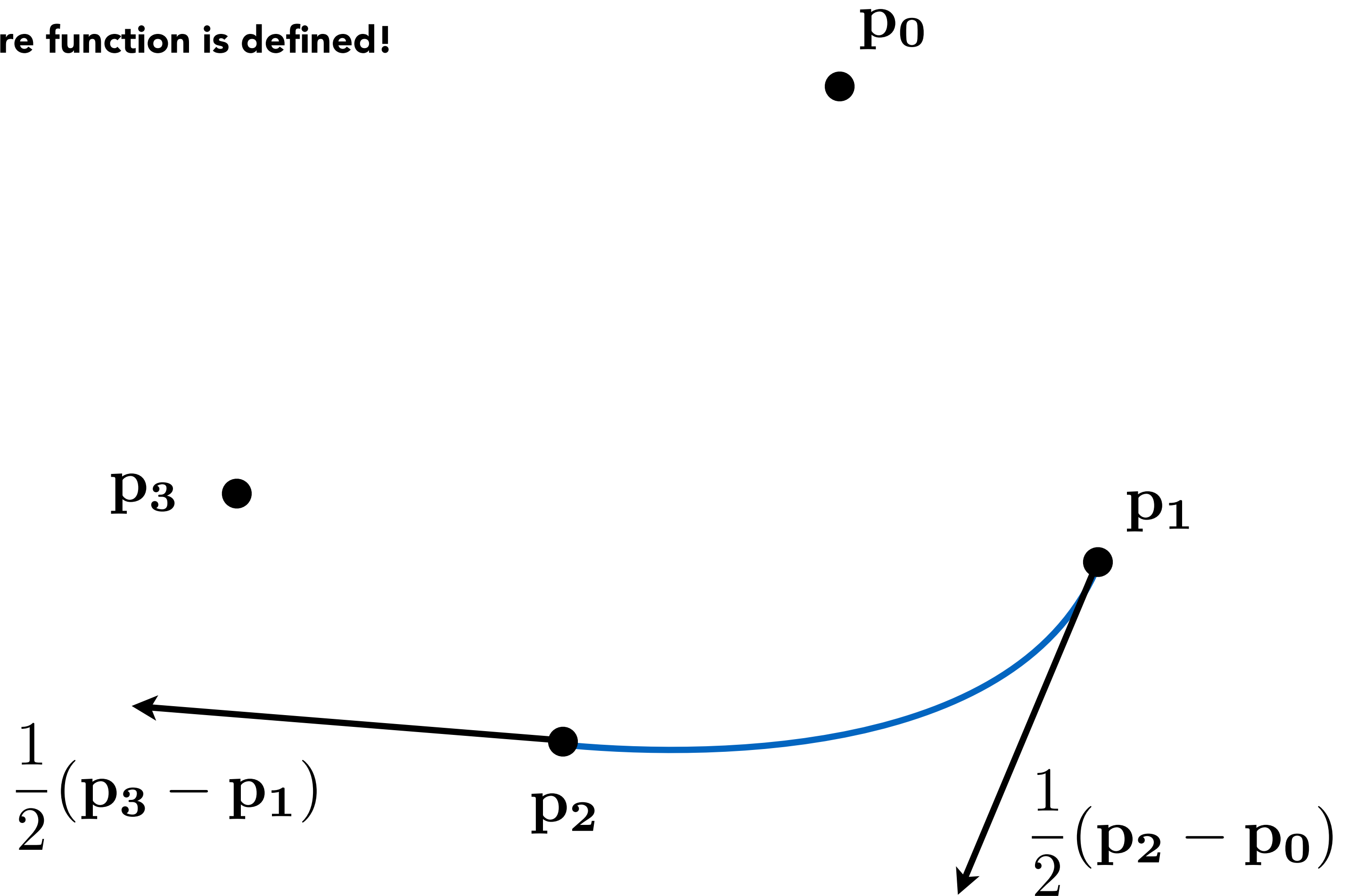
Can Interpolate Points As Easily As Values



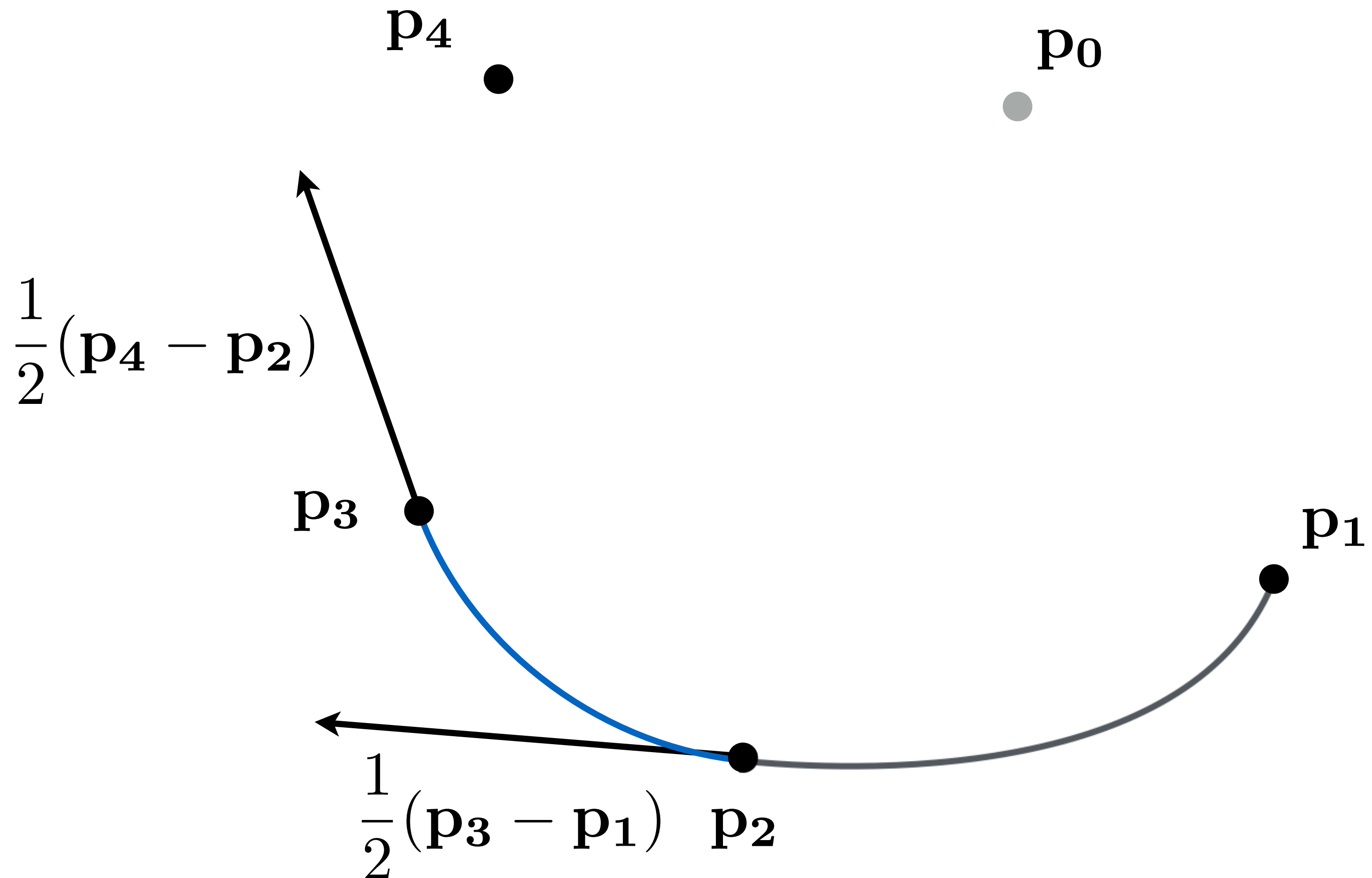
Catmull-Rom 3D tangent vectors

Can Interpolate Points As Easily As Values

Notice where function is defined!



Can Interpolate Points As Easily As Values



Catmull-Rom 3D space curve

Use Basis Functions to Define Curves

General formula for a particular interpolation scheme: $\mathbf{p}(t) = \sum_{i=0}^n \mathbf{p}_i F_i(t)$

$$x(t) = \sum_{i=0}^n x_i F_i(t) \quad y(t) = \sum_{i=0}^n y_i F_i(t) \quad z(t) = \sum_{i=0}^n z_i F_i(t)$$

Coefficients $\mathbf{p}_i$ can be points & vectors, not just values
 $F_i(t)$ are basis functions for the interpolation scheme.

Saw $H_i(t)$ for Hermite interpolation earlier. Will see $C_i(t)$ for Catmull-Rom shortly, and $B_i(t)$ for Bézier scheme later. The basis functions are properties of the interpolation scheme.

Matrix Form of Catmull-Rom Space Curve?

Use Hermite matrix form

- Points & tangents given by Catmull-Rom rules

Power Basis

Polynomial Coefficients

Hermite points

$$\mathbf{h}_0 = \mathbf{p}_1$$

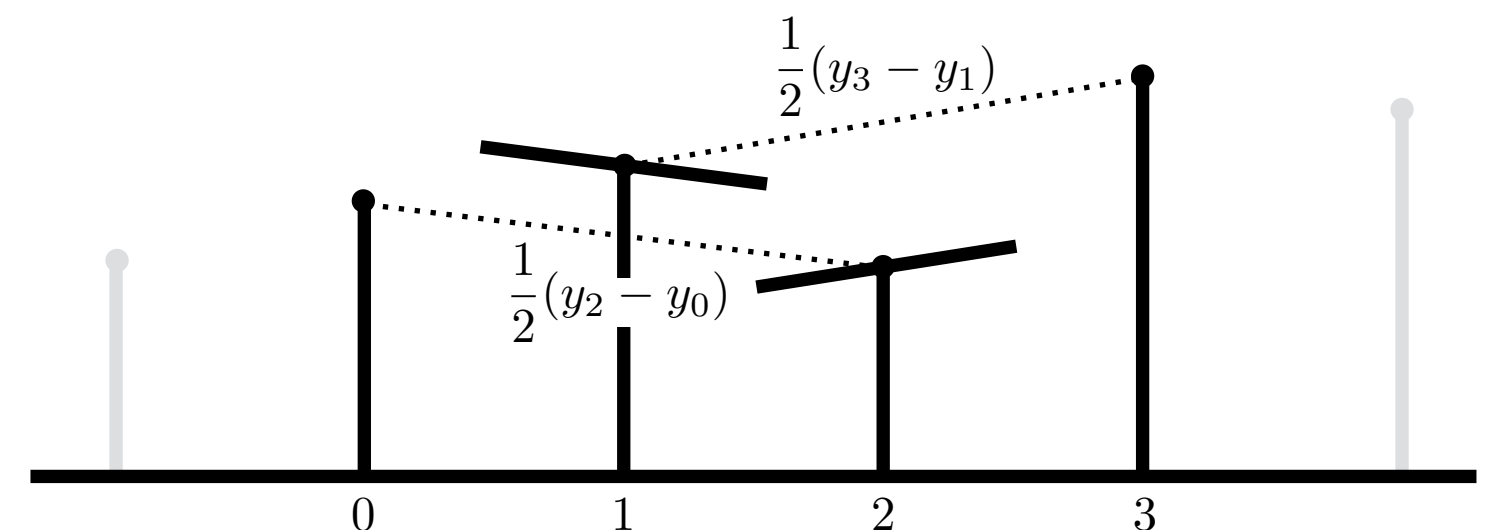
$$\mathbf{h}_1 = \mathbf{p}_2$$

$$\mathbf{h}_2 = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$$

$$\mathbf{h}_3 = \frac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$$

Hermite tangents

$$\mathbf{p}(u) = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}^T \cdot \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}$$



Matrix Form of Catmull-Rom Space Curve?

Use Hermite matrix form

- Points & tangents given by Catmull-Rom rules

Power Basis

Hermite Basis Matrix β_H

Hermite Control Values

Hermite points

$$\mathbf{h}_0 = \mathbf{p}_1$$

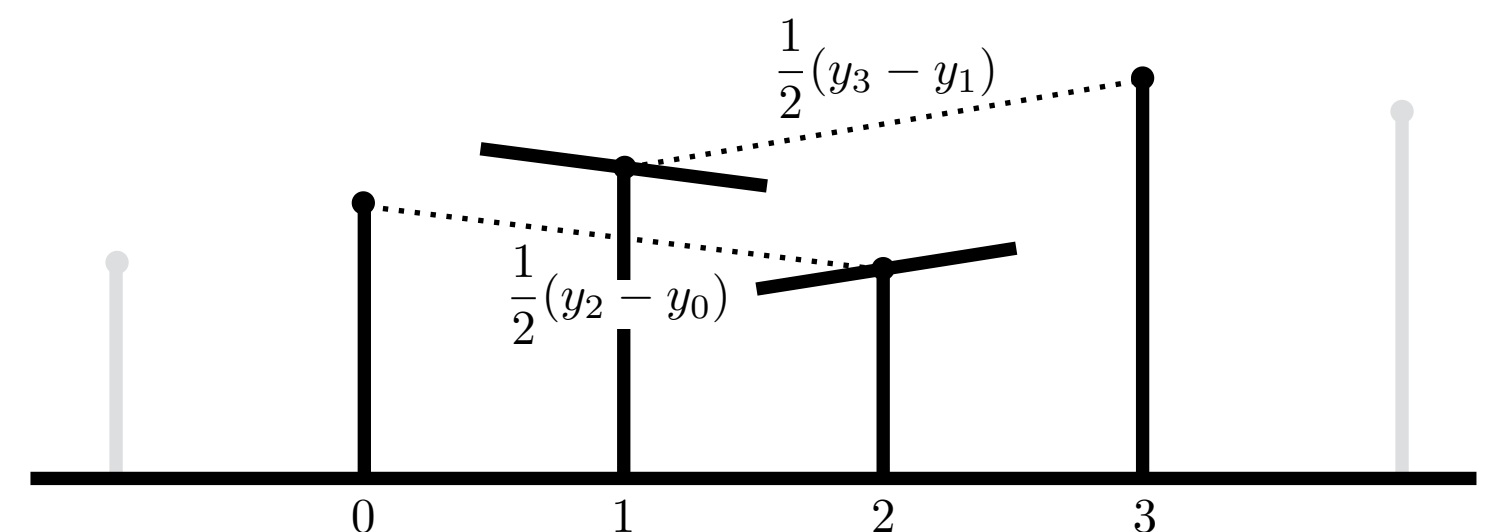
$$\mathbf{h}_1 = \mathbf{p}_2$$

$$\mathbf{h}_2 = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$$

$$\mathbf{h}_3 = \frac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$$

Hermite tangents

$$\mathbf{p}(u) = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}^T \cdot \beta_H \cdot \begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{bmatrix}$$



Matrix Form of Catmull-Rom Space Curve?

Use Hermite matrix form

- Points & tangents given by Catmull-Rom rules

Power Basis

Hermite points

$$\mathbf{h}_0 = \mathbf{p}_1$$

$$\mathbf{h}_1 = \mathbf{p}_2$$

$$\mathbf{h}_2 = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$$

$$\mathbf{h}_3 = \frac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$$

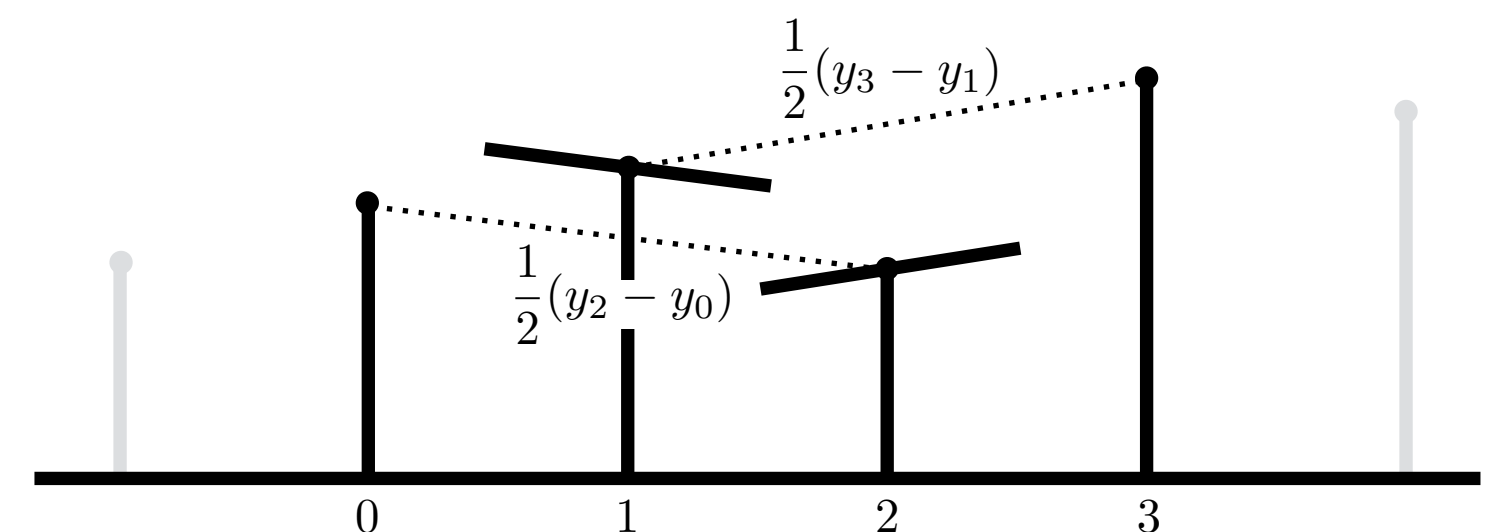
Hermite Basis Matrix β_H

Convert Catmull-Rom to Hermite

Catmull-Rom Control Points

Hermite tangents

$$\mathbf{p}(u) = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}^T \cdot \beta_H \cdot \mathbf{M}_{CR \rightarrow H} \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$



Matrix Form of Catmull-Rom Space Curve?

Use Hermite matrix form

- Points & tangents given by Catmull-Rom rules

Power Basis

Hermite points

$$\mathbf{h}_0 = \mathbf{p}_1$$

$$\mathbf{h}_1 = \mathbf{p}_2$$

$$\mathbf{h}_2 = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$$

$$\mathbf{h}_3 = \frac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$$

Hermite Basis Matrix β_H

Convert Catmull-Rom to Hermite

Catmull-Rom Control Points

Hermite tangents

$$\mathbf{p}(u) = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Matrix Form of Catmull-Rom Space Curve?

Use Hermite matrix form

- Points & tangents given by Catmull-Rom rules

Power Basis

Hermite points

$$\mathbf{h}_0 = \mathbf{p}_1$$

Catmull-Rom Basis Matrix β_{CR}

$$\mathbf{h}_1 = \mathbf{p}_2$$

Catmull-Rom Control Points

Hermite tangents

$$\mathbf{h}_2 = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$$

$$\mathbf{h}_3 = \frac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$$

$$\mathbf{p}(u) = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}^T \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 1 & -2\frac{1}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 1\frac{1}{2} & -1\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Matrix Form of Catmull-Rom Space Curve?

Use Hermite matrix form

- Points & tangents given by Catmull-Rom rules

Power Basis

Hermite points

$$\mathbf{h}_0 = \mathbf{p}_1$$

Catmull-Rom Basis Matrix β_{CR}

$$\mathbf{h}_1 = \mathbf{p}_2$$

Catmull-Rom Control Points

Hermite tangents

$$\mathbf{h}_2 = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$$

$$\mathbf{h}_3 = \frac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$$

$$\mathbf{p}(u) = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}^T \cdot \beta_{CR} \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{p}(u) = \mathcal{P}^3(u) \cdot \beta_{CR} \cdot \mathbf{p}$$

Matrix Form of Catmull-Rom Space Curve

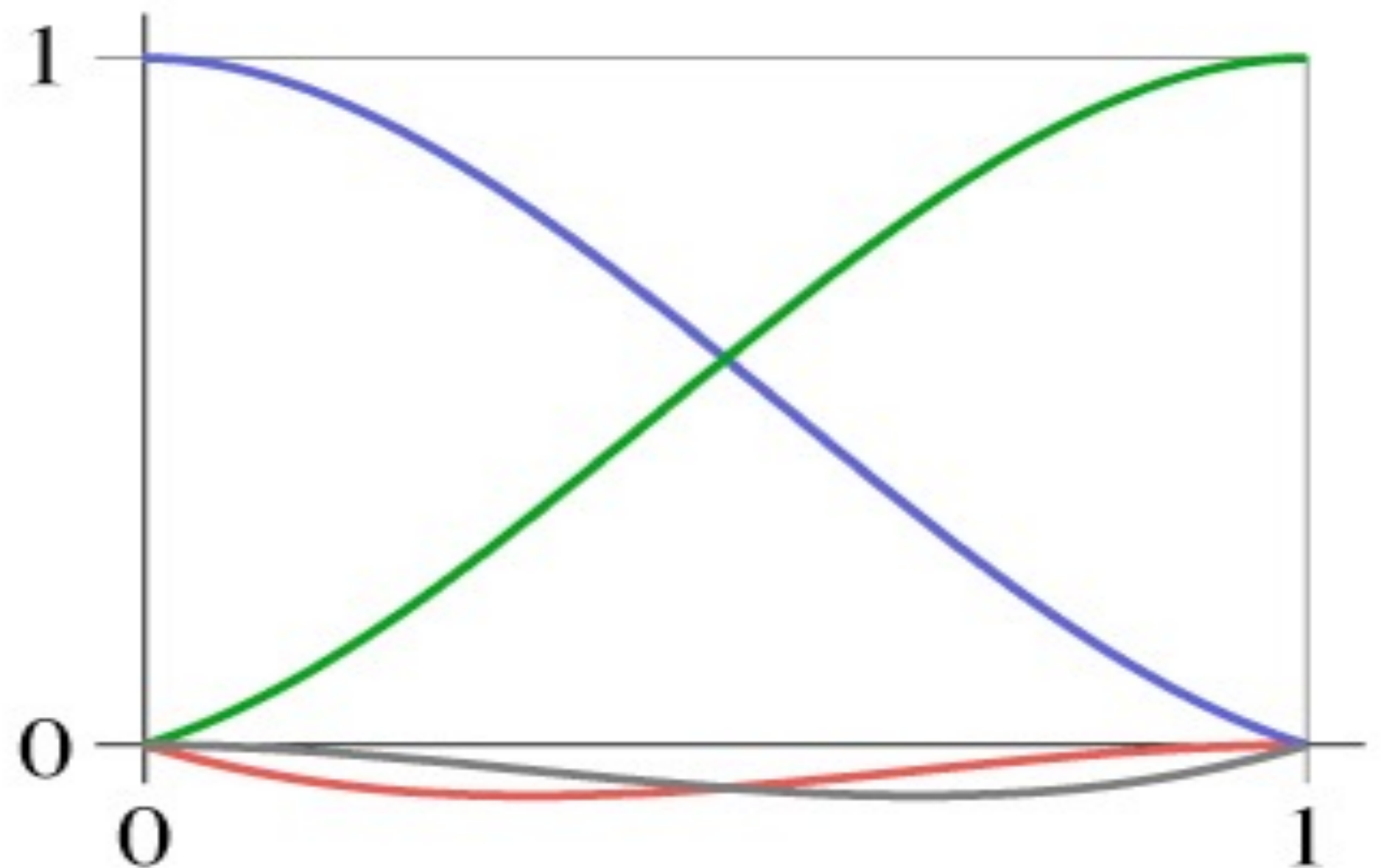
$$\mathbf{p}(u) = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}^T \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} & 0 \\ 1 & -2\frac{1}{2} & 2 & \frac{-1}{2} \\ \frac{-1}{2} & 1\frac{1}{2} & -1\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

↑
↑
↑
↑

$$\mathbf{p}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_{CR} \cdot \mathbf{p}$$

Matrix columns are
Catmull-Rom basis
functions

Compare to Hermite



Catmull-Rom Basis Functions

$$\mathbf{p}(u) = \begin{bmatrix} 0 - \frac{1}{2}u + 1u^2 - \frac{1}{2}u^3 \\ 1 + 0u - 2\frac{1}{2}u^2 + 1\frac{1}{2}u^3 \\ 0 + \frac{1}{2}u + 2u^2 - 1\frac{1}{2}u^3 \\ 0 + 0u - \frac{1}{2}u^2 + \frac{1}{2}u^3 \end{bmatrix}^T \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

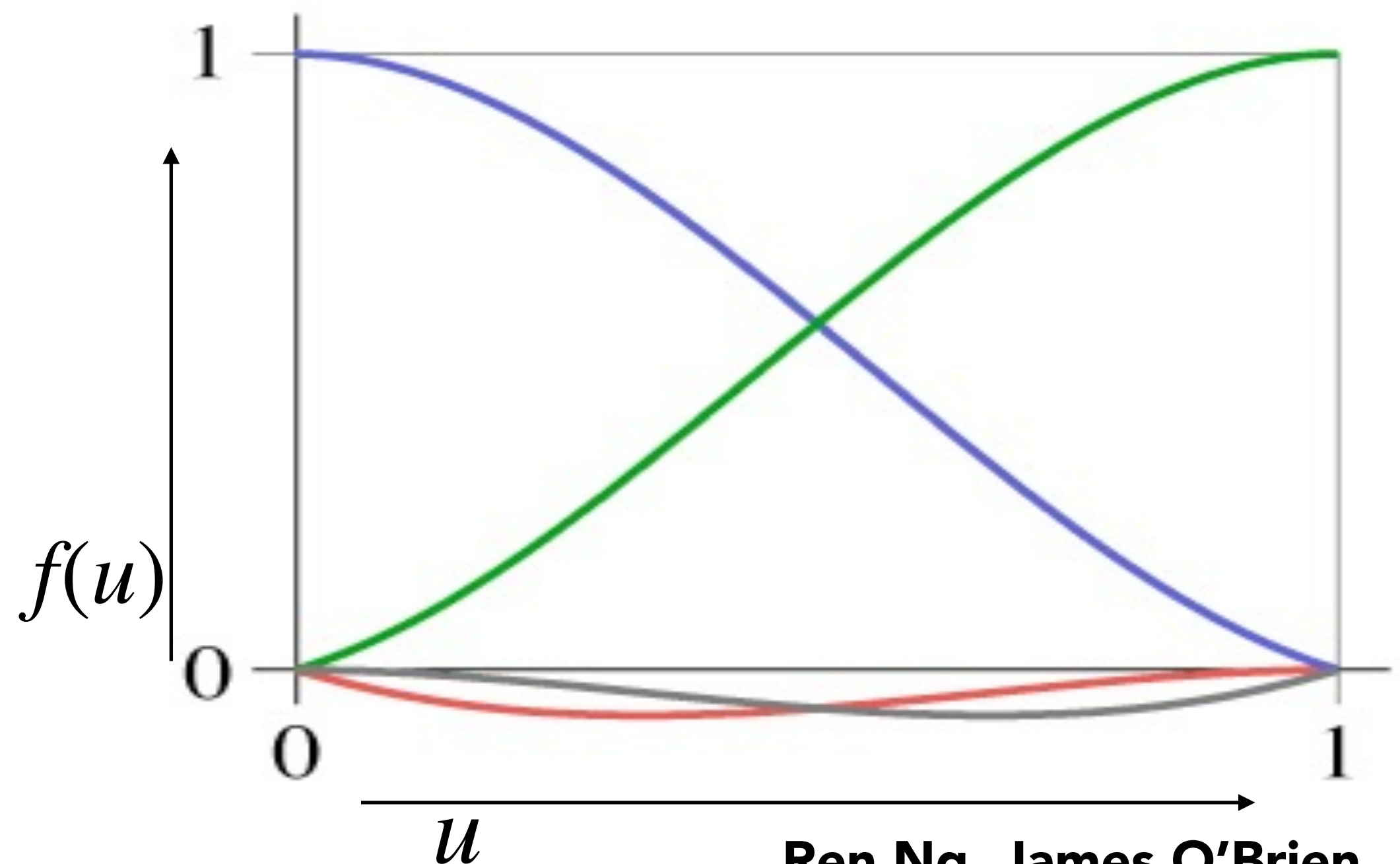
Diagram showing arrows pointing from the basis functions to the control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$:

- Red arrow points to the first row (from $\mathbf{p}_0$ and $\mathbf{p}_1$)
- Blue arrow points to the second row (from $\mathbf{p}_1$ and $\mathbf{p}_2$)
- Green arrow points to the third row (from $\mathbf{p}_2$ and $\mathbf{p}_3$)
- Grey arrow points to the fourth row (from $\mathbf{p}_2$ and $\mathbf{p}_3$)

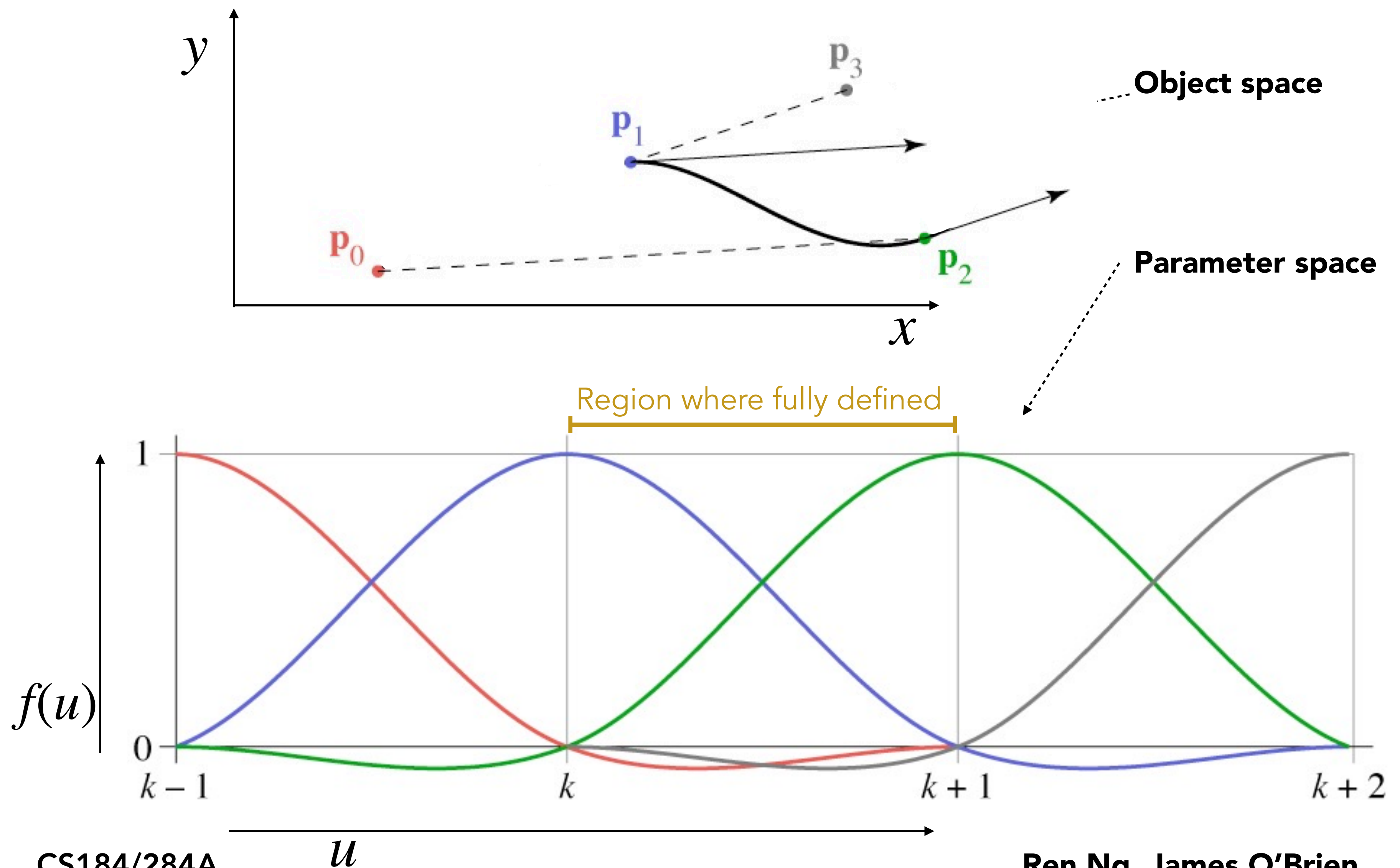
$$\mathbf{p}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_{CR} \cdot \mathbf{p}$$

Matrix columns are
Catmull-Rom basis
functions

Compare to Hermite



Catmull-Rom Basis in "Hump" form



Bézier Curves

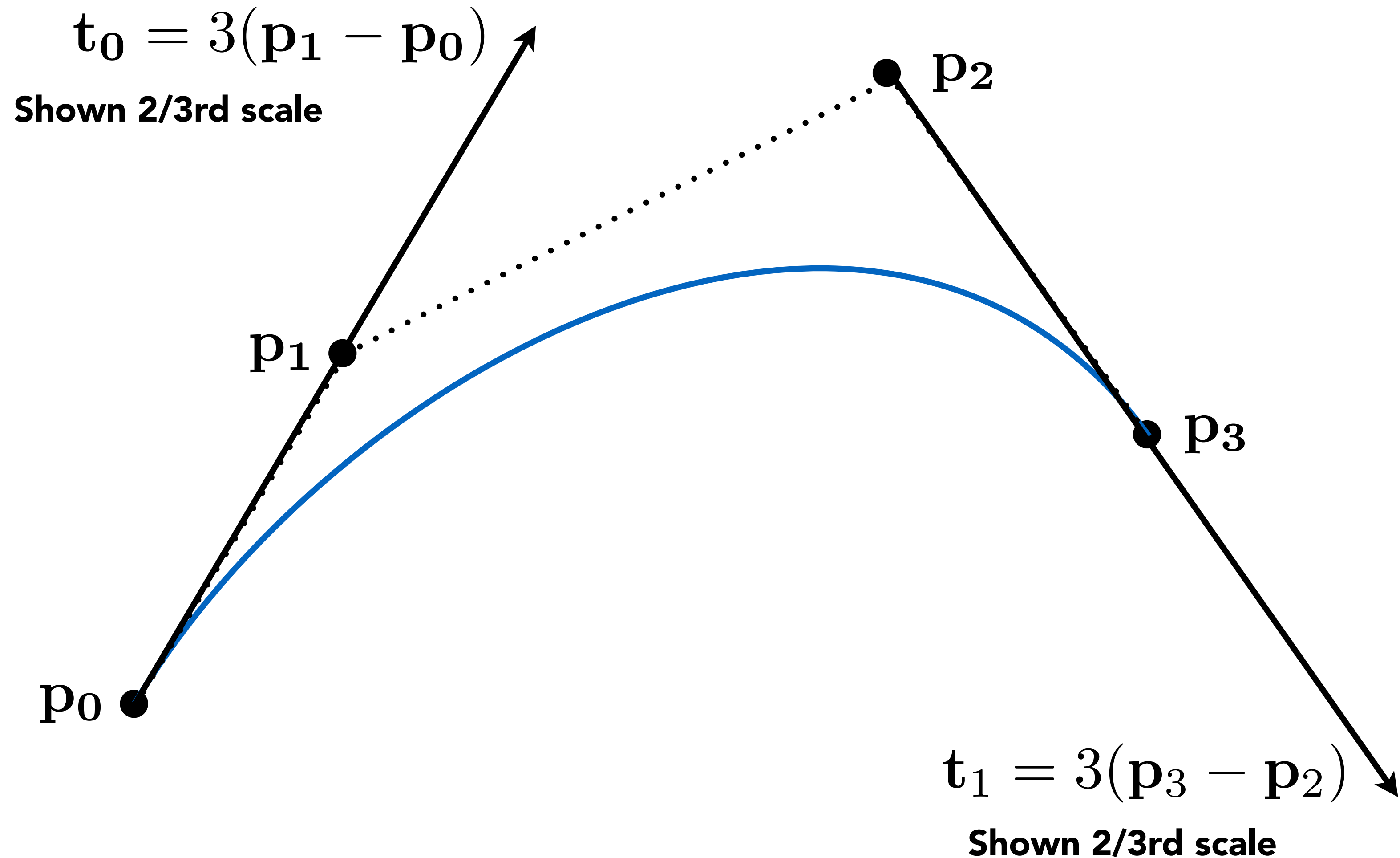
Examples of Geometry



CS184/284A

Ren Ng, James O'Brien

Defining Cubic Bézier Curve With Tangents



Cubic Bézier

Similar to Hermite, but specify tangents indirectly

$$x(u) = c_0 + c_1u + c_2u^2 + c_3u^3$$

$$x(u_0) = h_0 = p_0$$

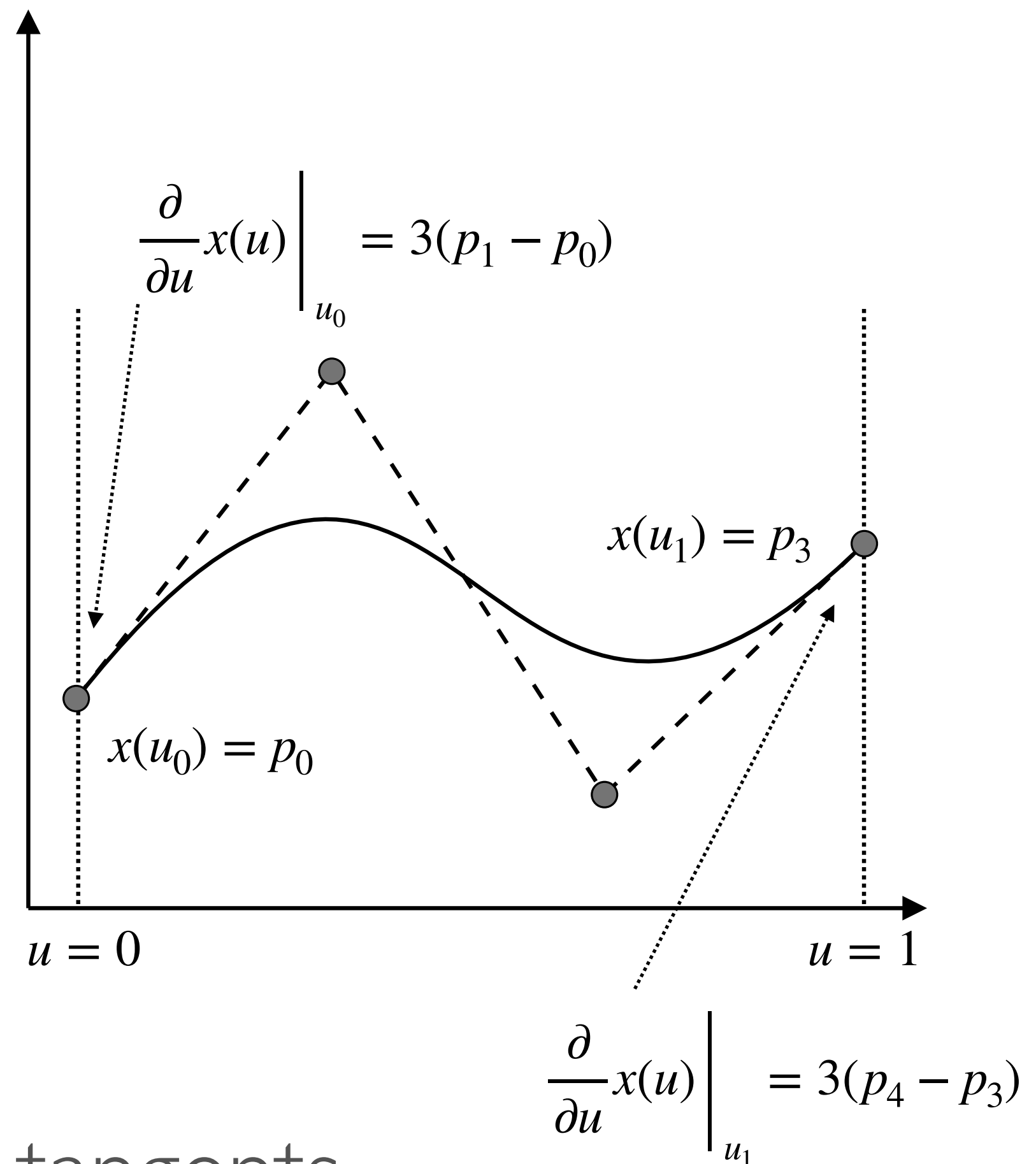
$$x(u_1) = h_1 = p_3$$

$$x'(u_0) = h_2 = 3(p_1 - p_0)$$

$$x'(u_1) = h_3 = 3(p_3 - p_2)$$

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_H \cdot \mathbf{h}$$

Note: all the control points are actually points in space, no tangents.



Cubic Bézier

Similar to Hermite, but specify tangents indirectly

$$x(u) = c_0 + c_1u + c_2u^2 + c_3u^3$$

$$x(u_0) = h_0 = p_0$$

$$x(u_1) = h_1 = p_3$$

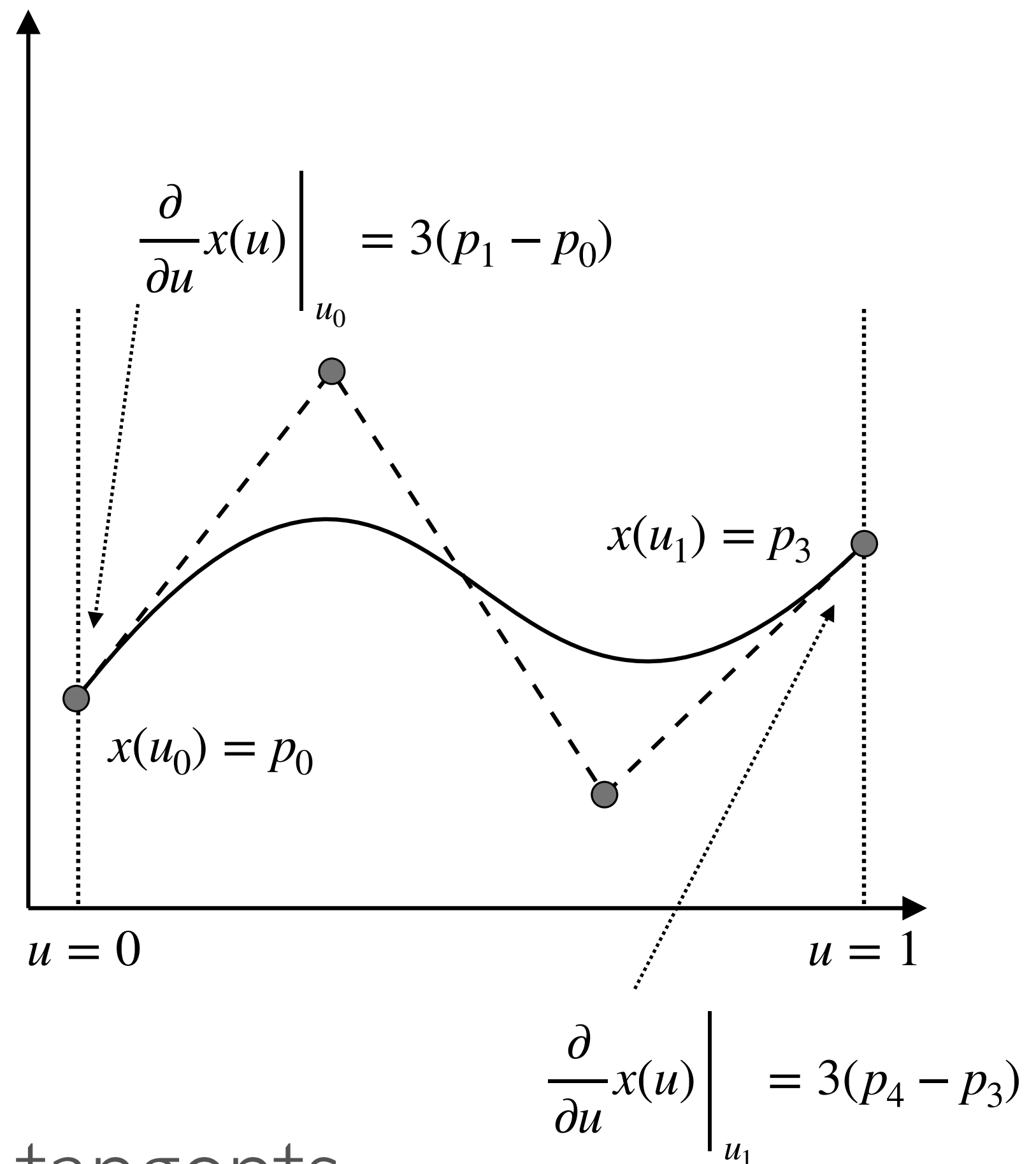
$$x'(u_0) = h_2 = 3(p_1 - p_0)$$

$$x'(u_1) = h_3 = 3(p_3 - p_2)$$

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_H \cdot \mathbf{M}_{Z \rightarrow H} \cdot \mathbf{p}$$

Bézier

Note: all the control points are actually points in space, no tangents.



Cubic Bézier

Similar to Hermite, but specify tangents indirectly

$$x(u) = c_0 + c_1u + c_2u^2 + c_3u^3$$

$$x(u_0) = h_0 = p_0$$

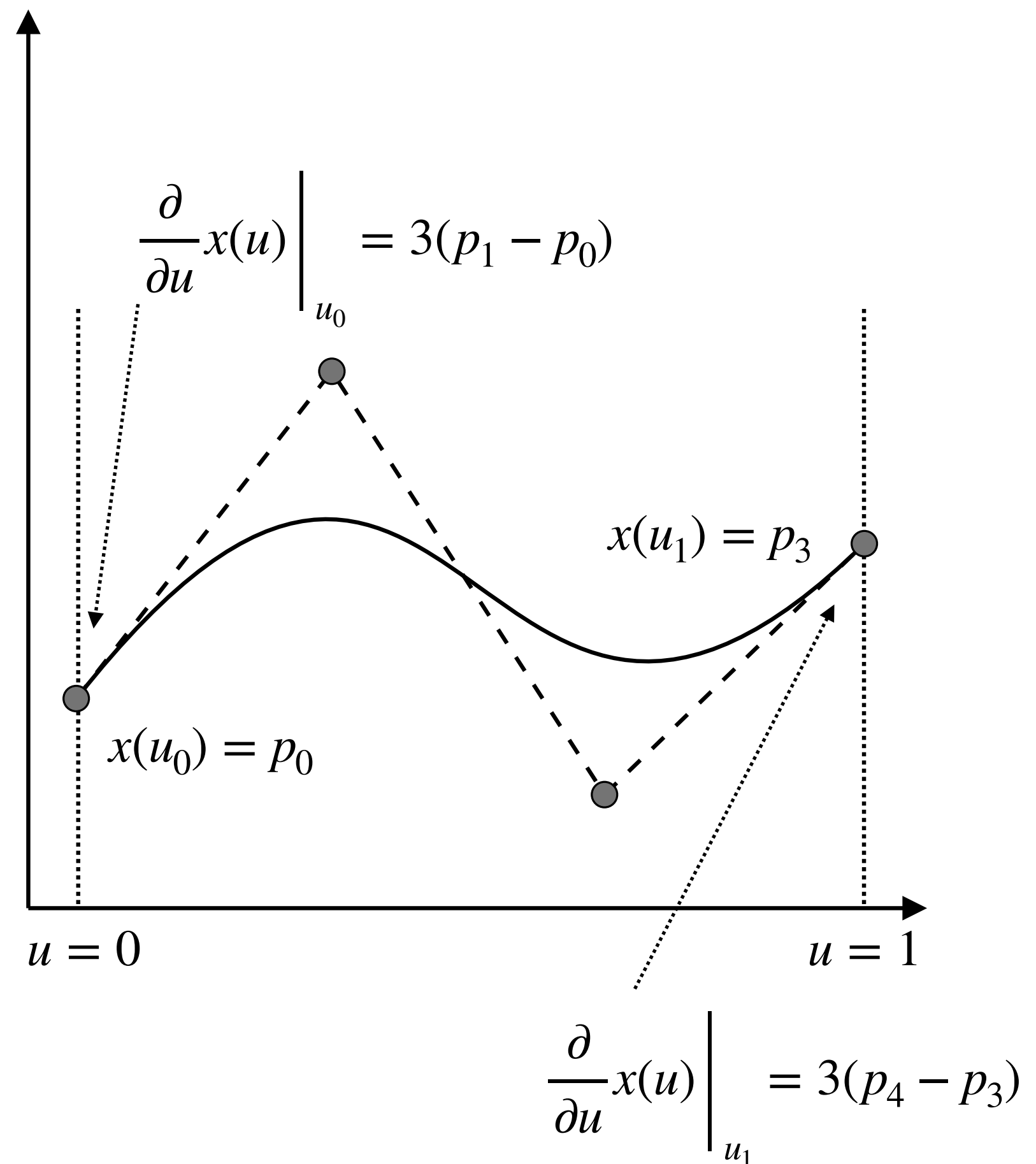
$$x(u_1) = h_1 = p_3$$

$$x'(u_0) = h_2 = 3(p_1 - p_0)$$

$$x'(u_1) = h_3 = 3(p_3 - p_2)$$

$$\mathbf{M}_{Z \rightarrow H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_H \cdot \mathbf{M}_{Z \rightarrow H} \cdot \mathbf{p}$$



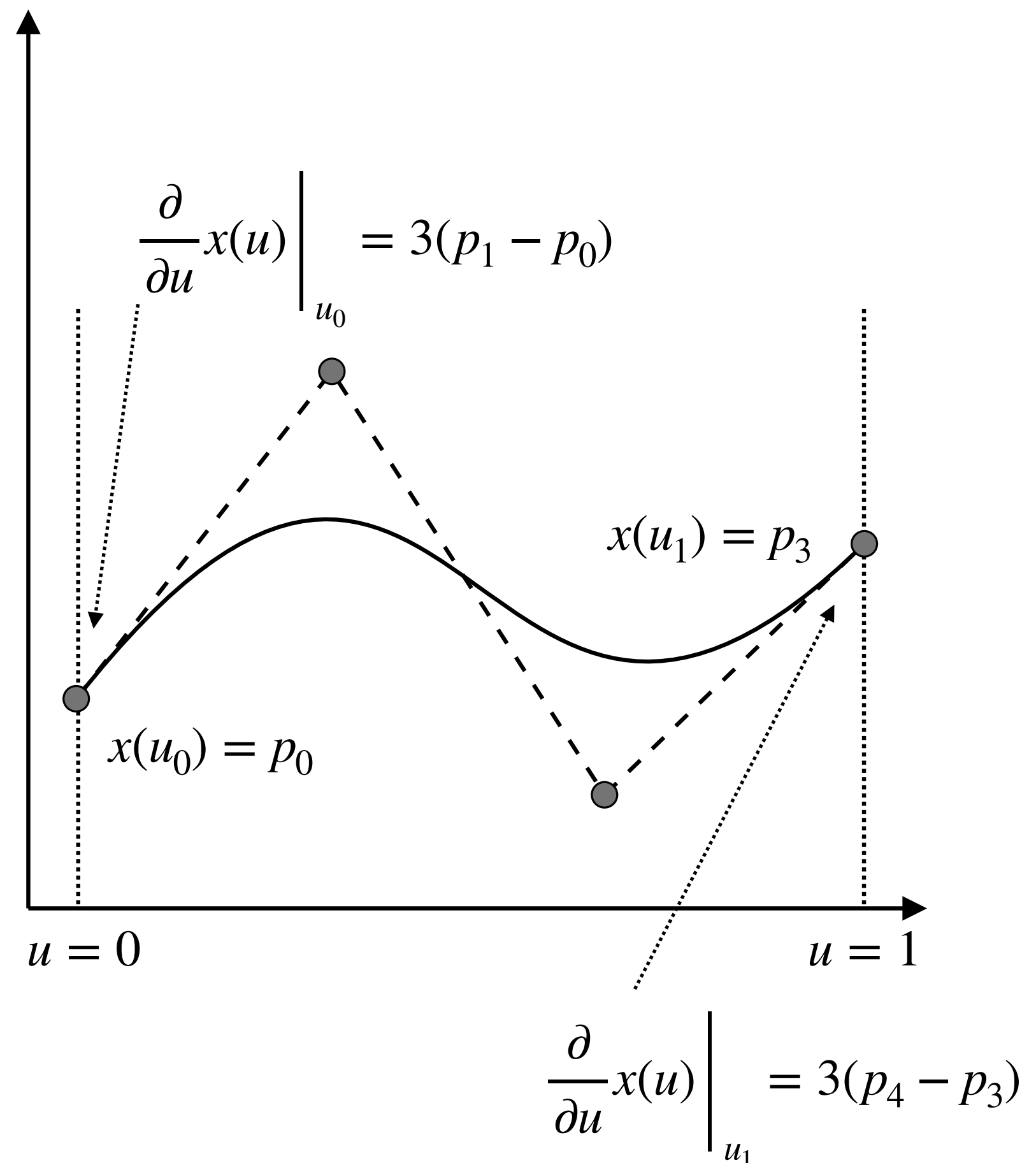
Cubic Bézier

Similar to Hermite, but specify tangents indirectly

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \beta_H \cdot \mathbf{M}_{Z \rightarrow H} \cdot \mathbf{p}$$

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \beta_Z \cdot \mathbf{p}$$

$$\beta_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$



Cubic Bézier

Bézier basis functions

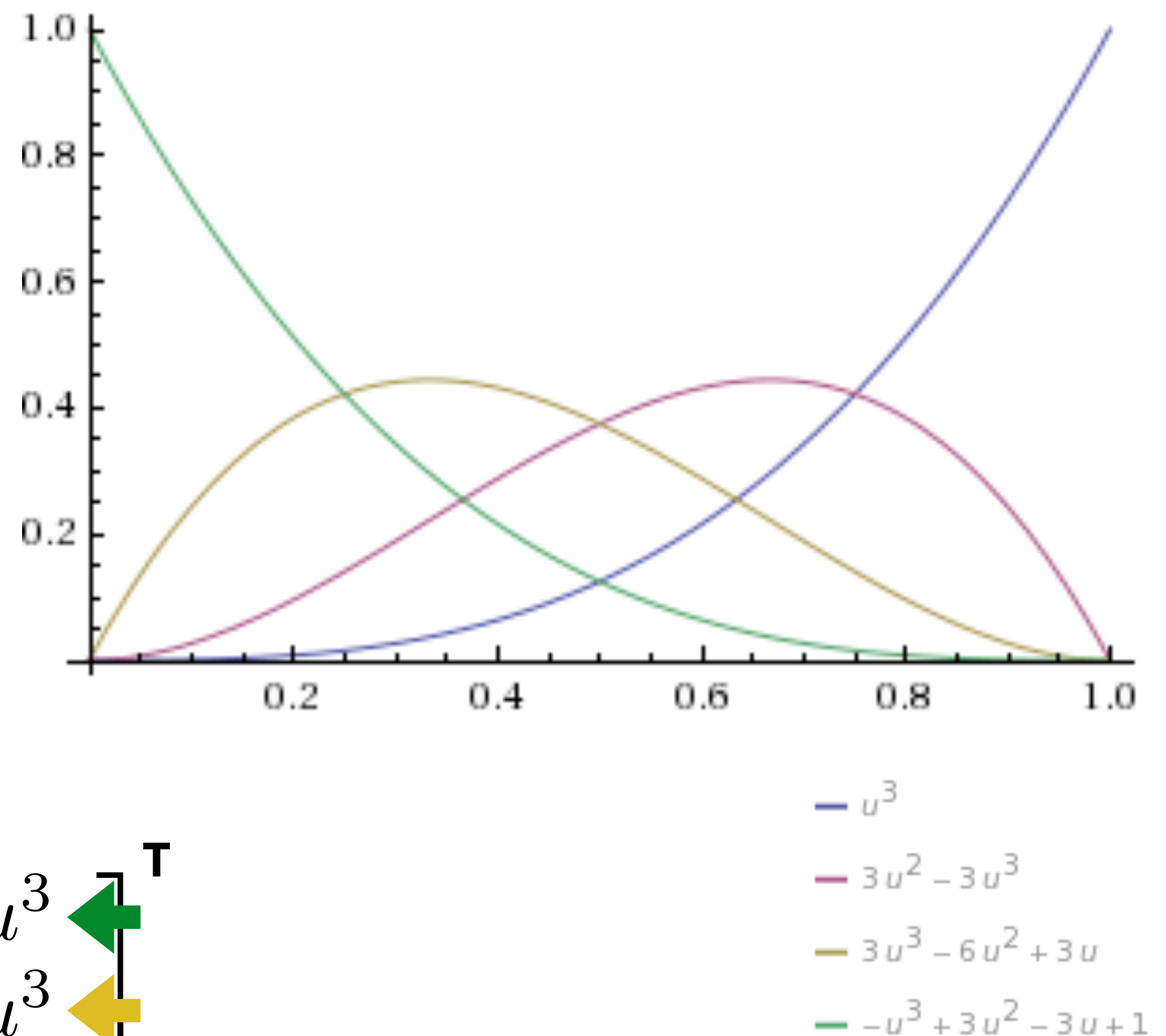
$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_Z \cdot \mathbf{p}$$

$$\boldsymbol{\beta}_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

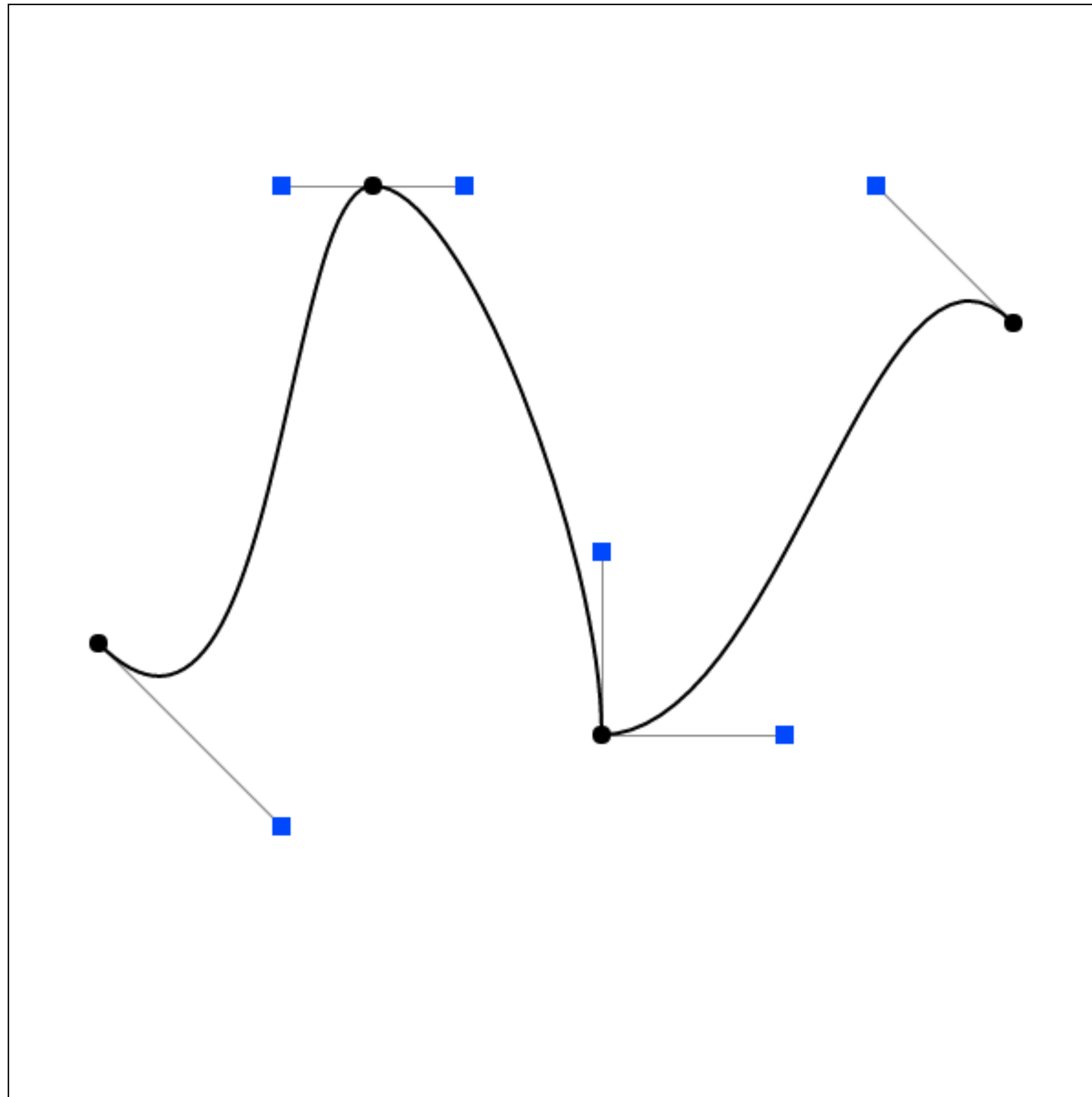
↑
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$$\mathbf{X}(u) = \begin{bmatrix} 1 - 3u + 3u^2 - 1u^3 \\ 0 + 3u - 6u^2 + 3u^3 \\ 0 + 0u + 3u^2 - 3u^3 \\ 0 + 0u + 0u^2 + 1u^3 \end{bmatrix}^T \mathbf{p}$$

←
←
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Demo – Piecewise Cubic Bézier Curve



Useful Properties of a Basis

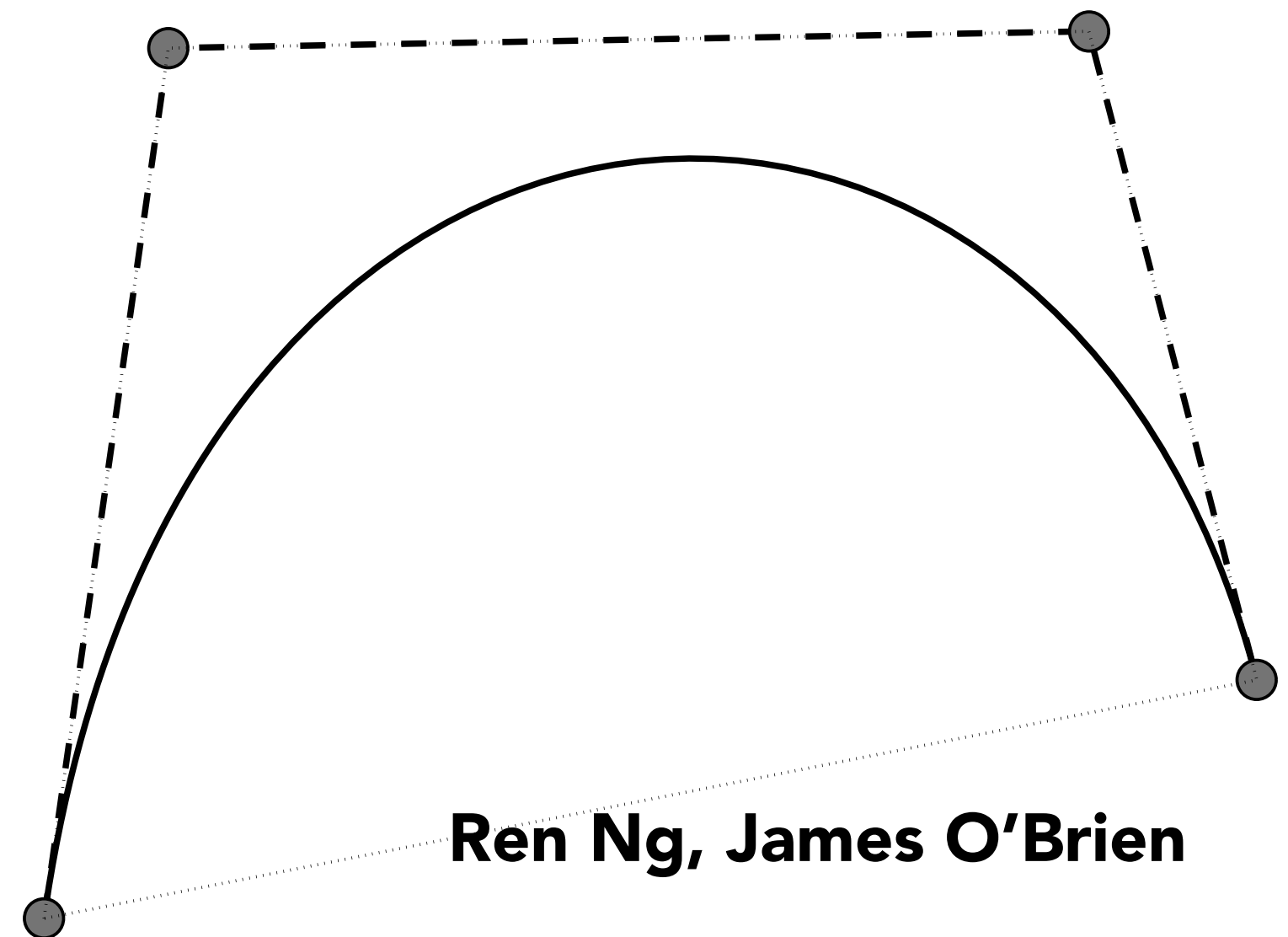
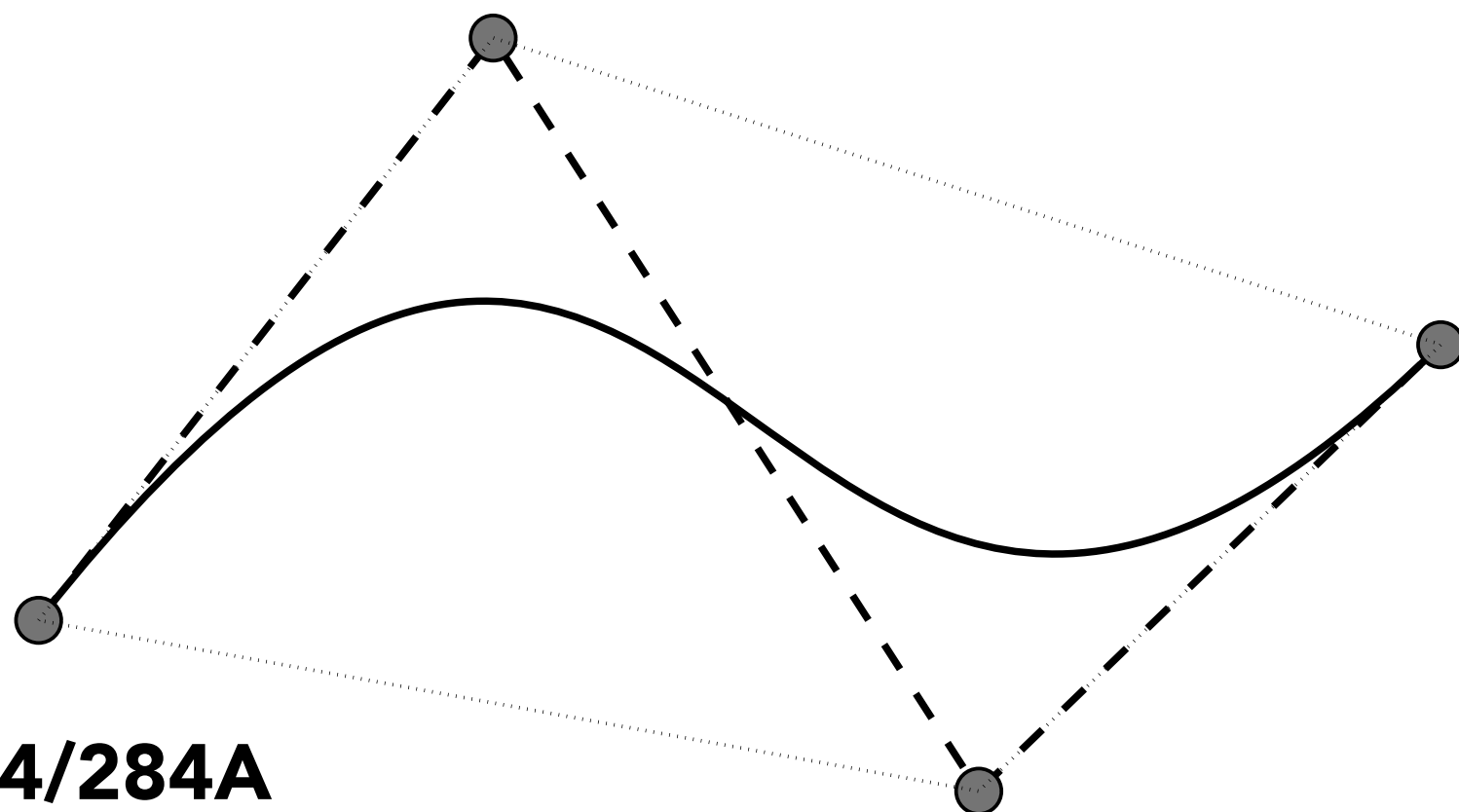
Convex Hull Property

- All points on curve inside convex hull of control points
- Bézier basis has convex hull property!
- Convex hull if the following are true:

- Basis functions sum to one $\sum_i b_i(u) = 1$

- Basis functions are positive over domain

$$b_i(u) \geq 0 \quad \forall u \in \Omega$$



Recall Cubic Bézier

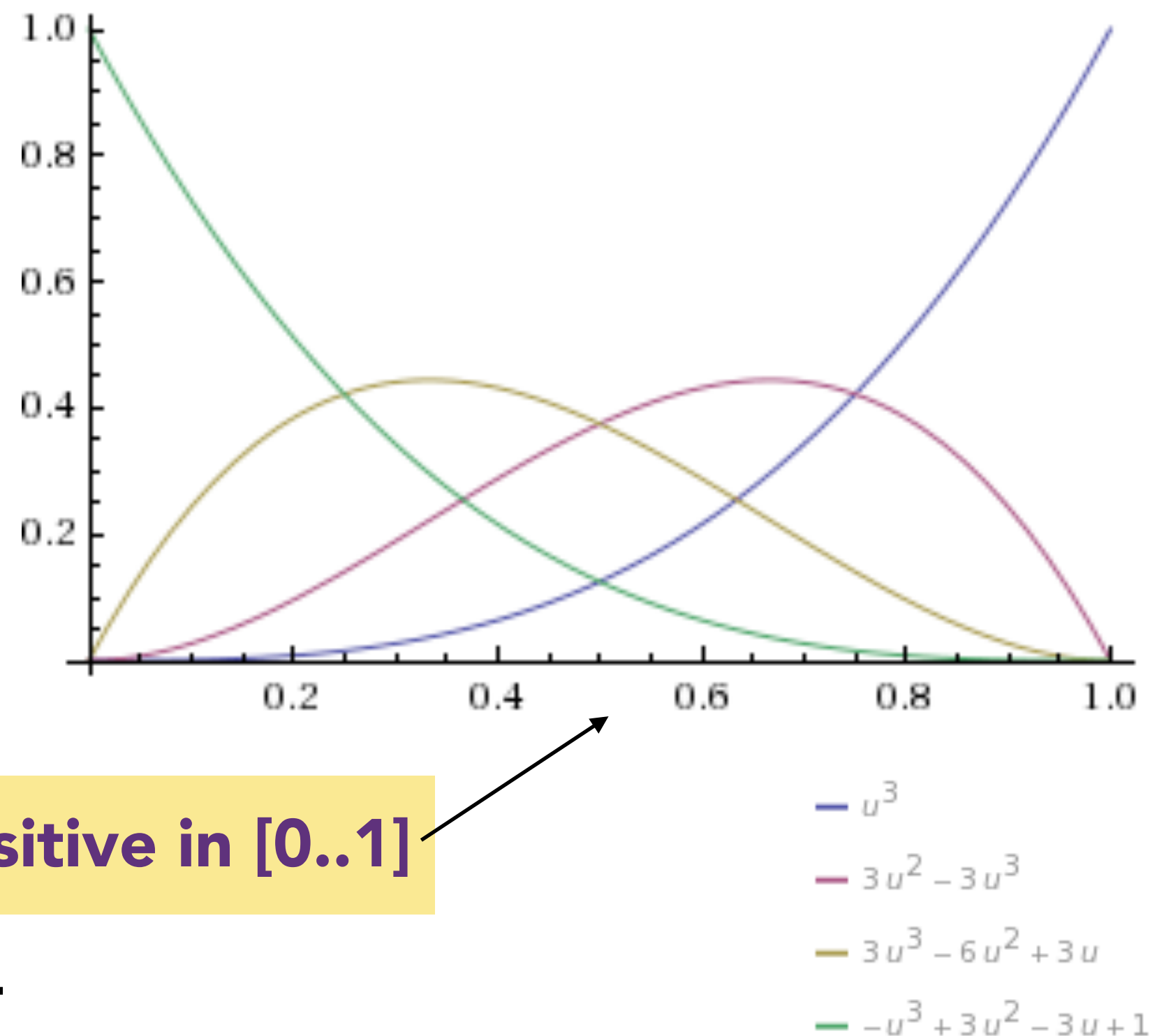
Bézier basis functions

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_Z \cdot \mathbf{p}$$

$$\boldsymbol{\beta}_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

↑
↑
↑
↑

$$x(u) = \begin{bmatrix} 1 - 3u + 3u^2 - 1u^3 \\ 0 + 3u - 6u^2 + 3u^3 \\ 0 + 0u + 3u^2 - 3u^3 \\ 0 + 0u + 0u^2 + 1u^3 \end{bmatrix}^T \mathbf{p}$$



These sum to 1

What if that 3 had been something else?

Useful Properties of a Basis

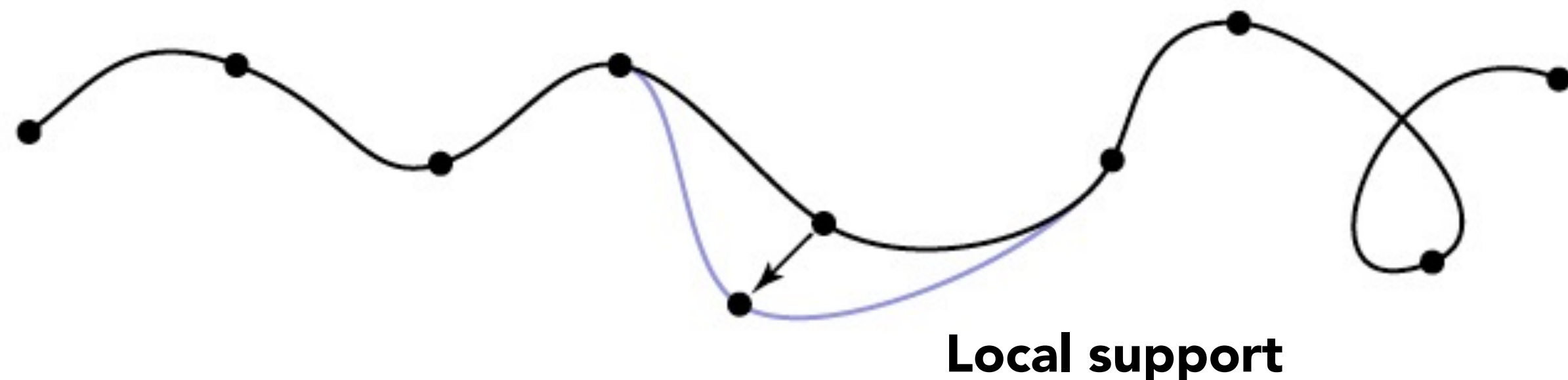
Invariance under class of transforms

- Transforming curve is same as transforming control points
- Bézier basis invariant w.r.t. *affine* transforms
- Bézier NOT invariant for perspective transforms
 - NURBS are invariant w.r.t. perspective transforms

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \iff \mathbf{T} \cdot \mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{T} \cdot \mathbf{p}_0 \\ \mathbf{T} \cdot \mathbf{p}_1 \\ \mathbf{T} \cdot \mathbf{p}_2 \\ \mathbf{T} \cdot \mathbf{p}_3 \end{bmatrix}$$

More Useful Properties of a Basis

- **Local support**
 - Changing one control point does not change the entire curve
- **Nice subdivision rules**
- **Orthogonality** ($\int_{\Omega} b_i(u)b_j(u)du = \delta_{ij}$)
- **Fast evaluation scheme**
- **Interpolation -vs- approximation**



Changing Basis

Switching between different basis functions is done by matrix multiplication:

- Example: $\mathbf{p}_Z = \beta_Z^{-1} \cdot \mathbf{c}$
- Example: $\mathbf{p}_Z = \beta_Z^{-1} \cdot \beta_H \cdot \mathbf{p}_H$
- Only works if two sets of basis functions spans the same space (*i.e.* cubic polynomials)
- Different sets of basis functions have different useful properties

Properties of Bézier Curves

Interpolates endpoints

- For cubic Bézier: $\mathbf{b}(0) = \mathbf{b}_0$; $\mathbf{b}(1) = \mathbf{b}_3$

Tangent to end segments

- Cubic case: $\mathbf{b}'(0) = 3(\mathbf{b}_1 - \mathbf{b}_0)$; $\mathbf{b}'(1) = 3(\mathbf{b}_3 - \mathbf{b}_2)$

Affine transformation property

- Transform curve by transforming control points

Convex hull property

- Curve is within convex hull of control points

Well behaved

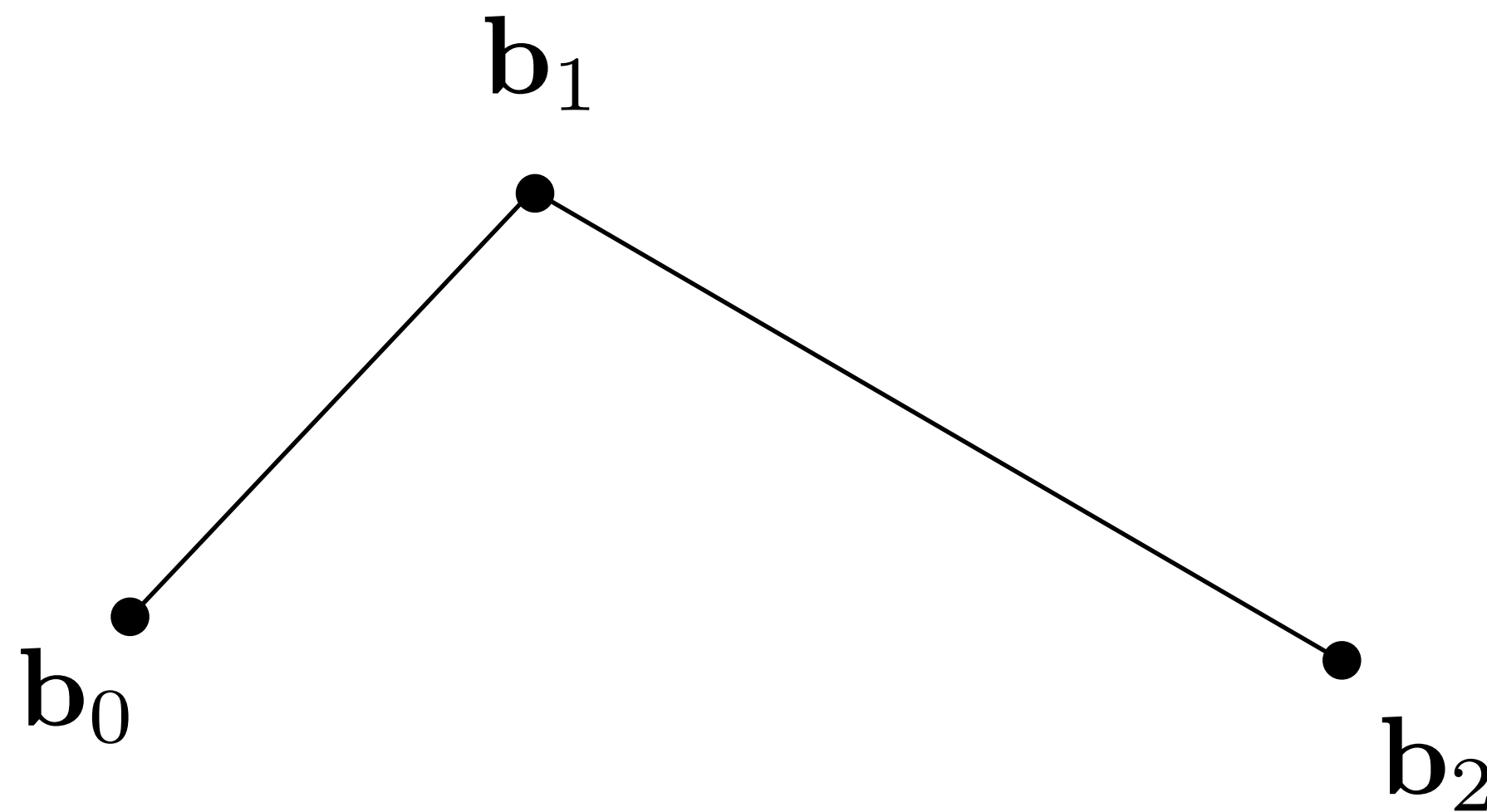
Local control (when joining)

Evaluating Bézier Curves

De Casteljau Algorithm

Bézier Curves – de Casteljau Algorithm

Consider three points (quadratic Bezier)



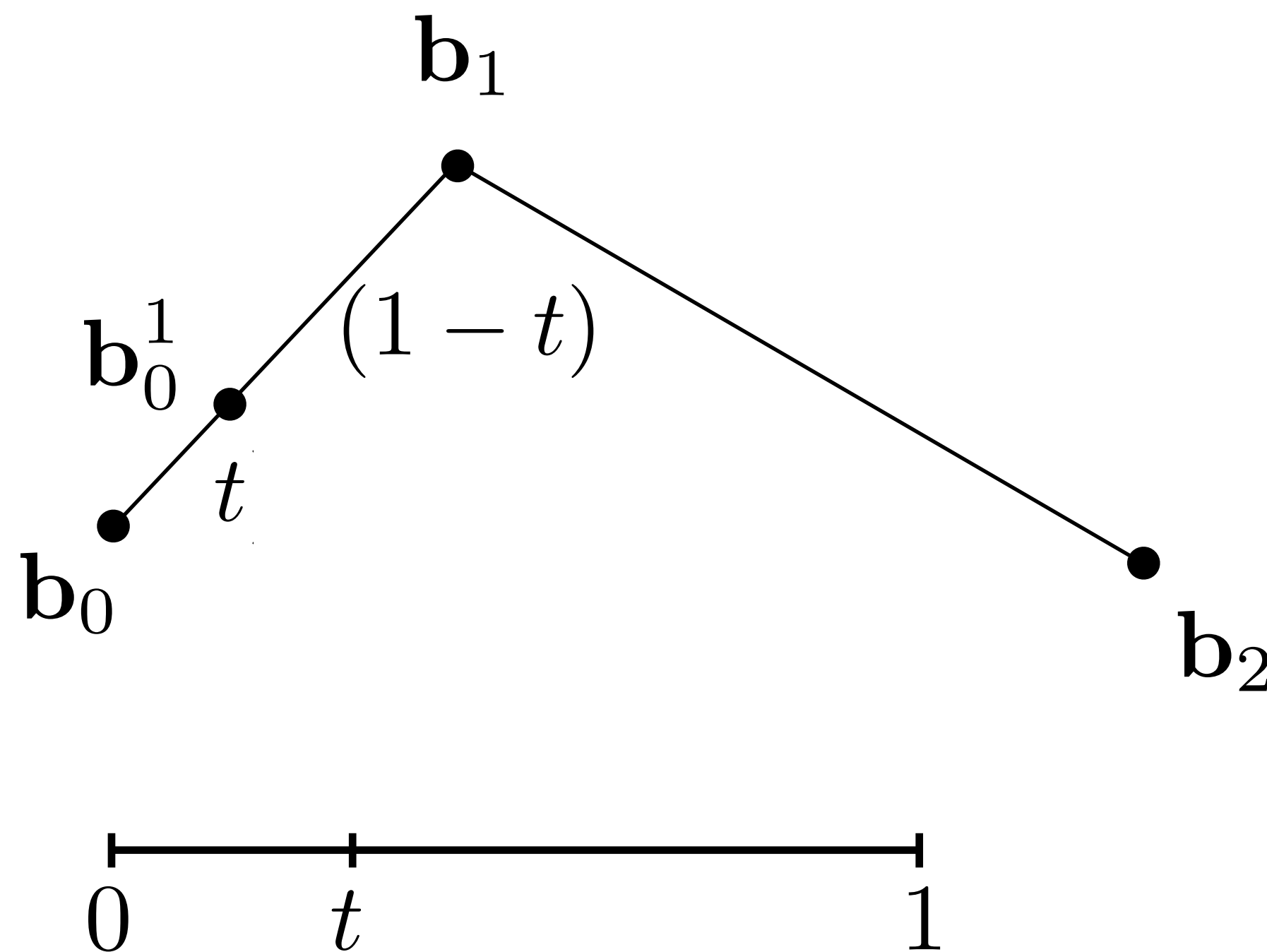
Pierre Bézier
1910 – 1999



Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Insert a point using linear interpolation



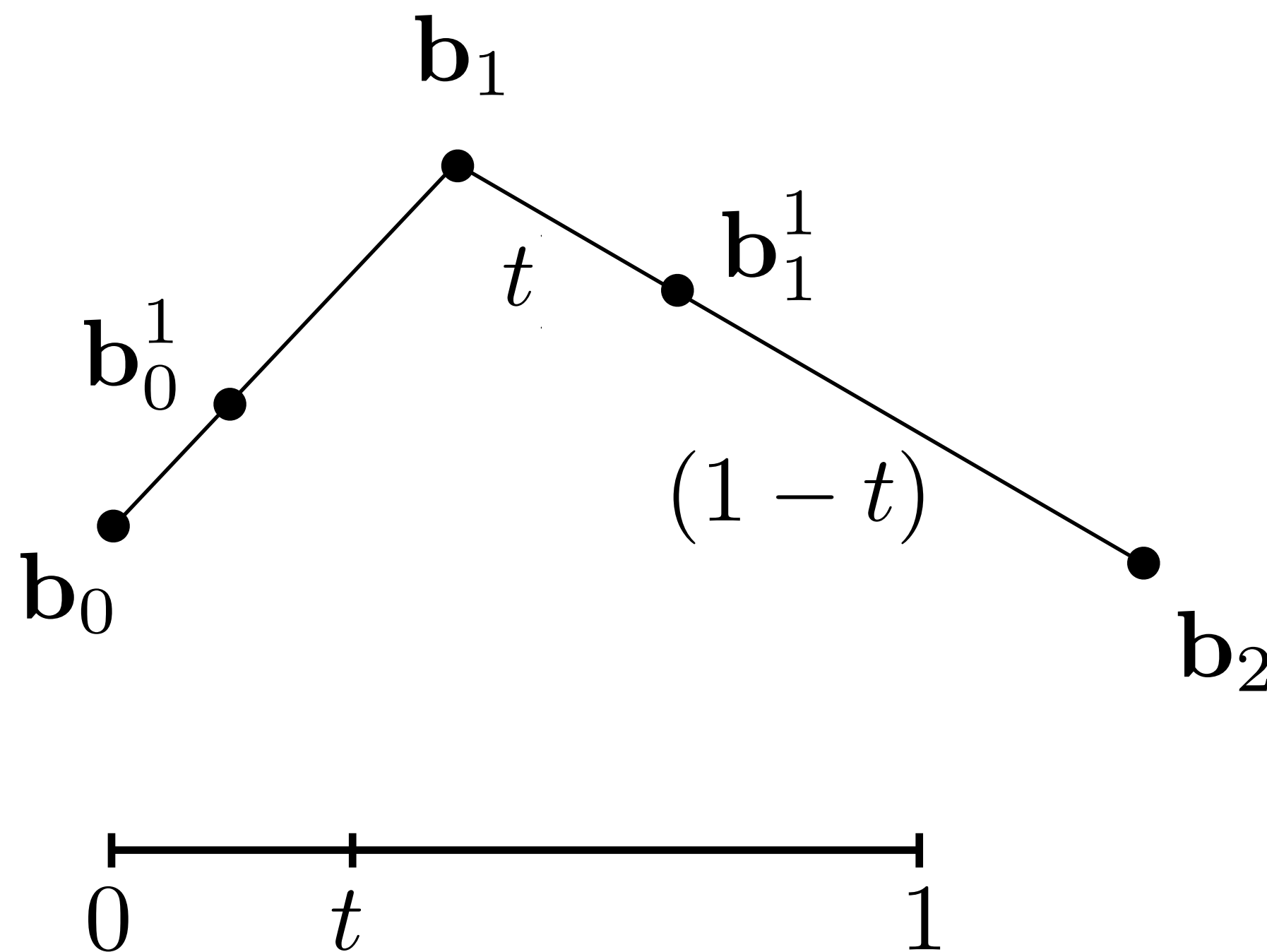
Pierre Bézier
1910 – 1999



Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Insert on both edges



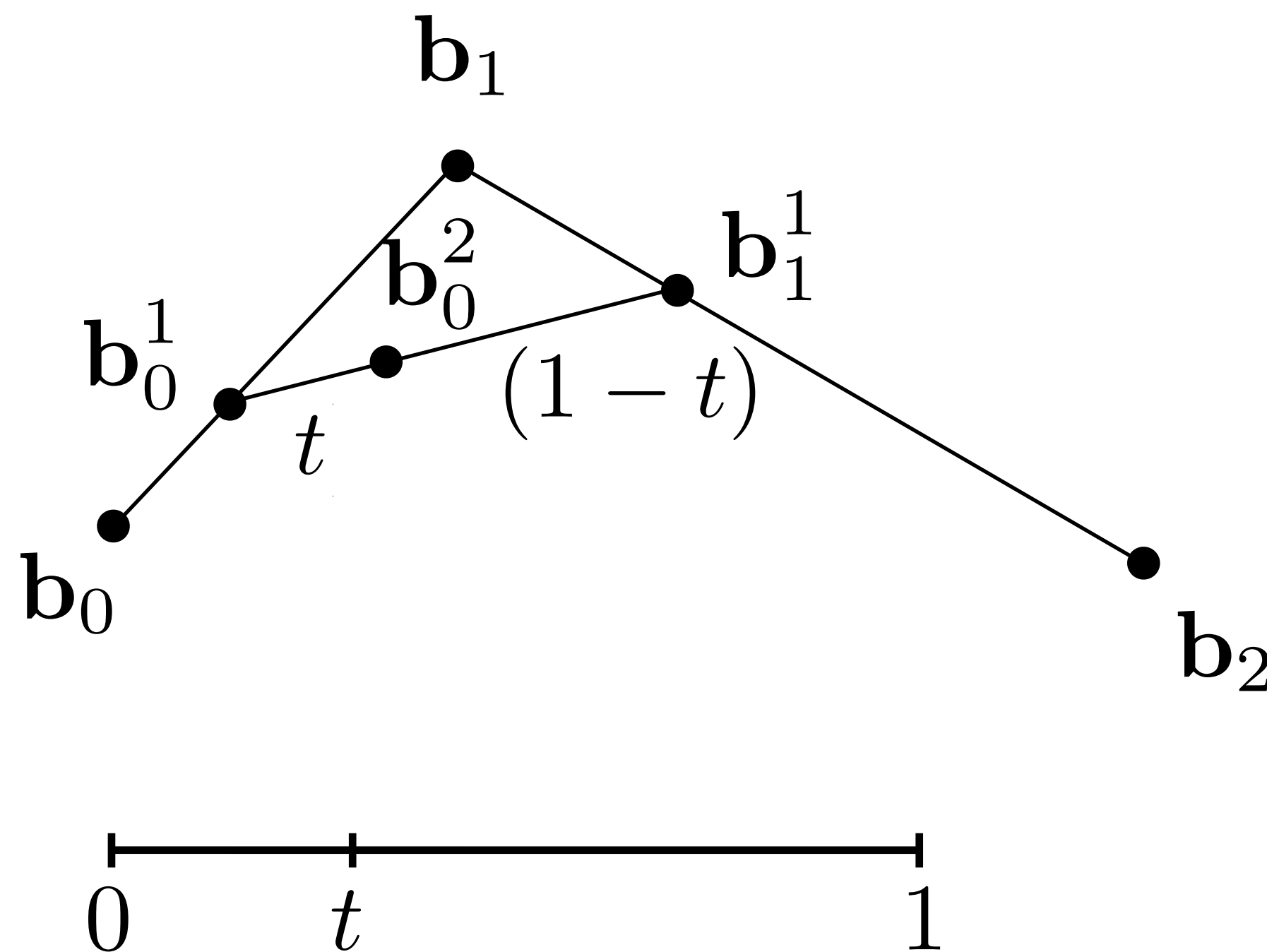
Pierre Bézier
1910 – 1999



Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Repeat recursively



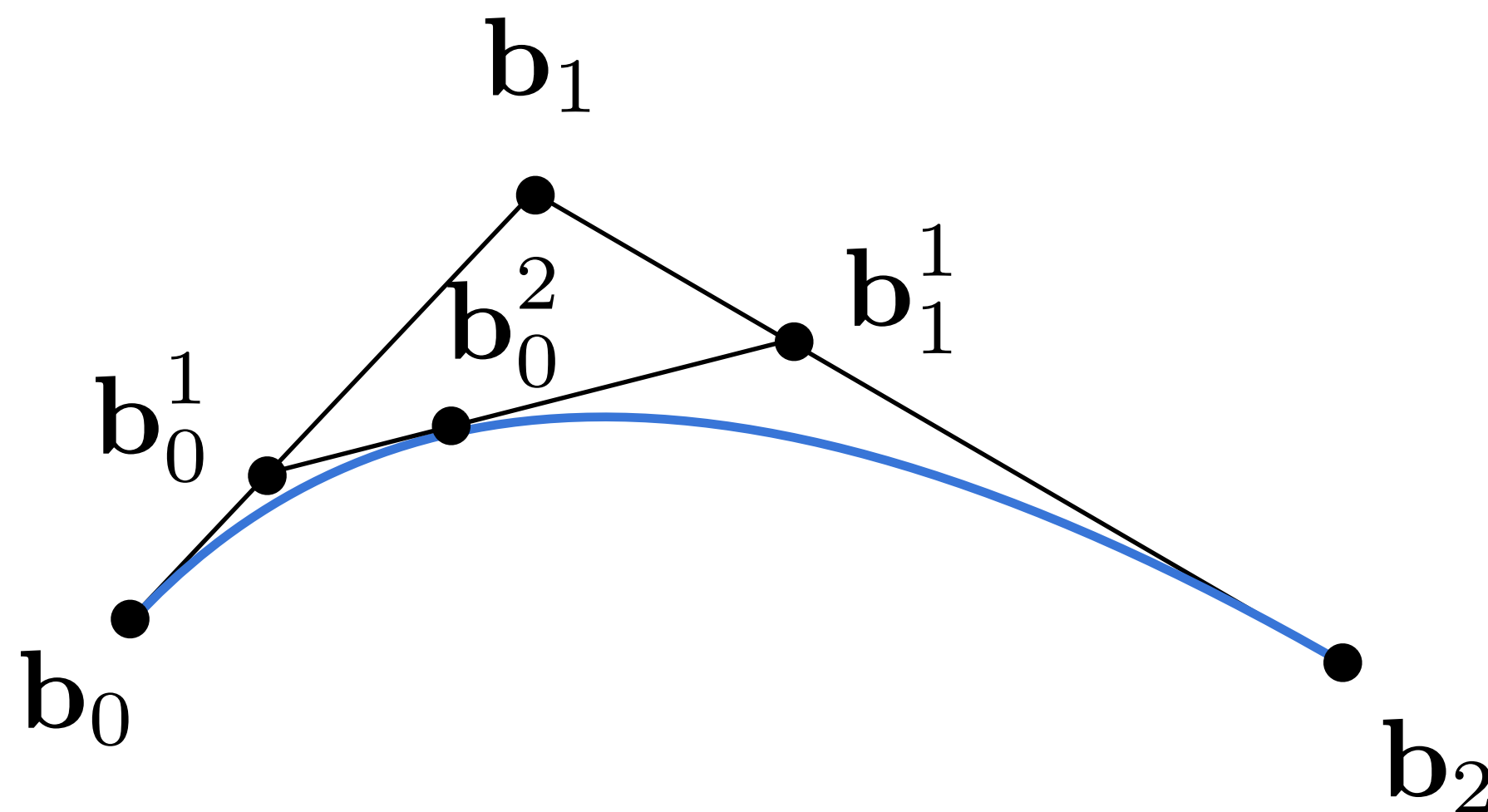
Pierre Bézier
1910 – 1999



Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Algorithm defines the curve



Pierre Bézier
1910 – 1999



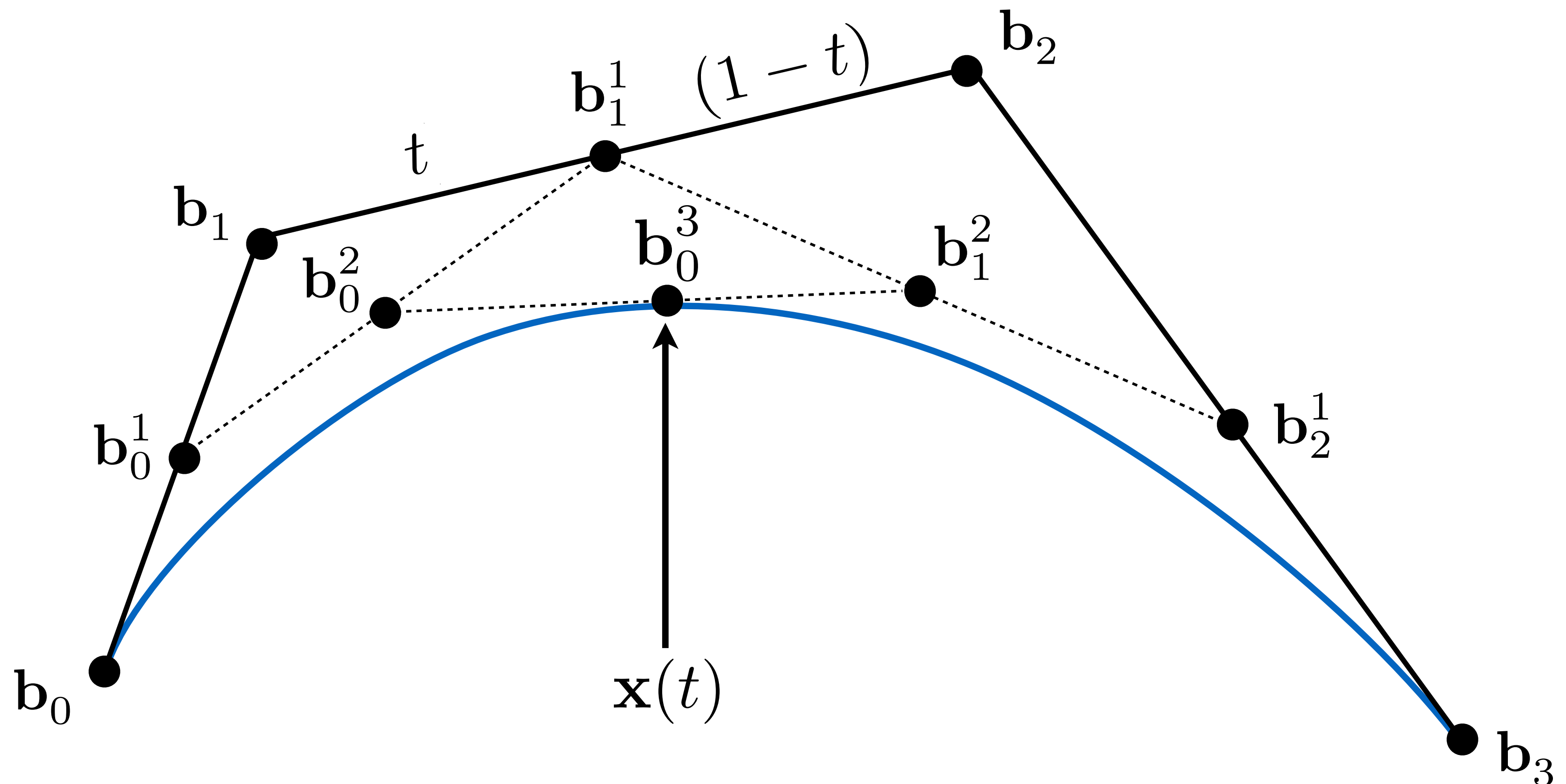
Paul de Casteljau
b. 1930

“Corner cutting” recursive subdivision

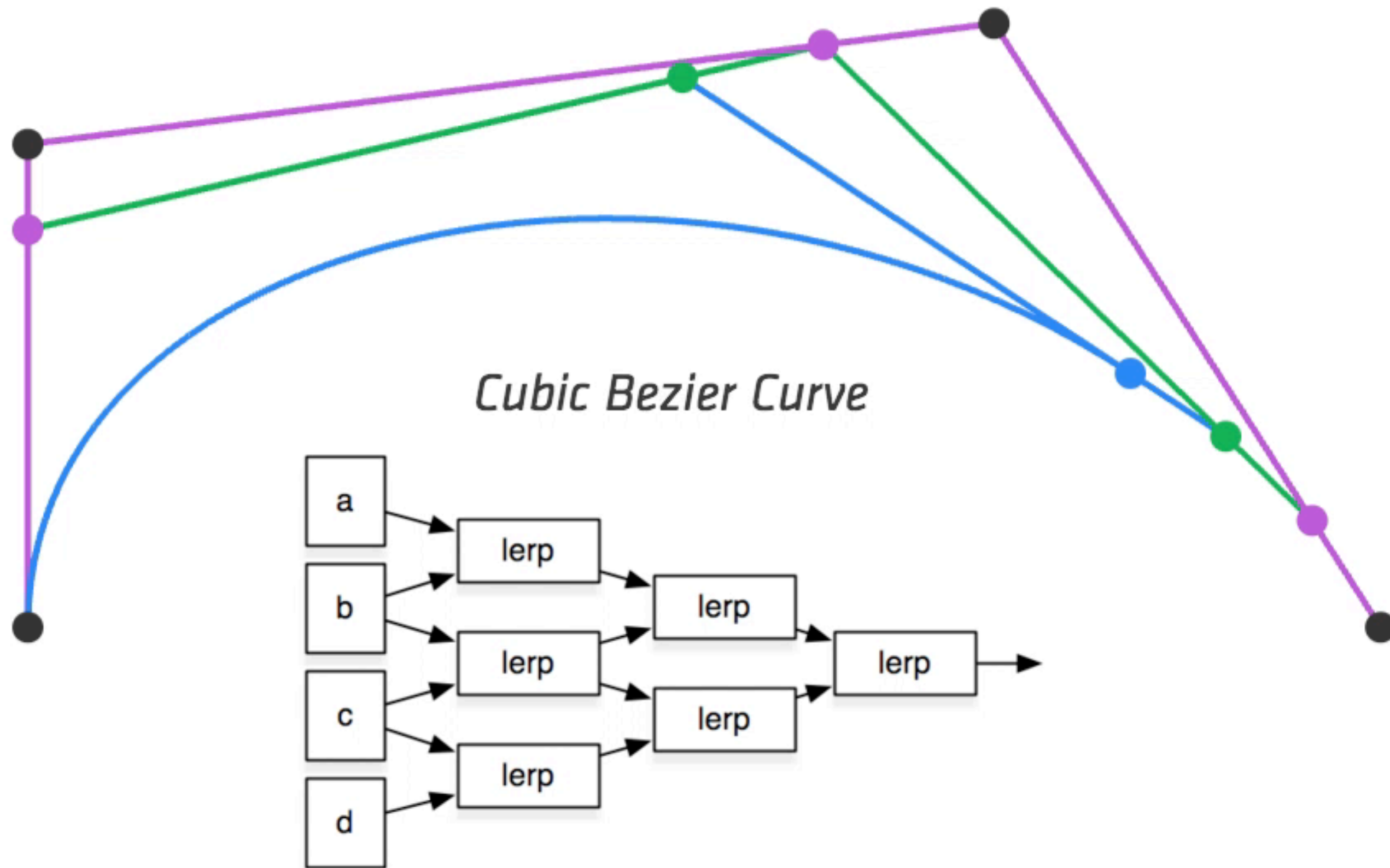
Successive linear interpolation

Cubic Bézier Curve – de Casteljau

Consider four points
Same recursive linear interpolations



Visualizing de Casteljau Algorithm

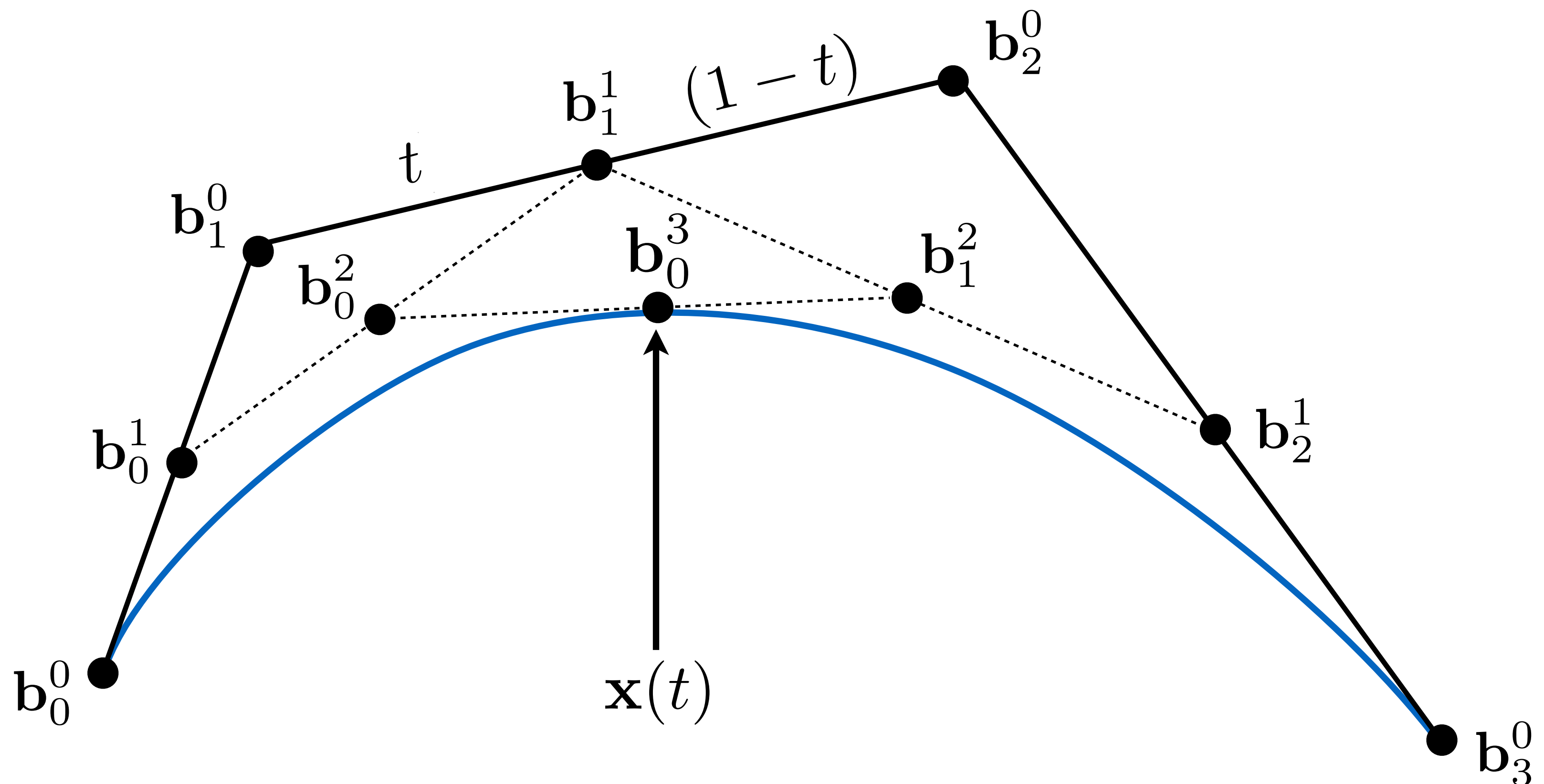


Animation: Steven Wittens, Making Things with Maths, <http://acko.net>

Cubic Bézier Curve – de Casteljau

Consider four points

Same recursive linear interpolations

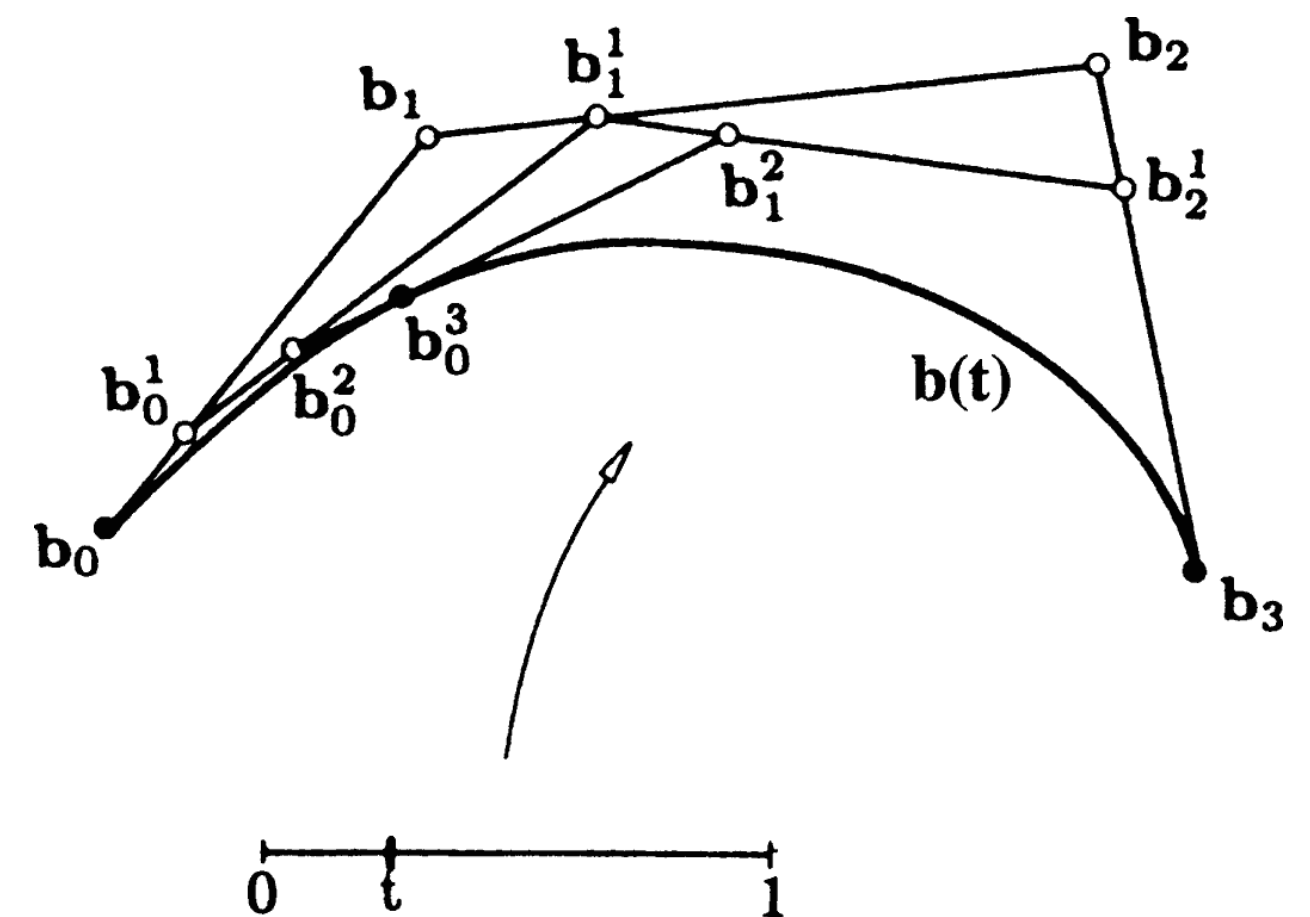
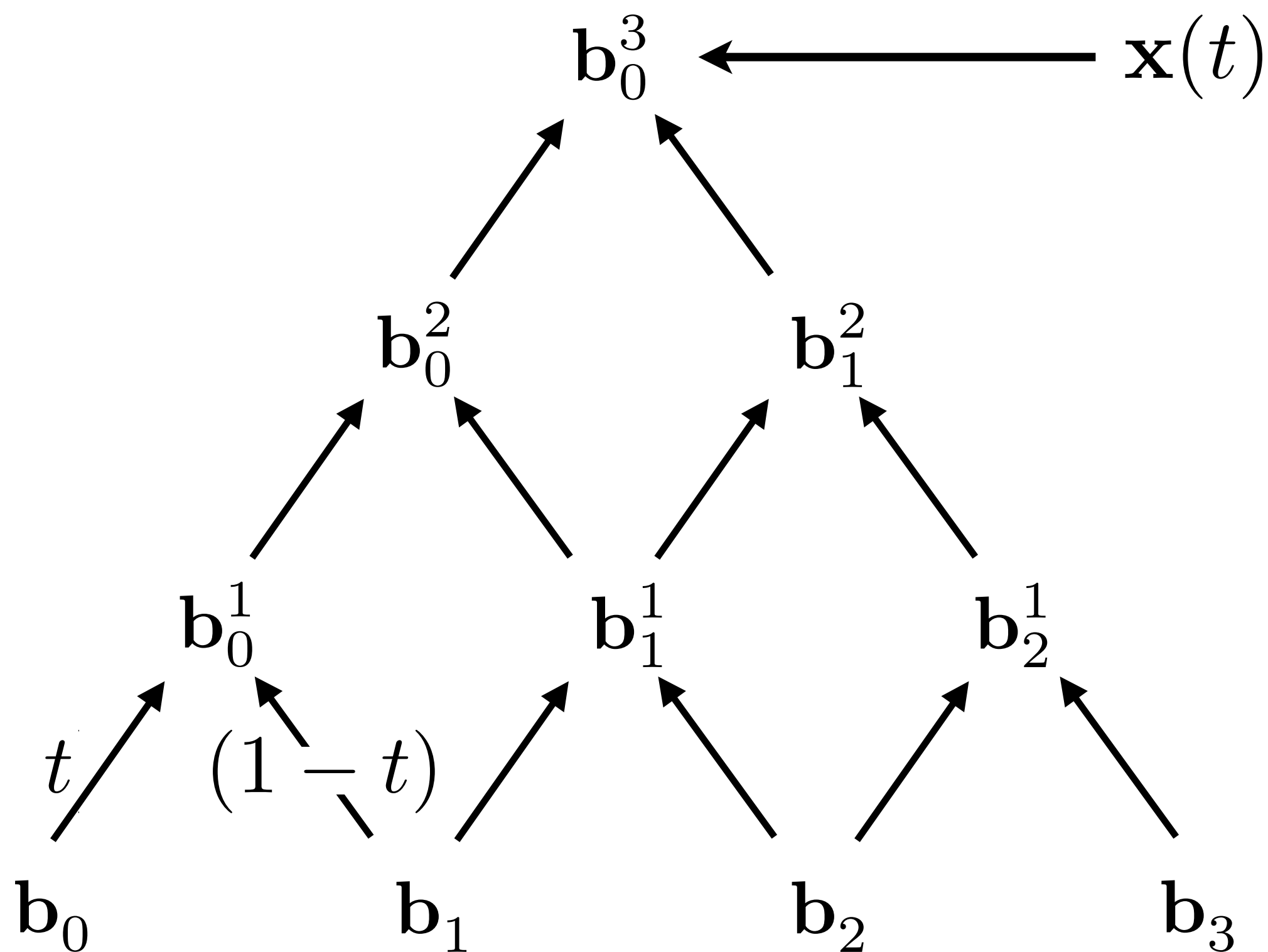


Evaluating Bézier Curves

Algebraic Formula

Bézier Curve – Algebraic Formula

de Casteljau algorithm gives a pyramid of coefficients



Every rightward arrow is multiplication by t ,
Every leftward arrow by $(1-t)$

Bézier Curve – General Algebraic Formula

Bernstein form of a Bézier curve of order n :

$$\mathbf{b}^n(t) = \mathbf{b}_0^n(t) = \sum_{j=0}^n \mathbf{b}_j B_j^n(t)$$

↑
Bézier curve order n
(vector polynomial of degree n)

↑
Bernstein polynomial
(scalar polynomial of degree n)

↑
Bézier control points
(vector in $\mathbb{R}^N$)

Bernstein polynomials:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Bézier Curve – Algebraic Formula: Example

Bernstein form of a Bézier curve of order n :

$$\mathbf{b}^n(t) = \mathbf{b}_0^n(t) = \sum_{j=0}^n \mathbf{b}_j B_j^n(t)$$

Example: assume $n = 3$ and we are in $\mathbb{R}^3$

i.e. we could have control points in 3D such as:

$$\mathbf{b}_0 = (0, 2, 3), \quad \mathbf{b}_1 = (2, 3, 5), \quad \mathbf{b}_2 = (6, 7, 9), \quad \mathbf{b}_3 = (3, 4, 5)$$

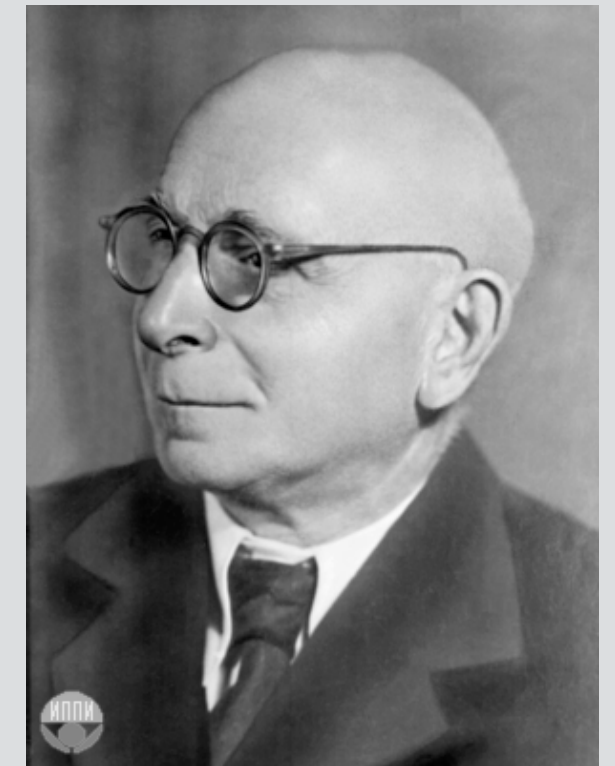
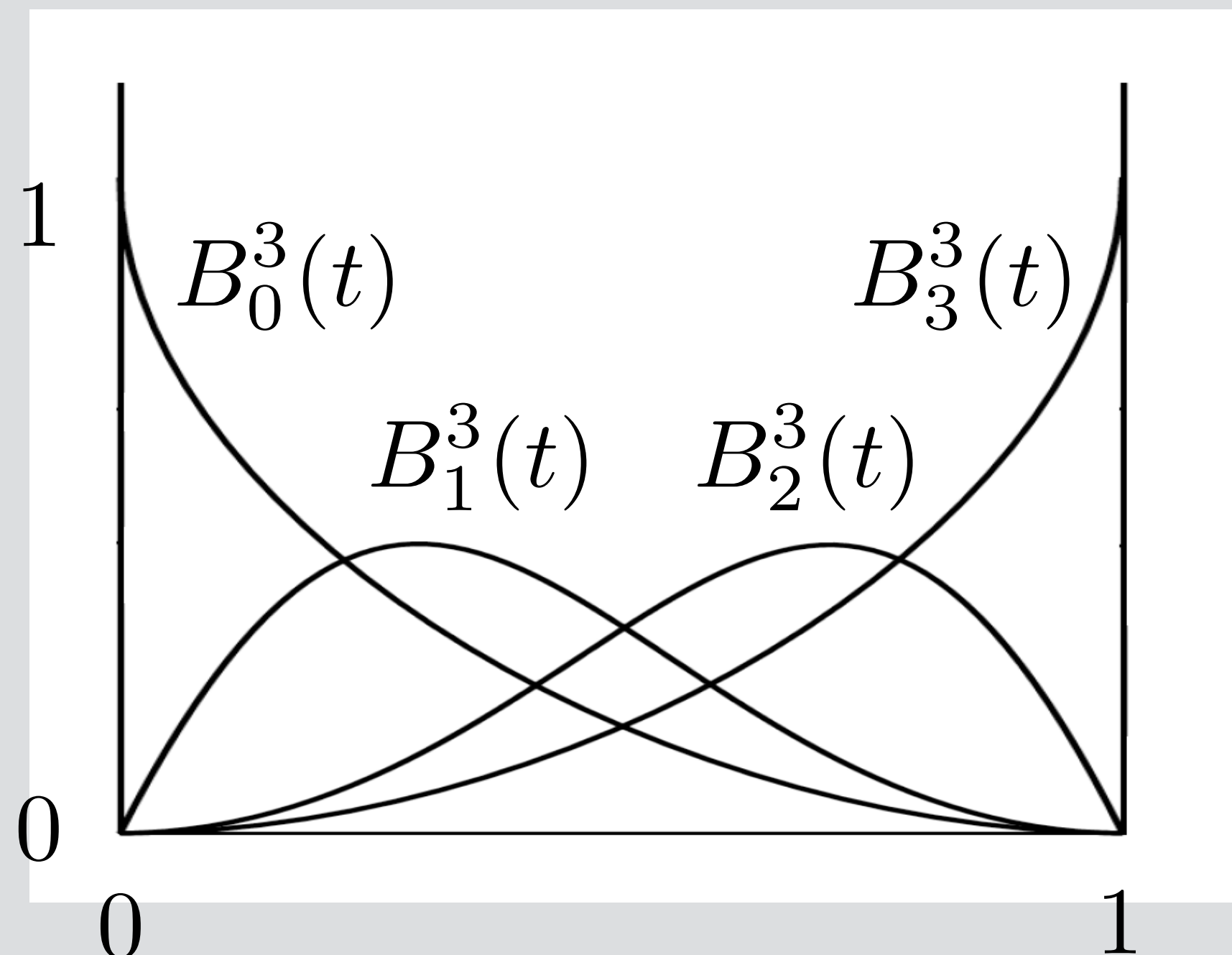
These points define a Bezier curve in 3D that is a cubic polynomial in t :

$$\mathbf{b}^n(t) = \mathbf{b}_0 (1 - t)^3 + \mathbf{b}_1 3t(1 - t)^2 + \mathbf{b}_2 3t^2(1 - t) + \mathbf{b}_3 t^3$$

Cubic Bézier Basis Functions

Bernstein polynomials:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



Sergei N. Bernstein
1880 – 1968

Evaluating Bézier Curves

Matrix Formula

Bézier Curve – Matrix Formula

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \quad \boldsymbol{\beta}_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Evaluating Bézier Curves Subdivision

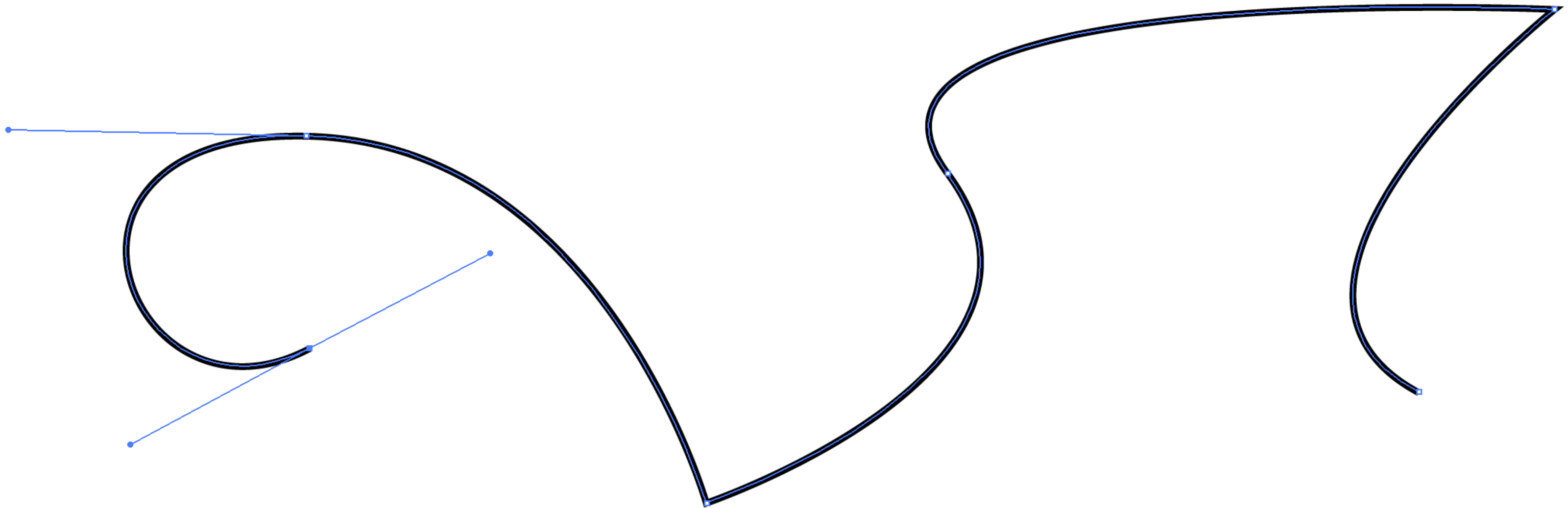
(See slides on using subdivision for Bézier Surfaces.)

Piecewise Bézier Curves (Bézier Spline)

Piecewise Bézier Curves

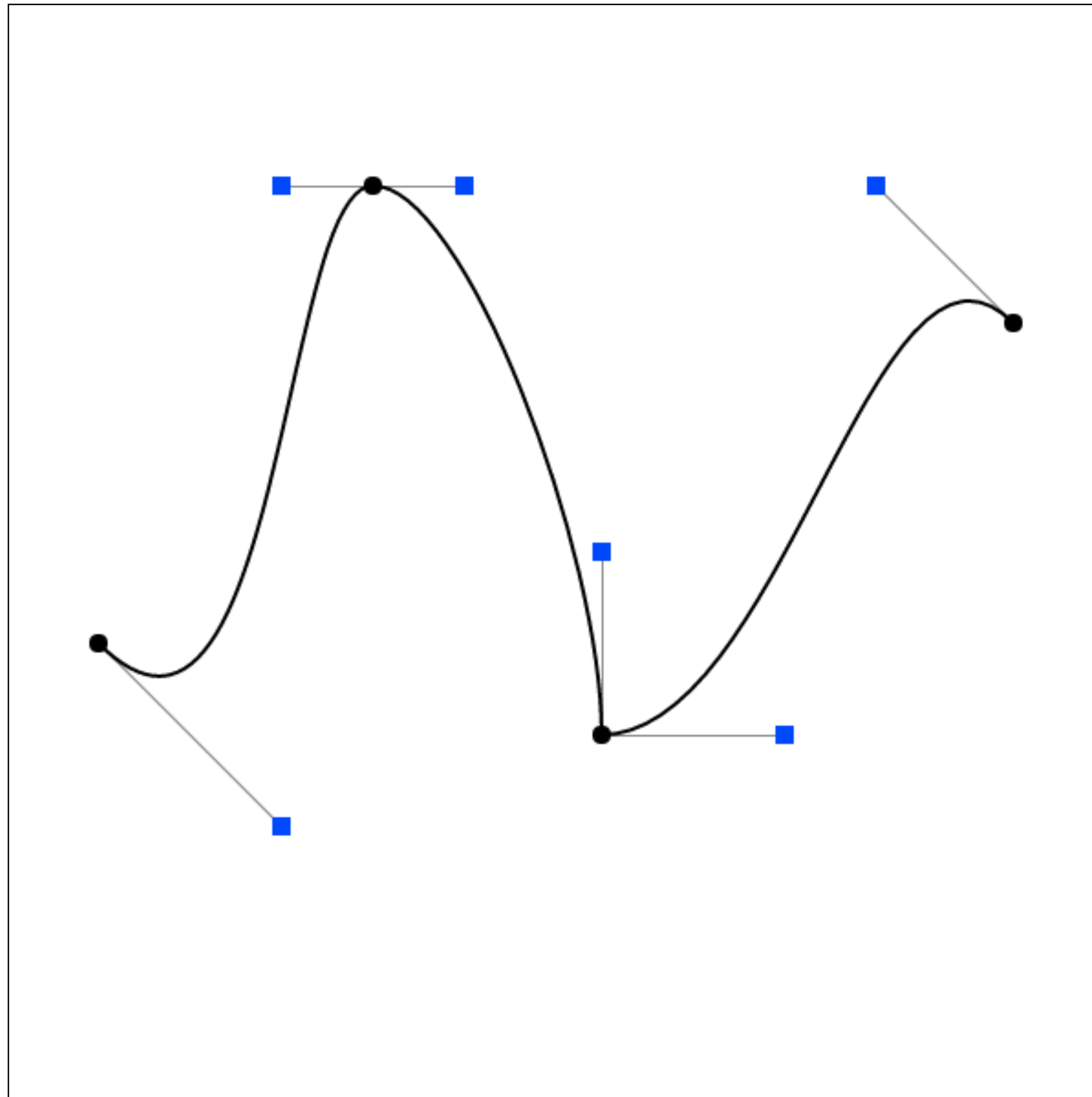
Instead, chain many low-order Bézier curve

Piecewise cubic Bézier the most common technique



Widely used (fonts, paths, Illustrator, Keynote, ...)

Demo – Piecewise Cubic Bézier Curve

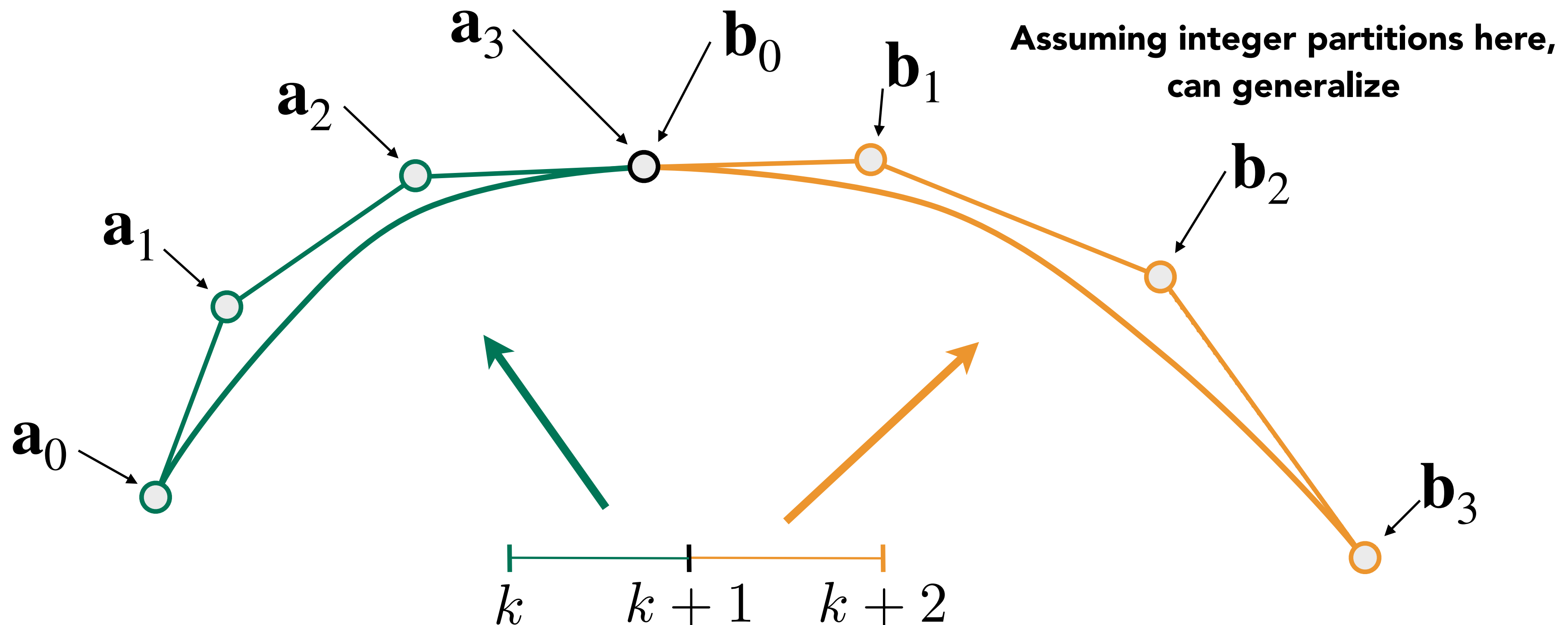


Piecewise Bézier Curve – Continuity

Two Bézier curves

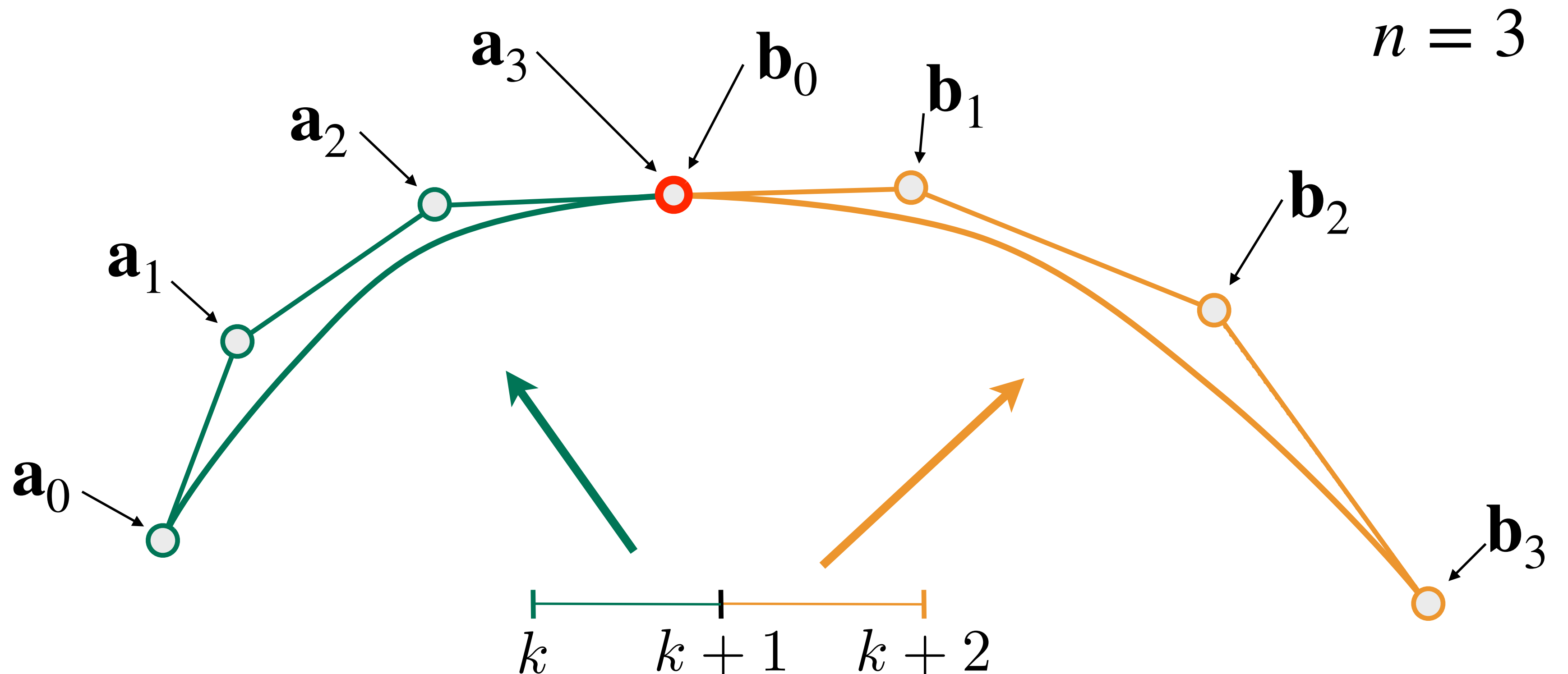
$$\mathbf{a} : [k, k + 1] \rightarrow \mathbb{R}^N$$

$$\mathbf{b} : [k + 1, k + 2] \rightarrow \mathbb{R}^N$$



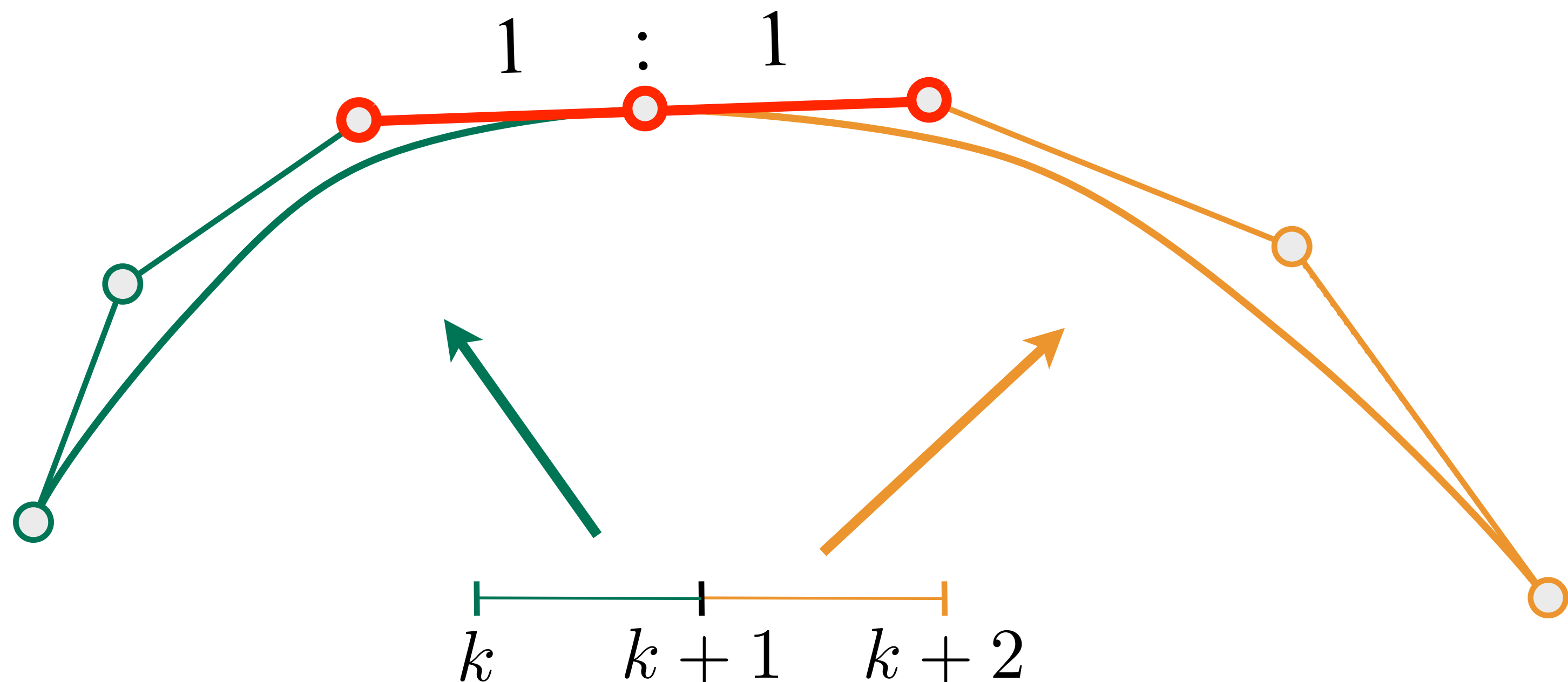
Piecewise Bézier Curve – Continuity

C^0 continuity: $\mathbf{a}_n = \mathbf{b}_0$



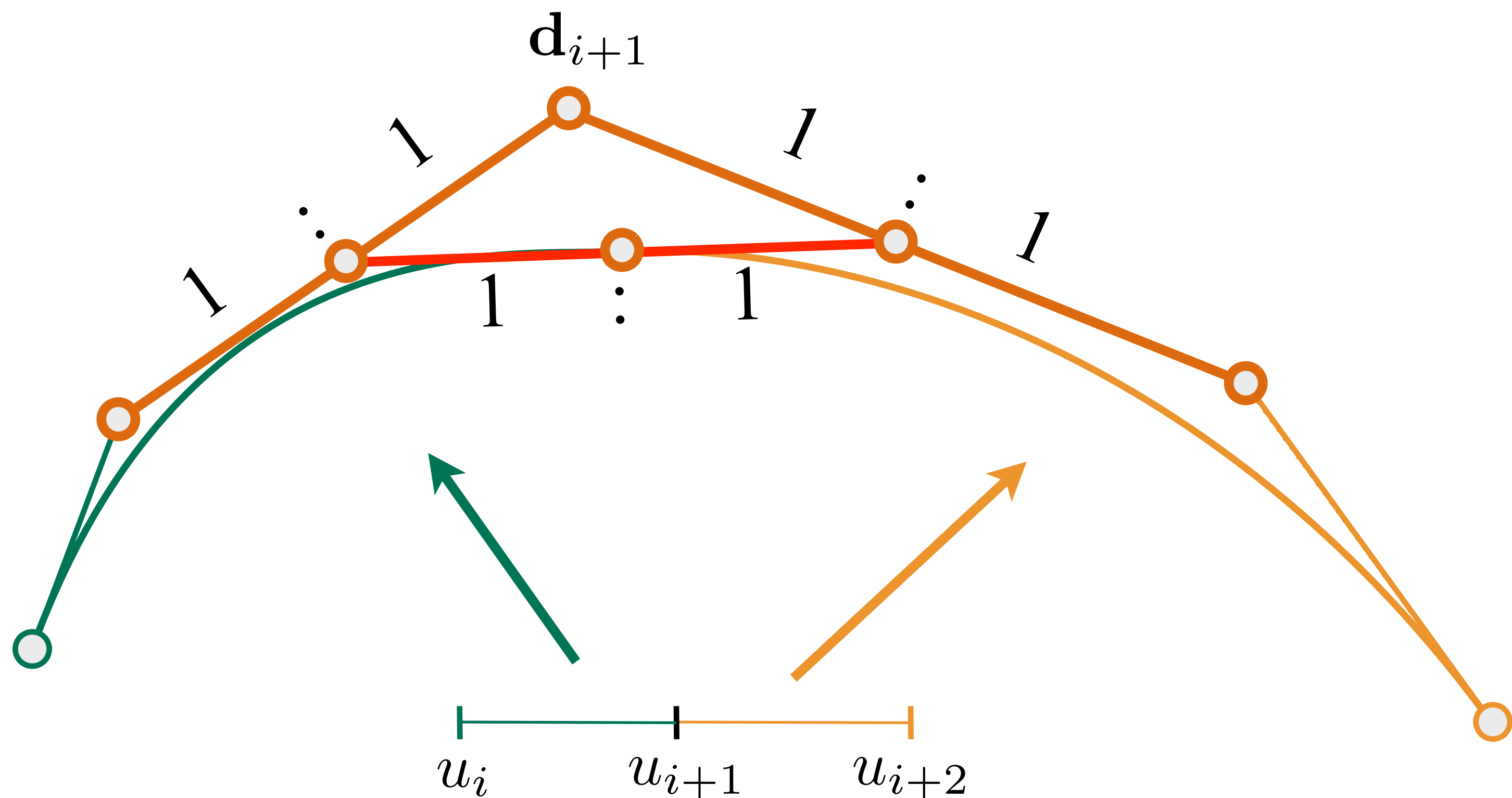
Piecewise Bézier Curve – Continuity

C¹ continuity: $\mathbf{a}_n = \mathbf{b}_0 = \frac{1}{2} (\mathbf{a}_{n-1} + \mathbf{b}_1)$



New Slide With Improvements [Ng]

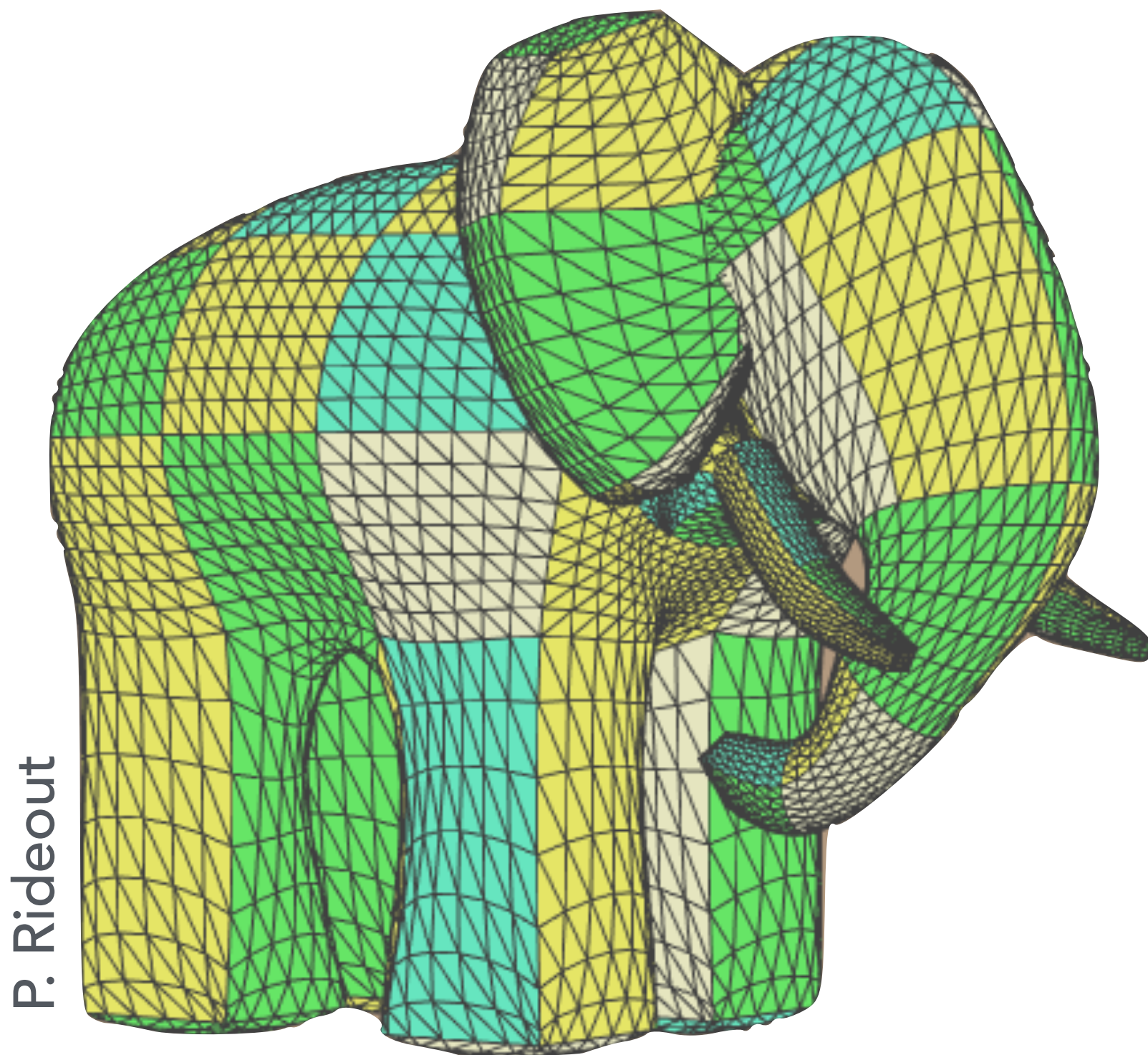
C^2 continuity: "A-frame" construction



Bézier Surfaces

Bézier Surfaces

Extend Bézier curves to surfaces



P. Rideout

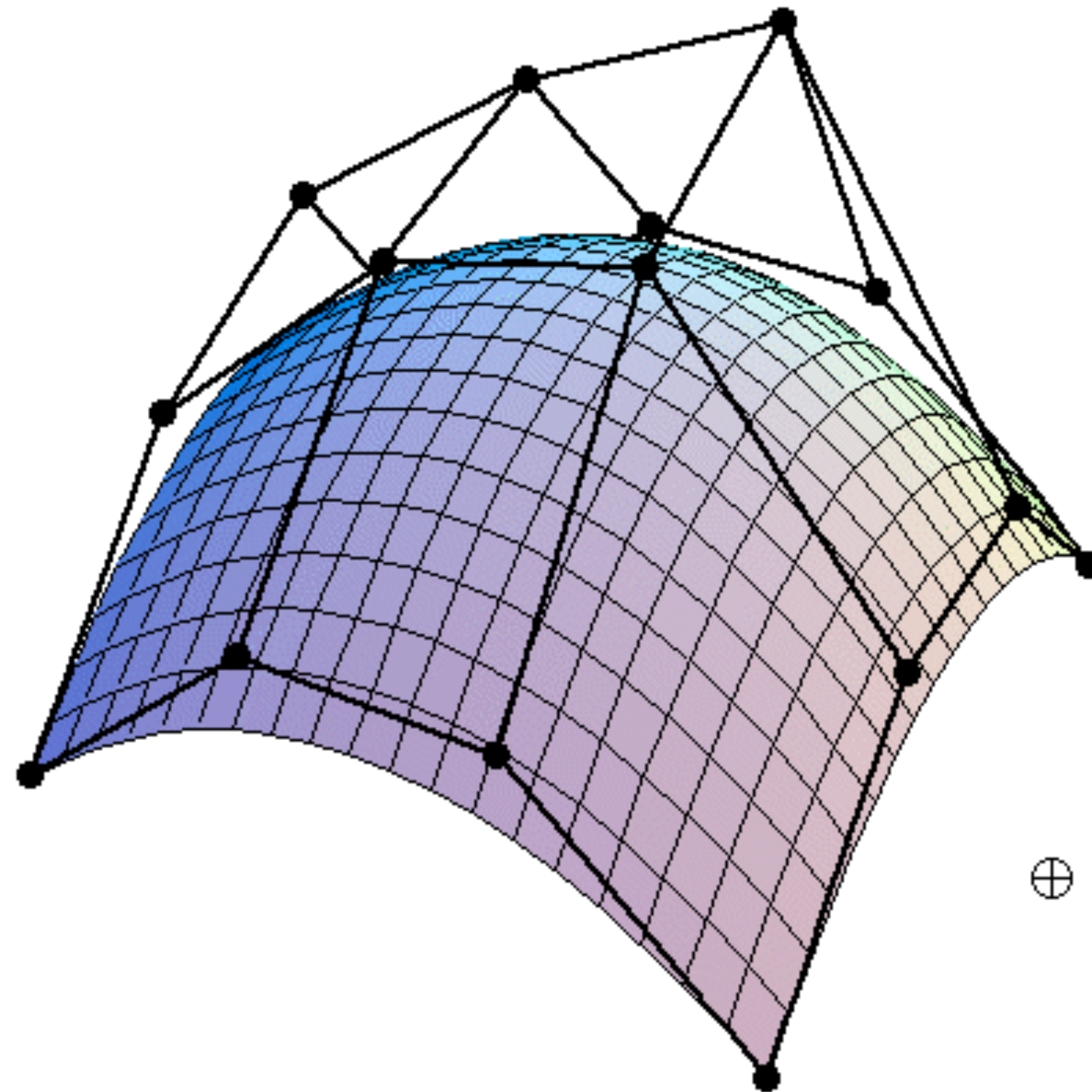
Ed Catmull's "Gumbo" model



renderspirit.com

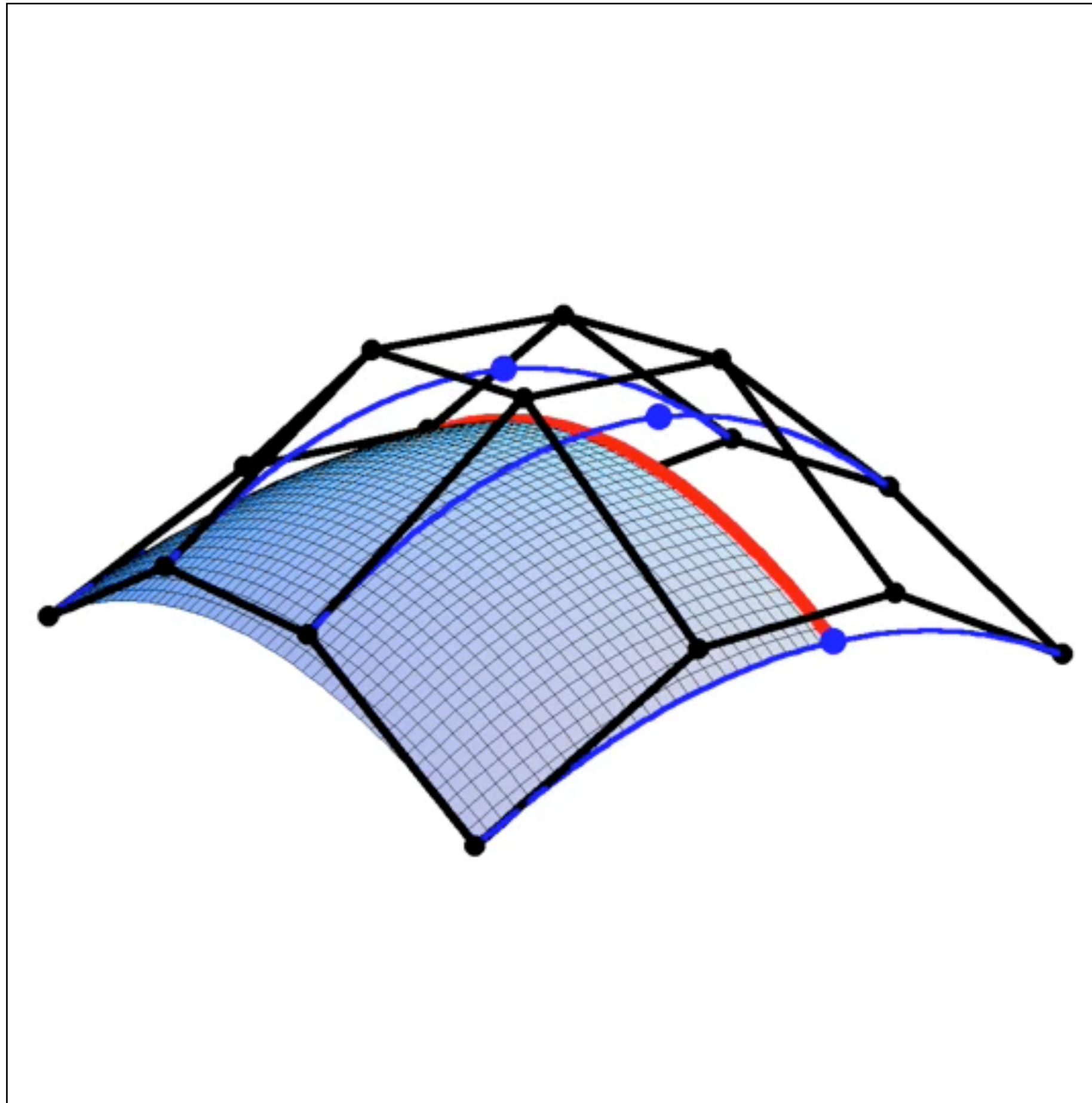
Utah Teapot

Bicubic Bézier Surface Patch



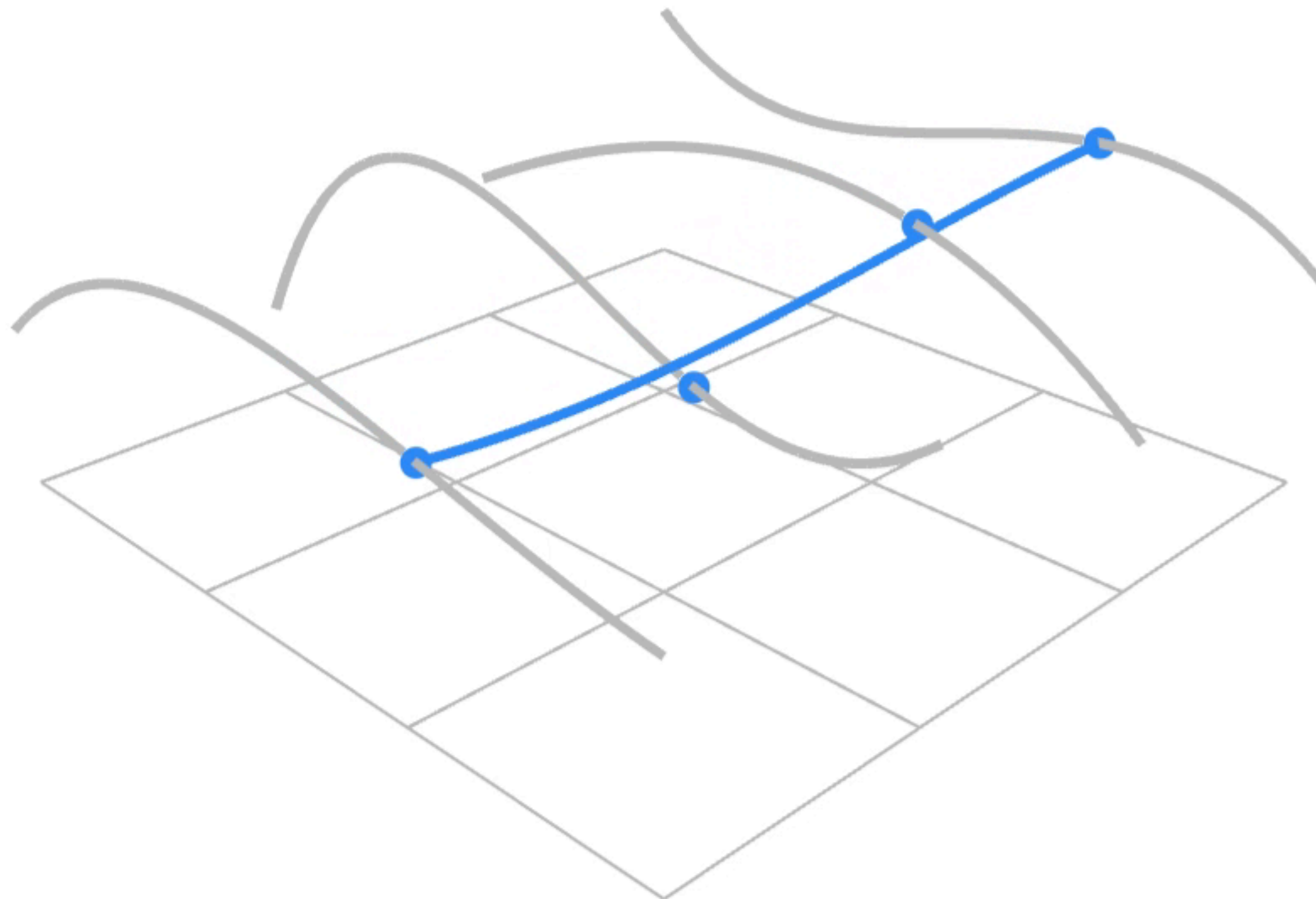
Bezier surface and 4 x 4 array of control points

Bicubic Bézier Surface Patch



Bezier surface and 4 x 4 array of control points

Visualizing Bicubic Bézier Surface Patch



Animation: Steven Wittens, Making Things with Maths, <http://acko.net>

Visualizing Bicubic Bézier Surface Patch

4x4 control points

- Each 4x1 control points in u define a Bezier curve
 - (4 Bezier curves in u)
- Corresponding points on these 4 Bezier curves define 4 control points for a “moving curve” in v
 - This “moving” curve sweeps out the 2D surface

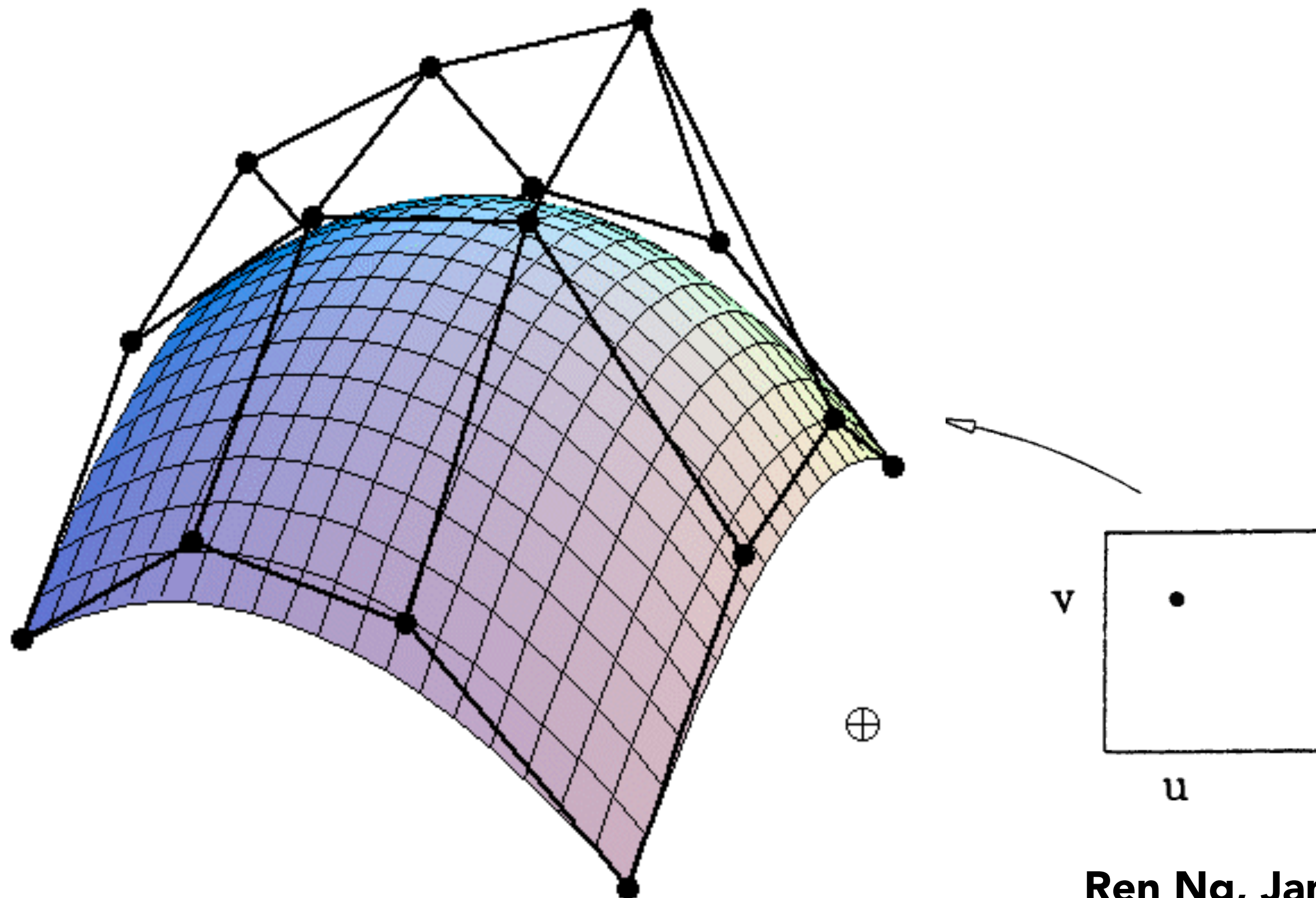
Evaluating Bézier Surfaces

Evaluating Surface Position For Parameters (u,v)

For bi-cubic Bezier surface patch,

Input: 4x4 control points

Output is 2D surface parameterized by (u,v) in $[0,1]^2$

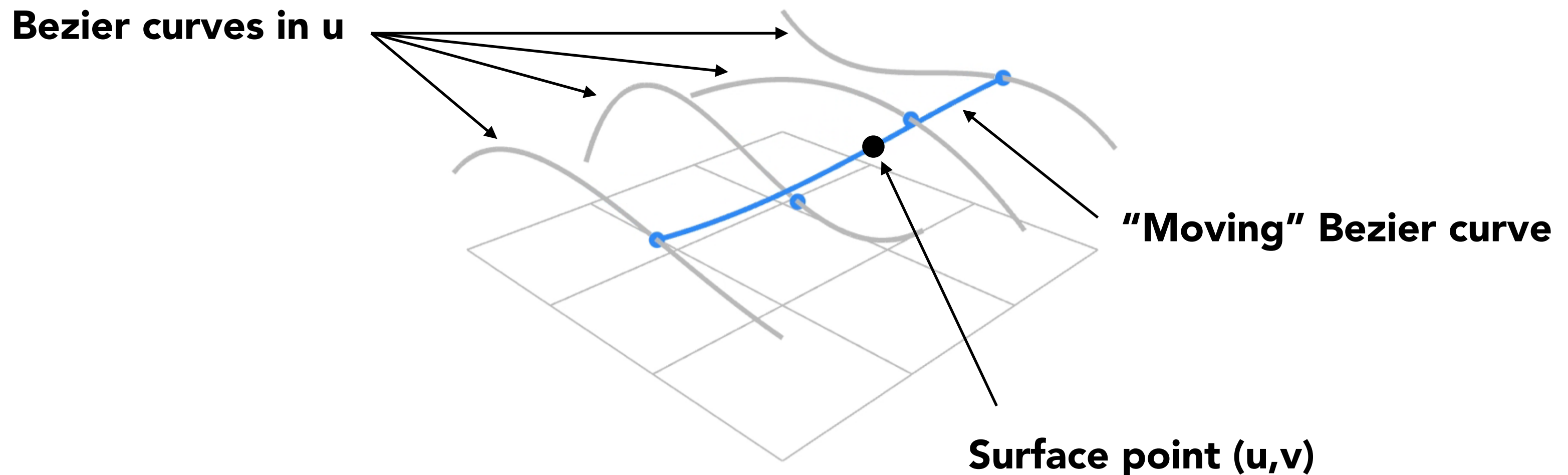


Method 1: Separable 1D de Casteljau Algorithm

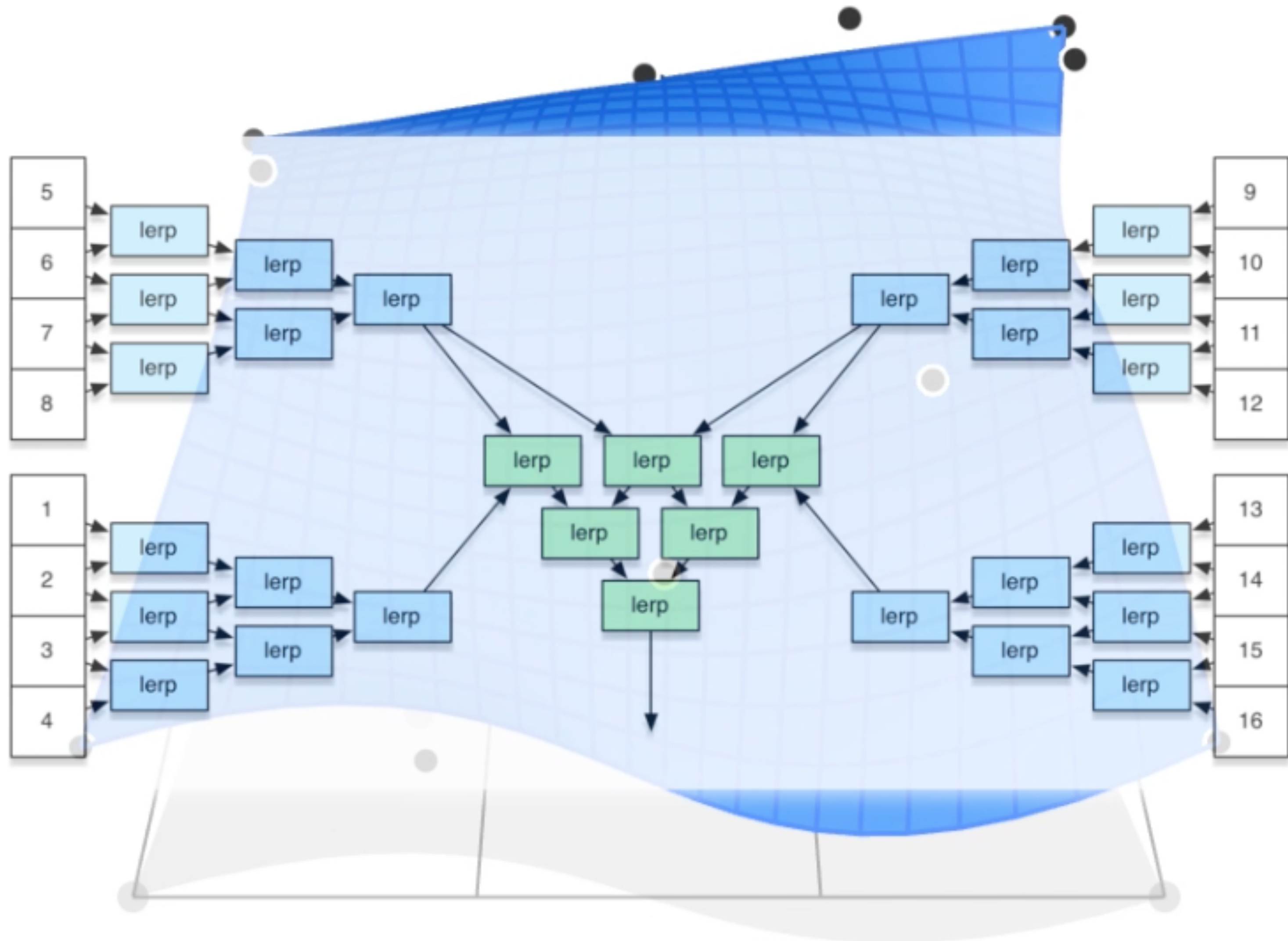
Goal: Evaluate surface position corresponding to (u,v)

(u,v) -separable application of de Casteljau algorithm

- Use de Casteljau to evaluate point u on each of the 4 Bezier curves in u . This gives 4 control points for the "moving" Bezier curve
- Use 1D de Casteljau to evaluate point v on the "moving" curve



Method 1: Separable 1D de Casteljau Algorithm



Method 2: Linear Algebra

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \beta_Z \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$\mathbf{p}_0(v) = \mathcal{P}^3(v) \cdot \beta_Z \cdot \begin{bmatrix} \mathbf{p}_{0,0} \\ \mathbf{p}_{0,1} \\ \mathbf{p}_{0,2} \\ \mathbf{p}_{0,3} \end{bmatrix}$

$\mathbf{p}_1(v) = \mathcal{P}^3(v) \cdot \beta_Z \cdot \begin{bmatrix} \mathbf{p}_{1,0} \\ \mathbf{p}_{1,1} \\ \mathbf{p}_{1,2} \\ \mathbf{p}_{1,3} \end{bmatrix}$

$\mathbf{p}_2(v) = \mathcal{P}^3(v) \cdot \beta_Z \cdot \begin{bmatrix} \mathbf{p}_{2,0} \\ \mathbf{p}_{2,1} \\ \mathbf{p}_{2,2} \\ \mathbf{p}_{2,3} \end{bmatrix}$

$\mathbf{p}_3(v) = \mathcal{P}^3(v) \cdot \beta_Z \cdot \begin{bmatrix} \mathbf{p}_{3,0} \\ \mathbf{p}_{3,1} \\ \mathbf{p}_{3,2} \\ \mathbf{p}_{3,3} \end{bmatrix}$

Method 2: Linear Algebra

$$\mathbf{x}(u) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{p}_0(v) = \mathcal{P}^3(v) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{p}_{0,0} \\ \mathbf{p}_{0,1} \\ \mathbf{p}_{0,2} \\ \mathbf{p}_{0,3} \end{bmatrix}$$

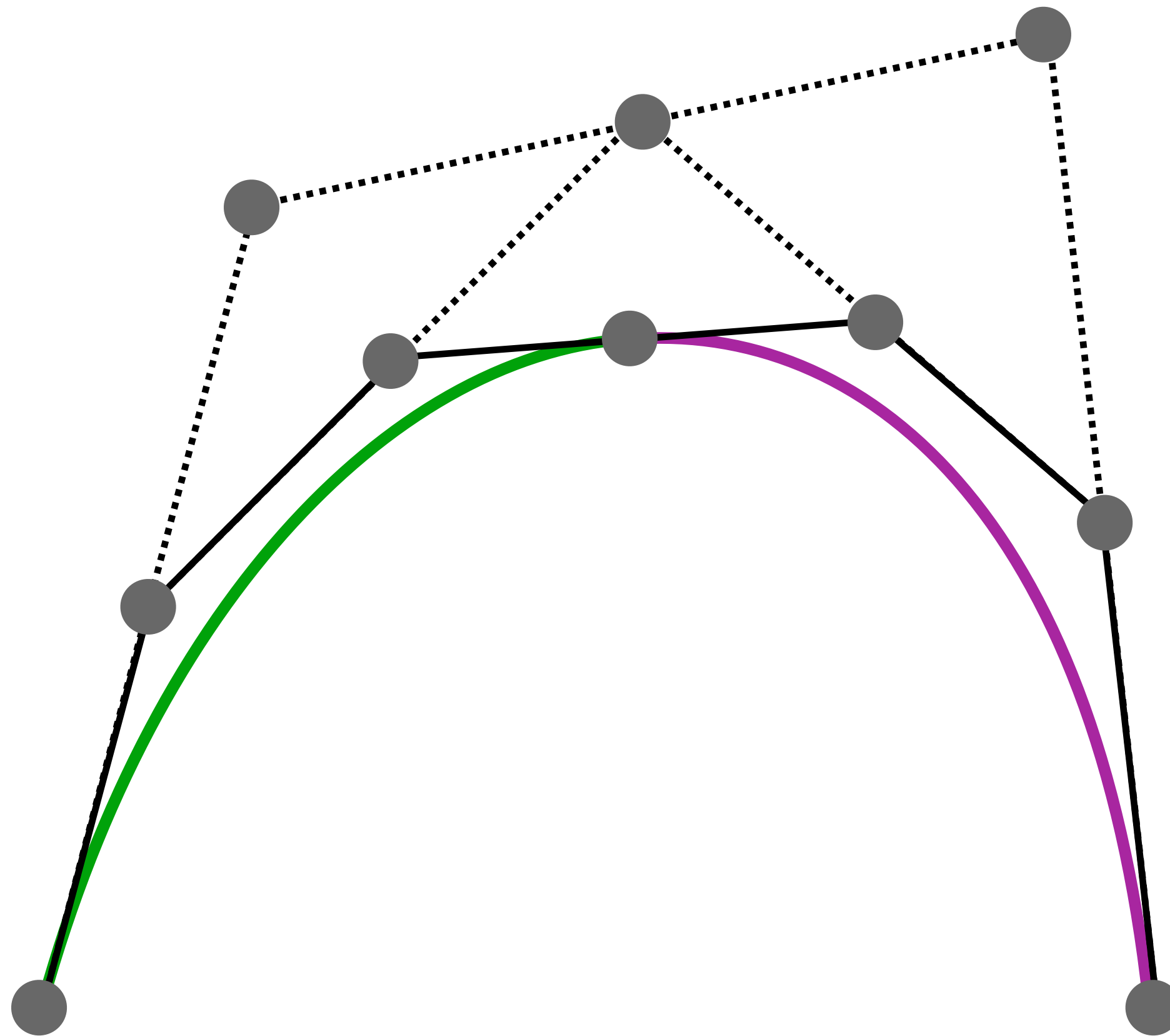
$$\mathbf{p}_1(v) = \mathcal{P}^3(v) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{p}_{1,0} \\ \mathbf{p}_{1,1} \\ \mathbf{p}_{1,2} \\ \mathbf{p}_{1,3} \end{bmatrix}$$

$$\mathbf{p}_2(v) = \mathcal{P}^3(v) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{p}_{2,0} \\ \mathbf{p}_{2,1} \\ \mathbf{p}_{2,2} \\ \mathbf{p}_{2,3} \end{bmatrix}$$

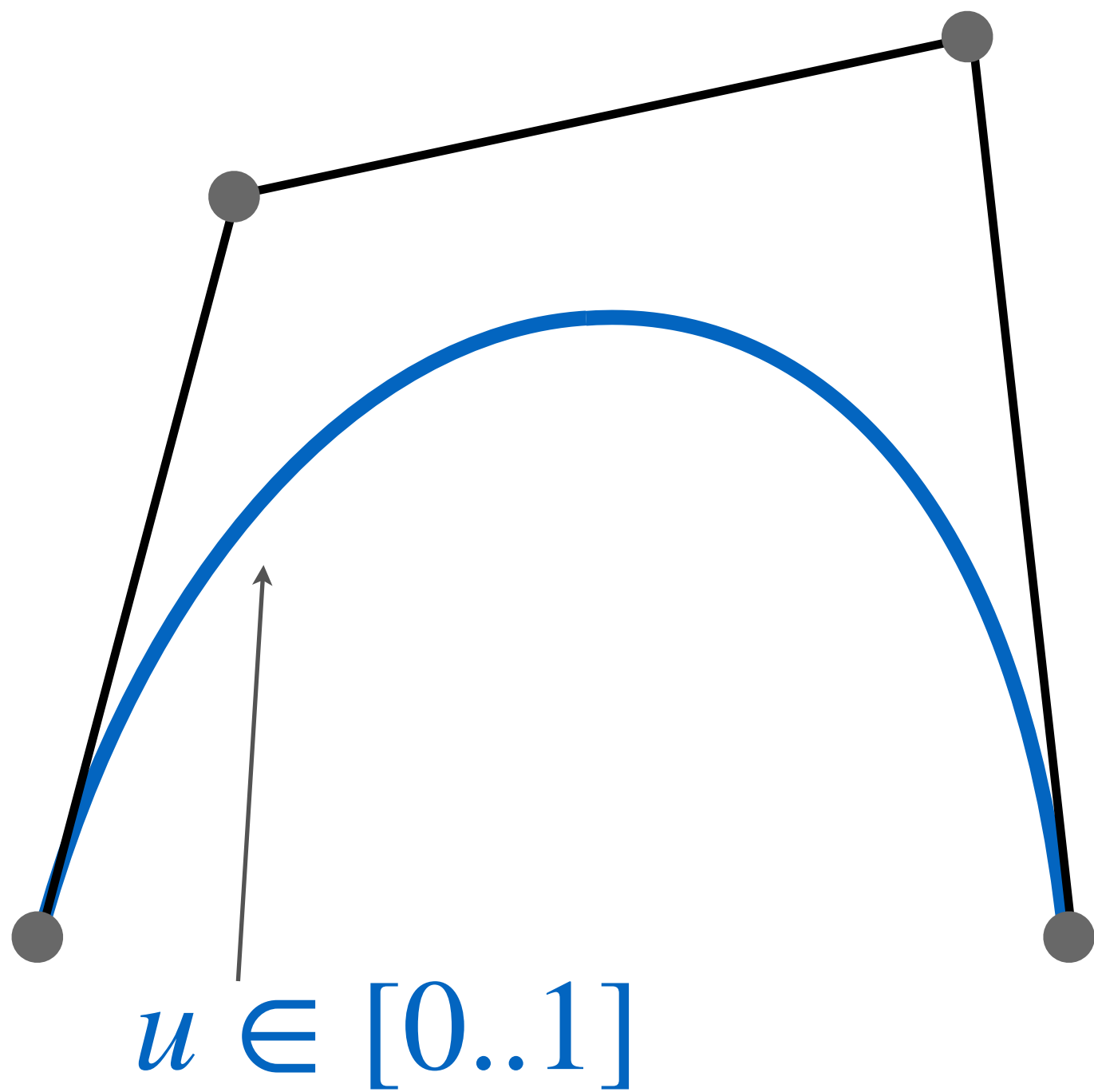
$$\mathbf{p}_3(v) = \mathcal{P}^3(v) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{p}_{3,0} \\ \mathbf{p}_{3,1} \\ \mathbf{p}_{3,2} \\ \mathbf{p}_{3,3} \end{bmatrix}$$

$$\mathbf{x}(u, v) = \mathcal{P}^3(u) \cdot \boldsymbol{\beta}_Z \cdot \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,1} & \mathbf{p}_{0,2} & \mathbf{p}_{0,3} \\ \mathbf{p}_{1,0} & \mathbf{p}_{1,1} & \mathbf{p}_{1,2} & \mathbf{p}_{1,3} \\ \mathbf{p}_{2,0} & \mathbf{p}_{2,1} & \mathbf{p}_{2,2} & \mathbf{p}_{2,3} \\ \mathbf{p}_{3,0} & \mathbf{p}_{3,1} & \mathbf{p}_{3,2} & \mathbf{p}_{3,3} \end{bmatrix} \cdot \boldsymbol{\beta}_Z^\top \cdot \mathcal{P}^3(v)^\top$$

Method 3: Bézier Subdivision



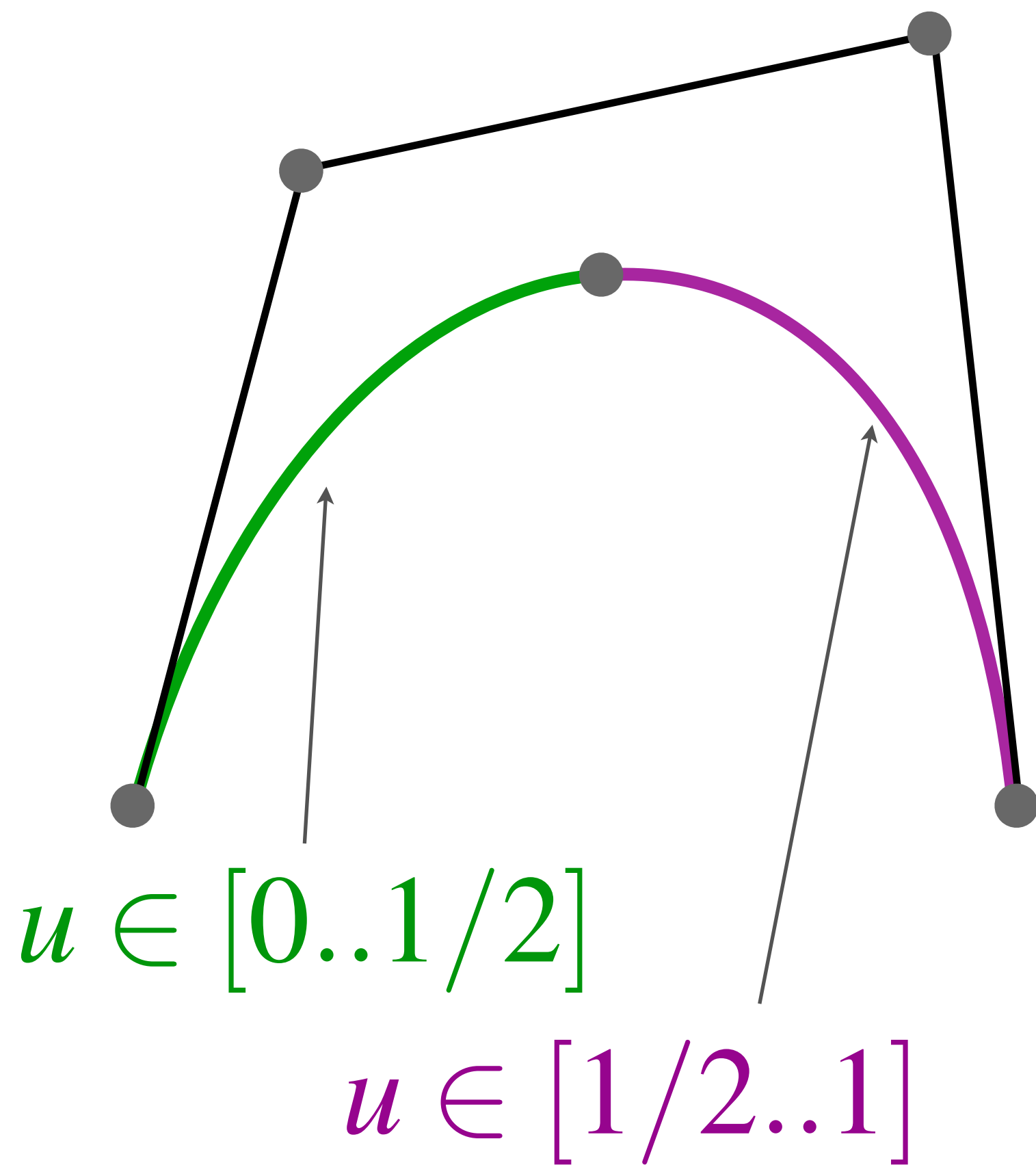
Bézier Subdivision



$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \mathbf{P}$$

$$\beta_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

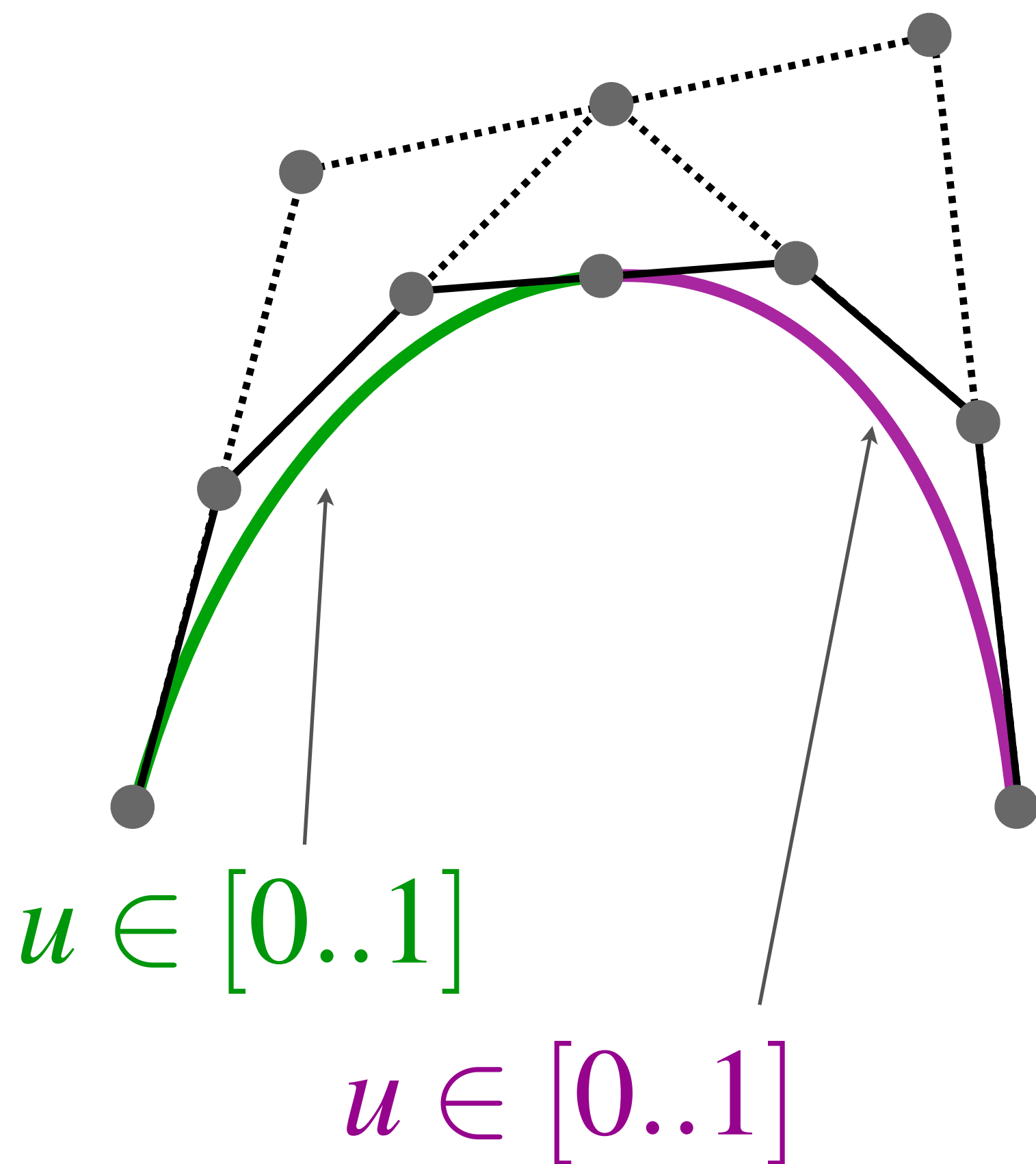
Bézier Subdivision



$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \mathbf{P}$$

$$\beta_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Bézier Subdivision

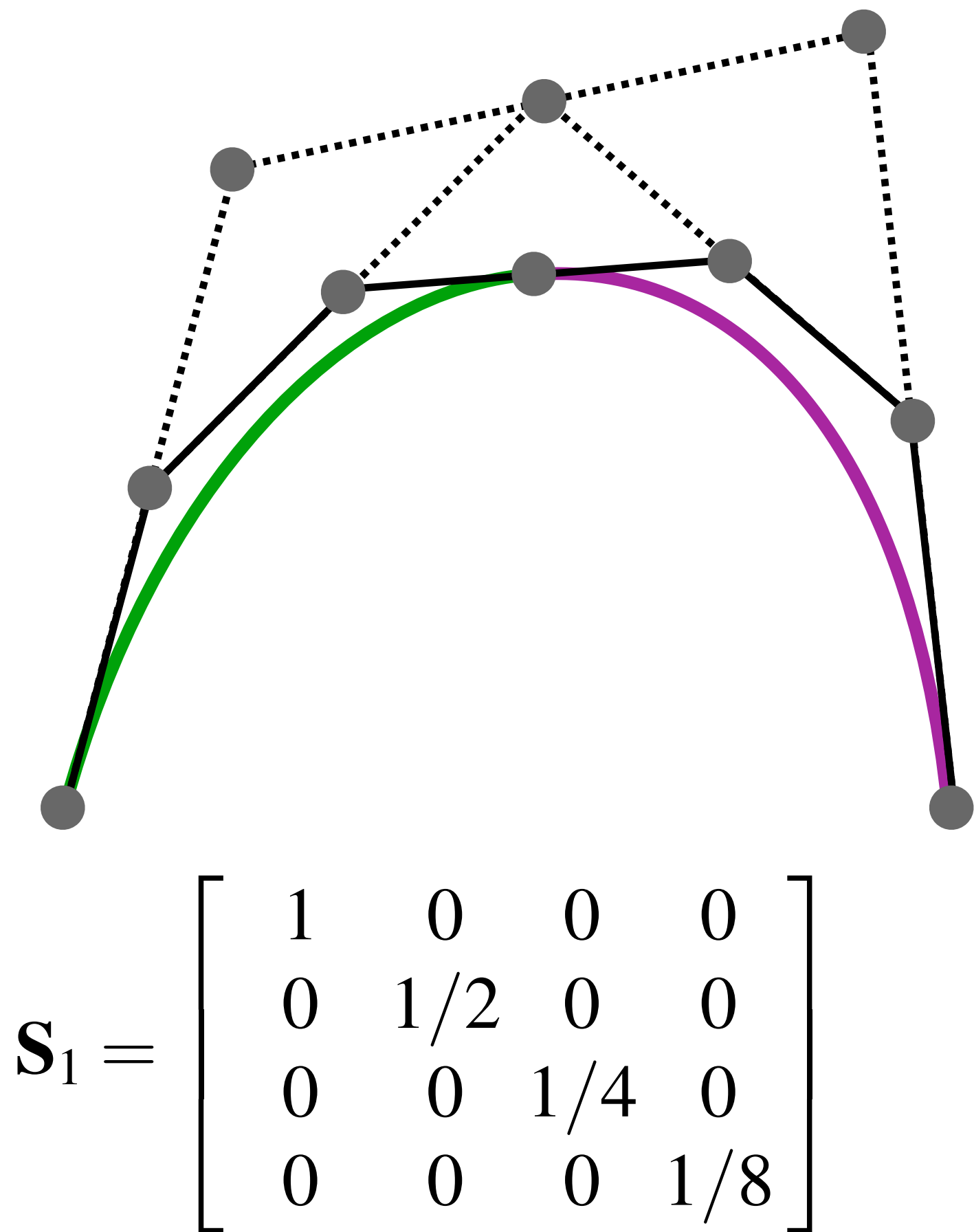


$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_z \mathbf{P}$$

Can't change these....

$$\beta_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Bézier Subdivision



$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \mathbf{P} \quad u \in [0..1/2]$$

$$\mathbf{x}(u) = [1, \frac{u}{2}, \frac{u^2}{4}, \frac{u^3}{8}] \beta_Z \mathbf{P} \quad u \in [0..1]$$

$$\mathbf{x}(u) = [1, u, u^2, u^3] \mathbf{S}_1 \beta_Z \mathbf{P}$$

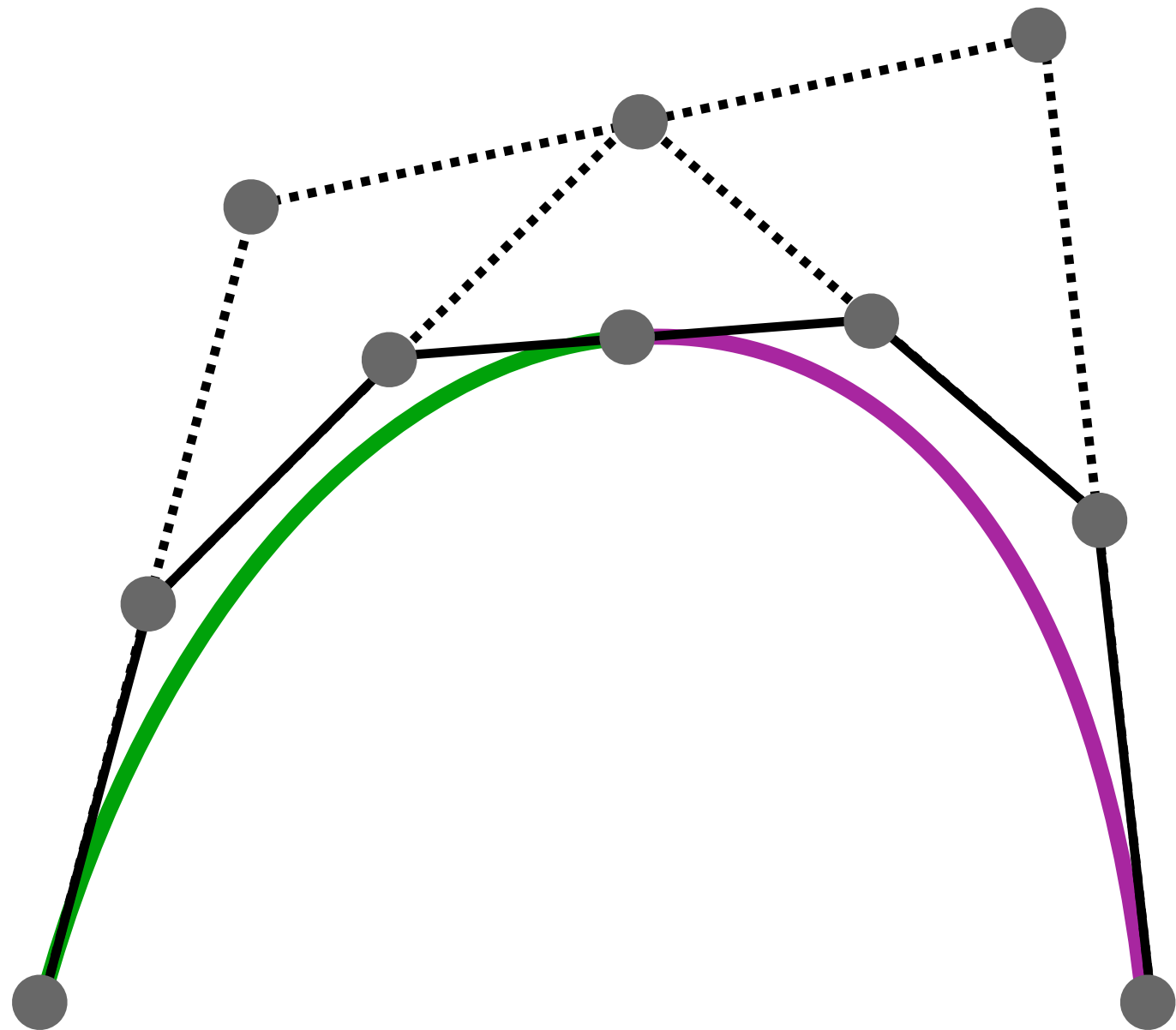
$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \beta_Z^{-1} \mathbf{S}_1 \beta_Z \mathbf{P}$$

$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \mathbf{H}_{Z1} \mathbf{P}$$

Bézier Subdivision

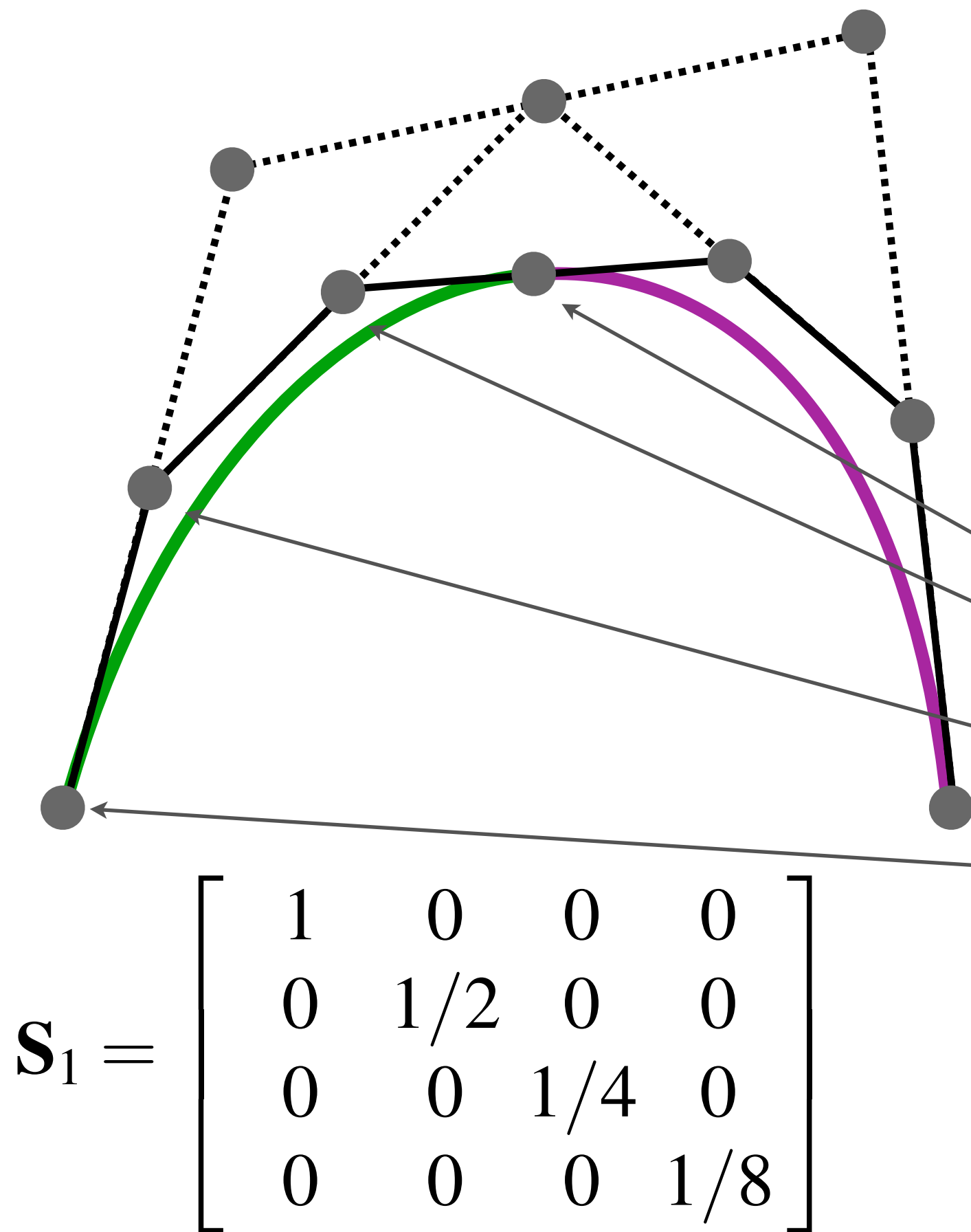
$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \beta_Z^{-1} \mathbf{S}_1 \beta_Z \mathbf{P}$$

$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \mathbf{H}_{Z1} \mathbf{P}$$



$$\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix}$$

Bézier Subdivision



$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \beta_Z^{-1} S_1 \beta_Z \mathbf{P}$$

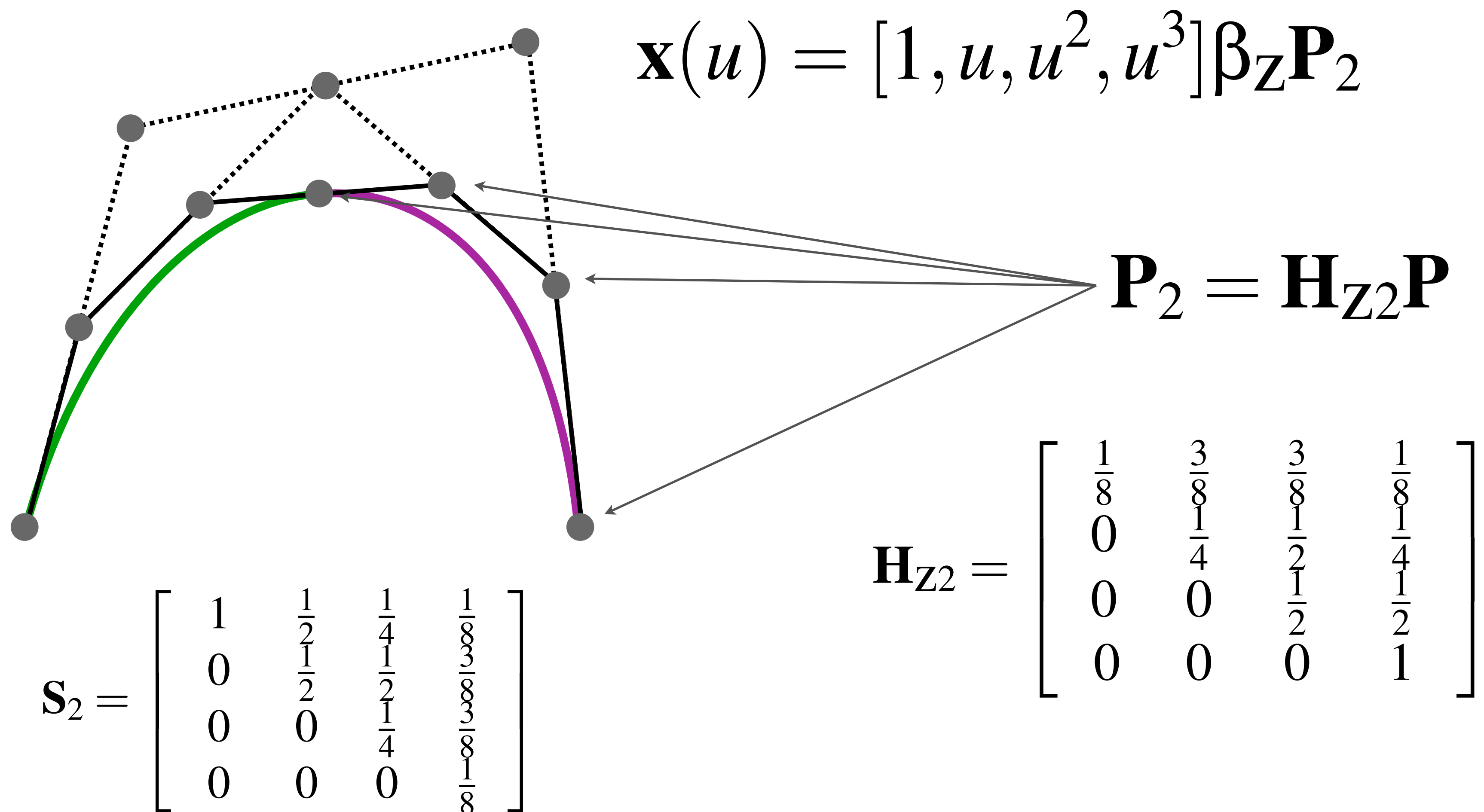
$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \mathbf{H}_{Z1} \mathbf{P}$$

$$\mathbf{x}(u) = [1, u, u^2, u^3] \beta_Z \mathbf{P}_1$$

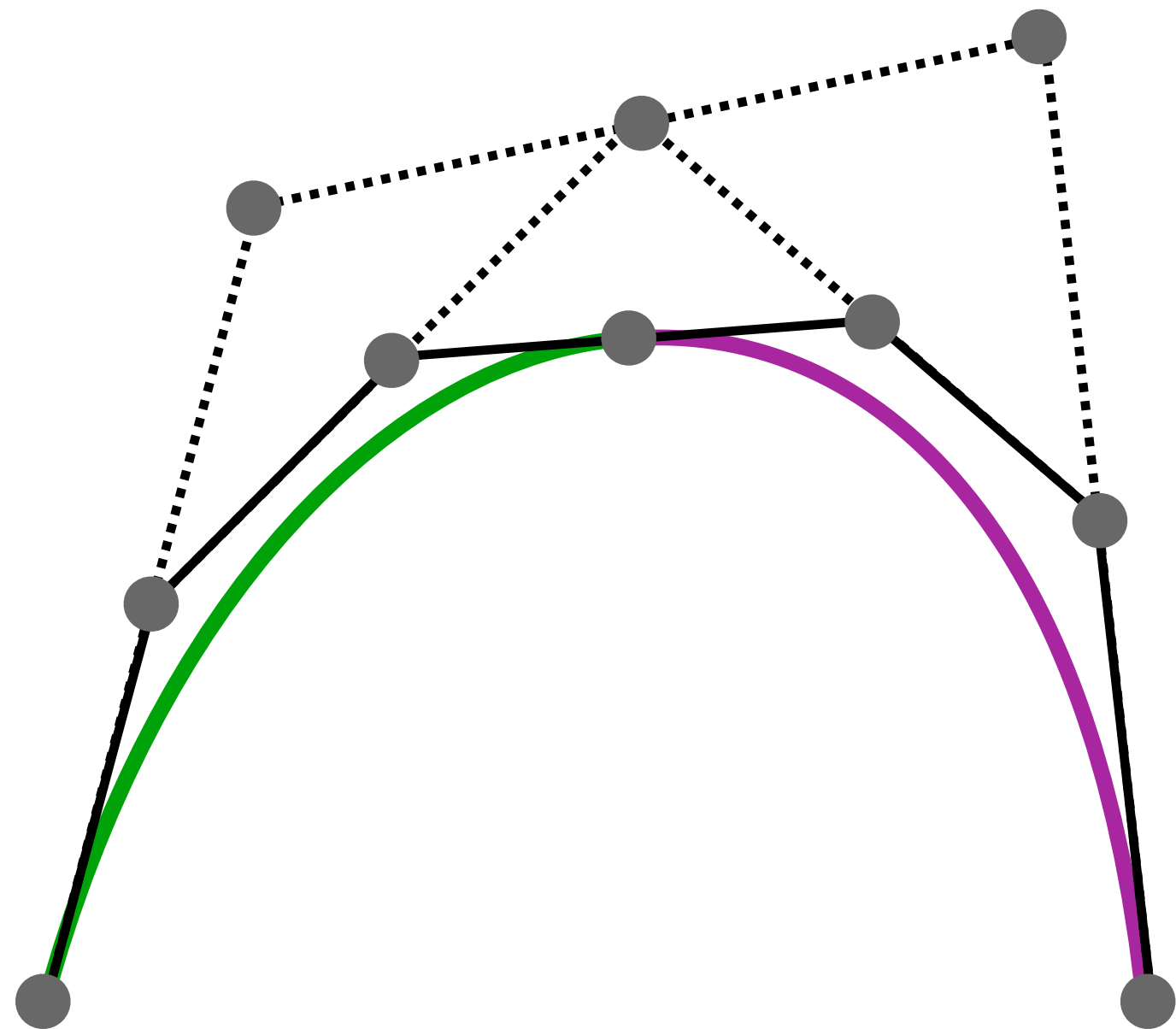
$$\mathbf{H}_{Z1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

$$\mathbf{P}_1 = \mathbf{H}_{Z1} \mathbf{P}$$

Bézier Subdivision



Bézier Subdivision



$$\mathbf{P}_1 = \mathbf{H}_{Z1}\mathbf{P}$$

$$\mathbf{P}_2 = \mathbf{H}_{Z2}\mathbf{P}$$

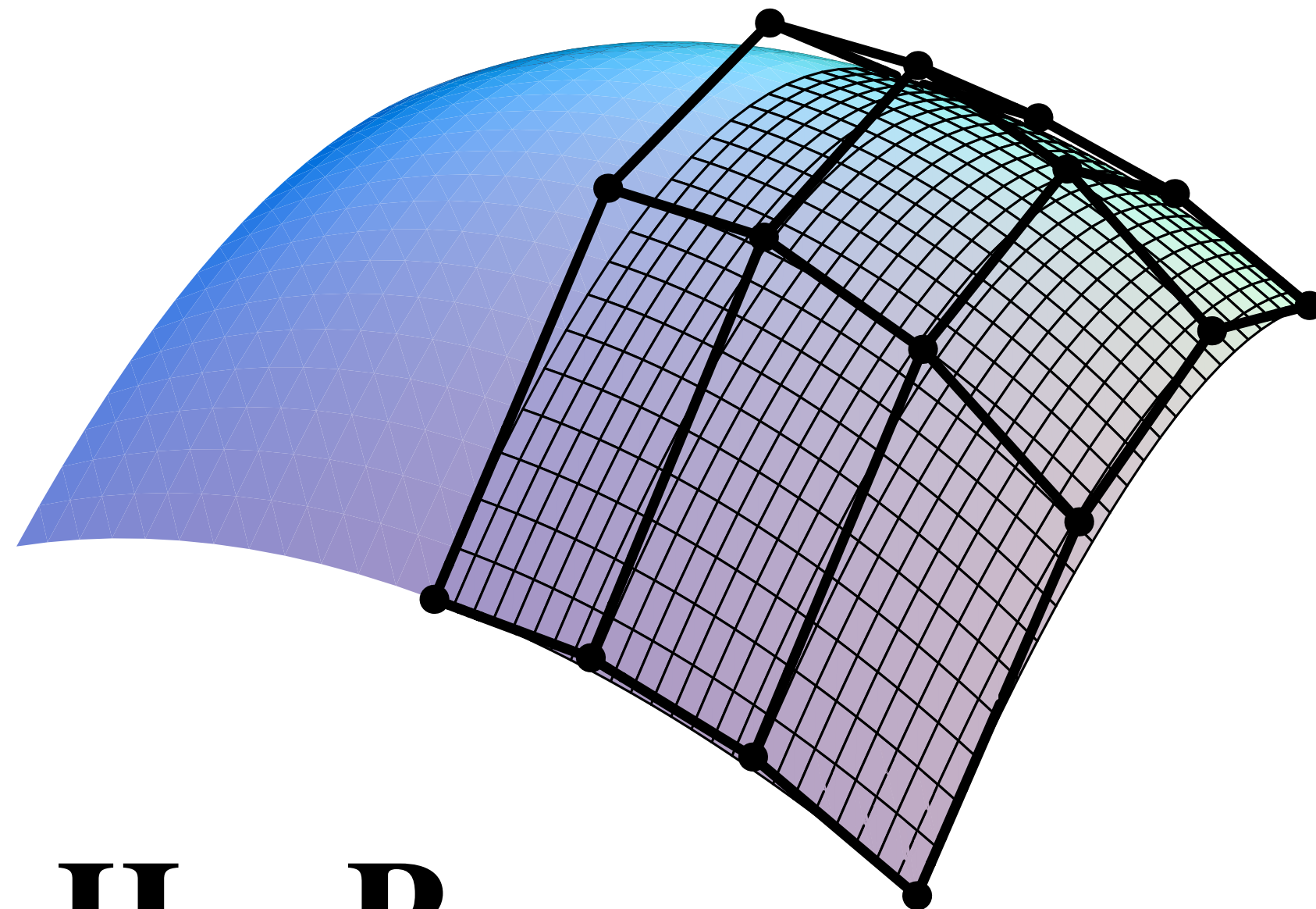
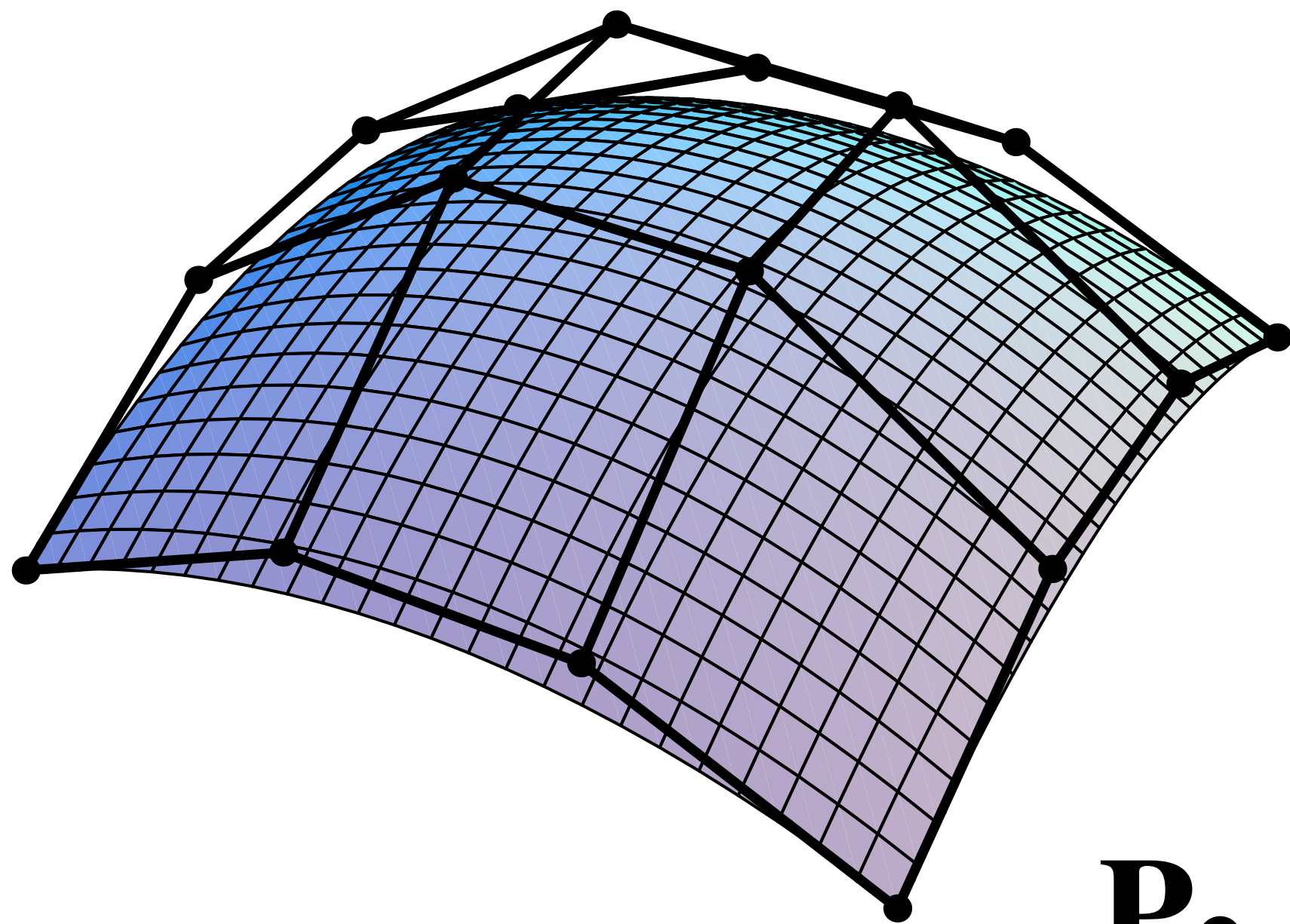
$$\mathbf{H}_{Z1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

$$\mathbf{H}_{Z2} = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rows of $\mathbf{H}_Z$ sum to 1 and $\geq 0 \Rightarrow$ Convex Hull $\Rightarrow \mathbf{P}_1$ and $\mathbf{P}_2 \in \text{ConvexHull}(\mathbf{P})$

Curve remains constant $\Rightarrow$ Control Mesh converges to surface!!

Bézier Subdivision

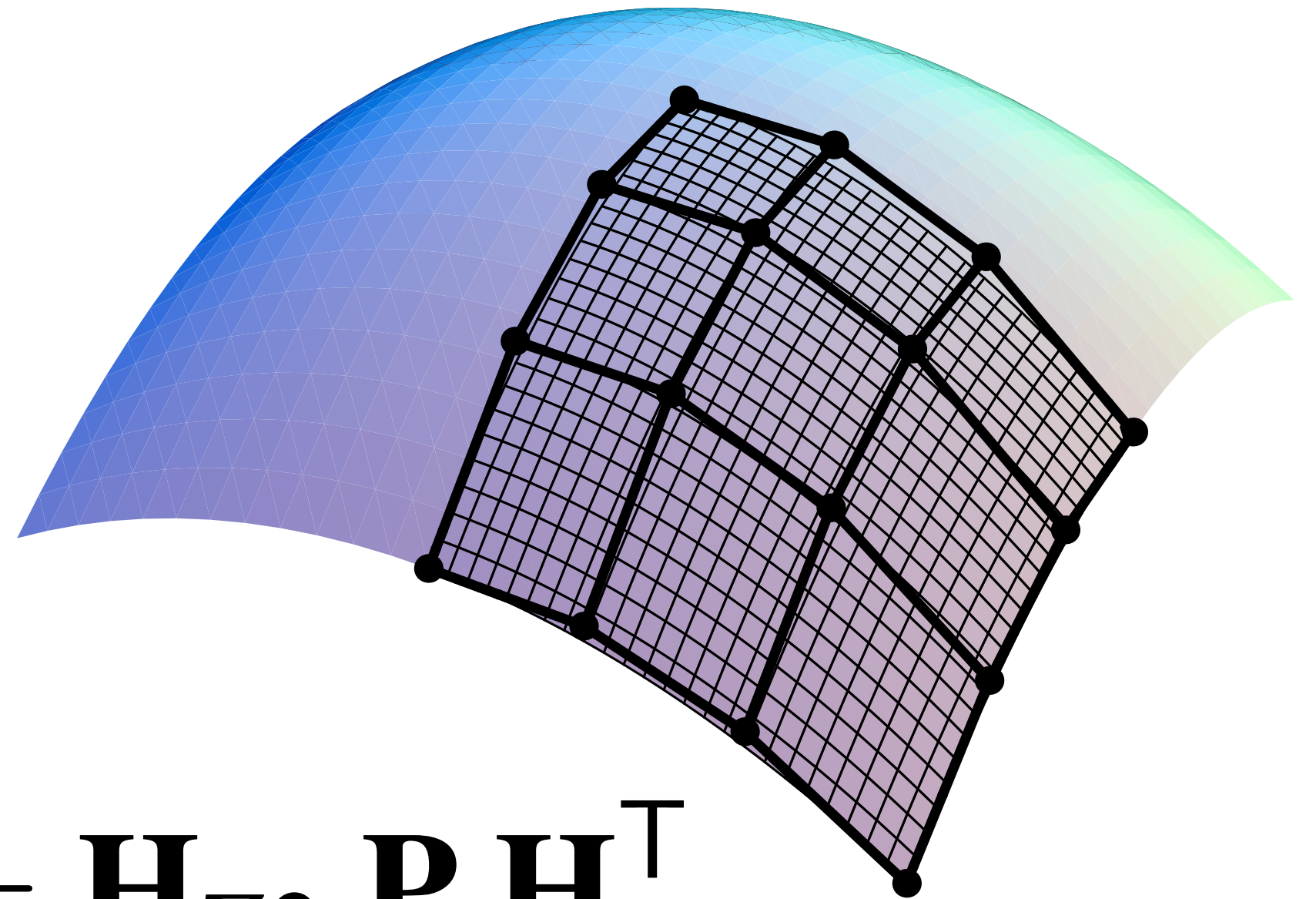
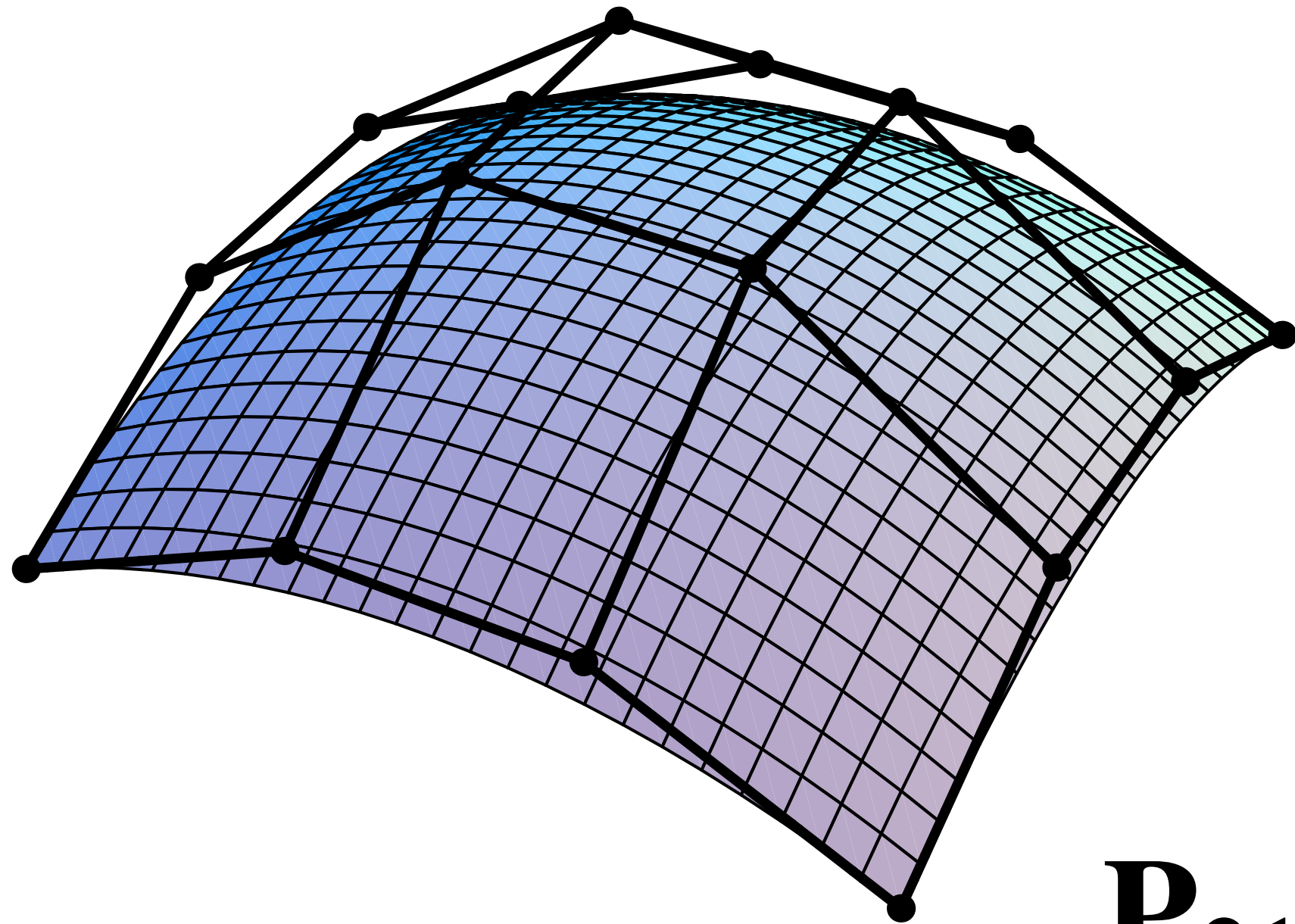


$$\mathbf{P}_{2.} = \mathbf{H}_{Z2} \mathbf{P}$$

$$\mathbf{x}(u, v) = [1, u, u^2, u^3] \beta_Z \mathbf{P} \beta_Z^T [1, v, v^2, v^3]^T$$

4 x 4 matrix of control points

Bézier Subdivision



$$\mathbf{P}_{21} = \mathbf{H}_{Z2} \mathbf{P} \mathbf{H}_{Z1}^T$$

$$\mathbf{x}(u, v) = [1, u, u^2, u^3] \beta_Z \mathbf{P} \beta_Z^T [1, v, v^2, v^3]^T$$

4 x 4 matrix of control points

Method 4: Algebraic Evaluation

Let the moving curve be a degree m Bézier curve

$$\mathbf{b}^m(u) = \sum_{i=0}^m \mathbf{b}_i B_i^m(u) \quad \text{(remember, Bernstein polynomials)}$$
$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Let each control point $\mathbf{b}_i$ be moving along a Bézier curve of degree n

$$\mathbf{b}_i = \mathbf{b}_i(v) = \sum_{j=0}^n \mathbf{b}_{i,j} B_j^n(v)$$

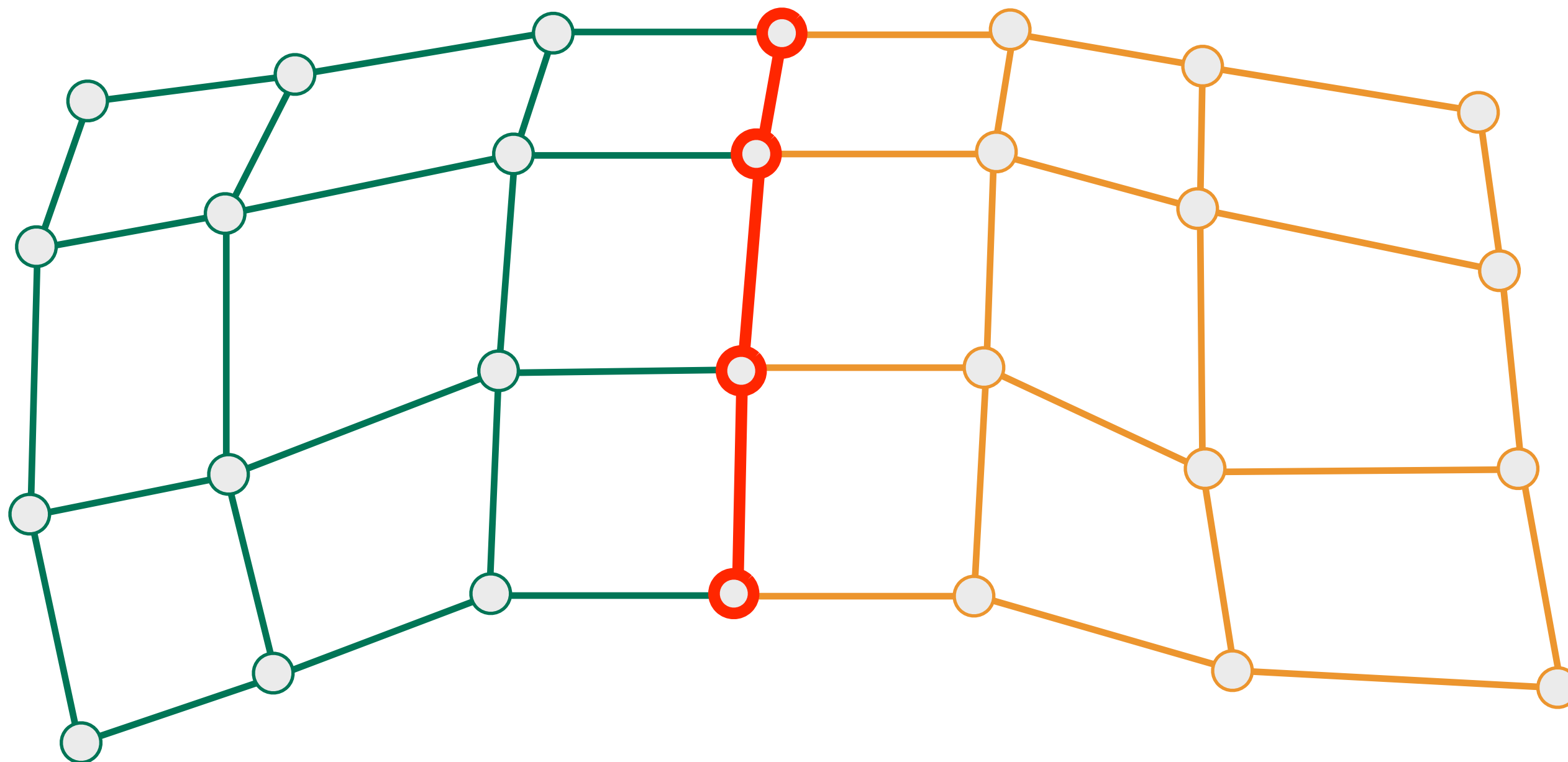
Tensor product Bézier patch

$$\mathbf{b}^{m,n}(u, v) = \sum_{i=0}^m \sum_{j=0}^n \mathbf{b}_{i,j} B_i^m(u) B_j^n(v)$$

Bézier Surface Continuity

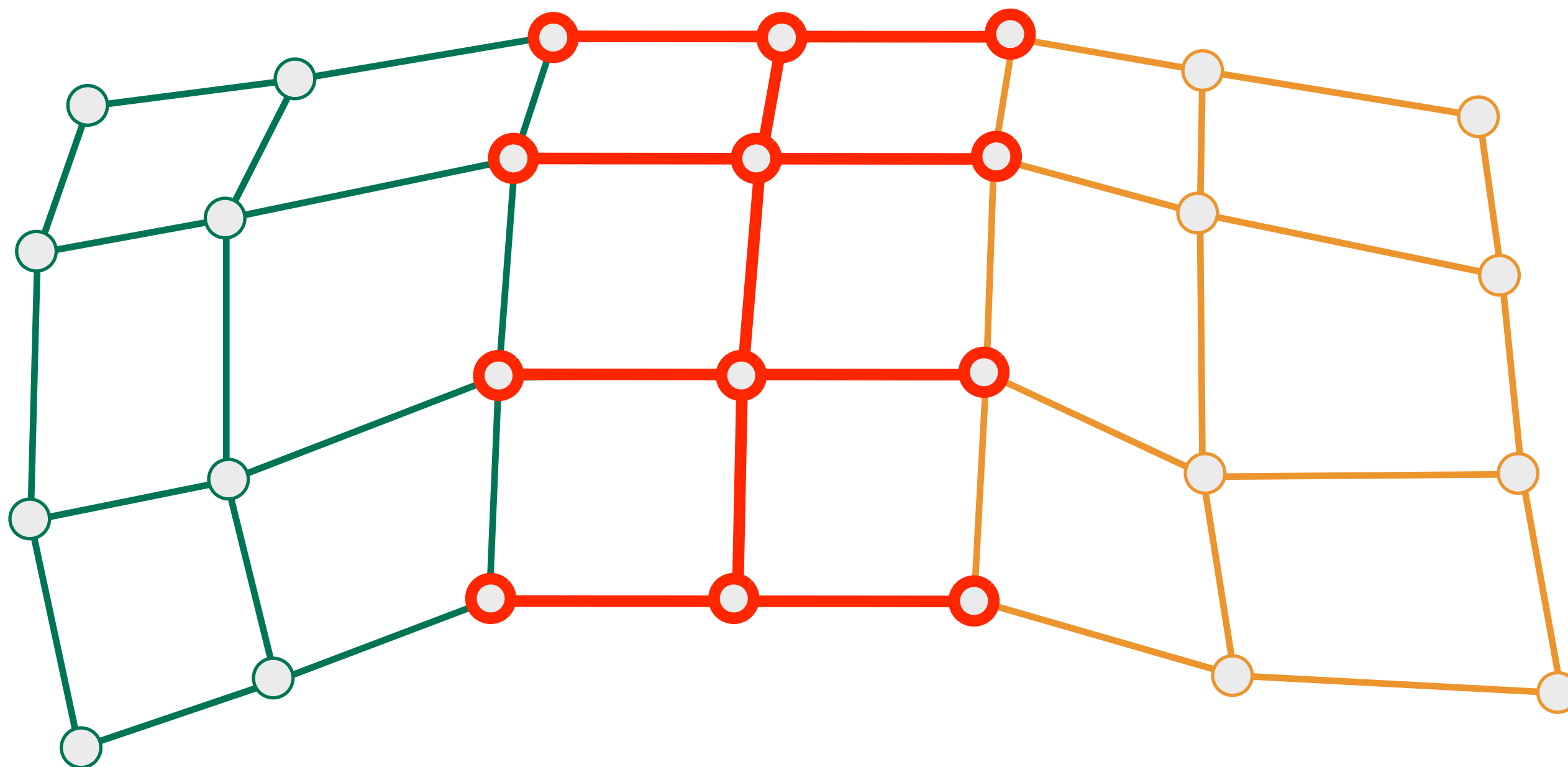
Piecewise Bézier Surfaces

C^0 continuity: Boundary curves



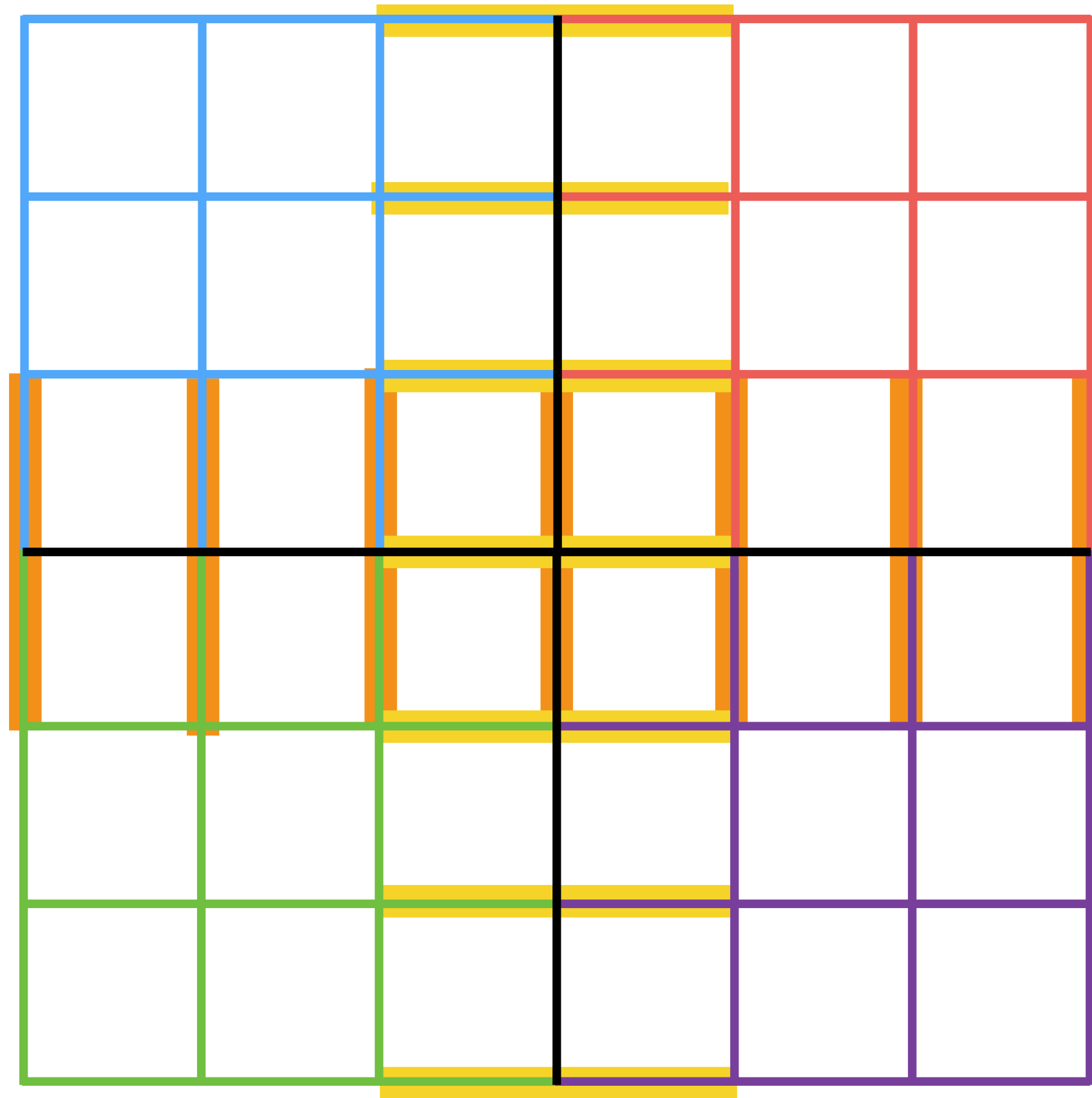
Piecewise Bézier Surfaces

C^1 continuity: Collinearity



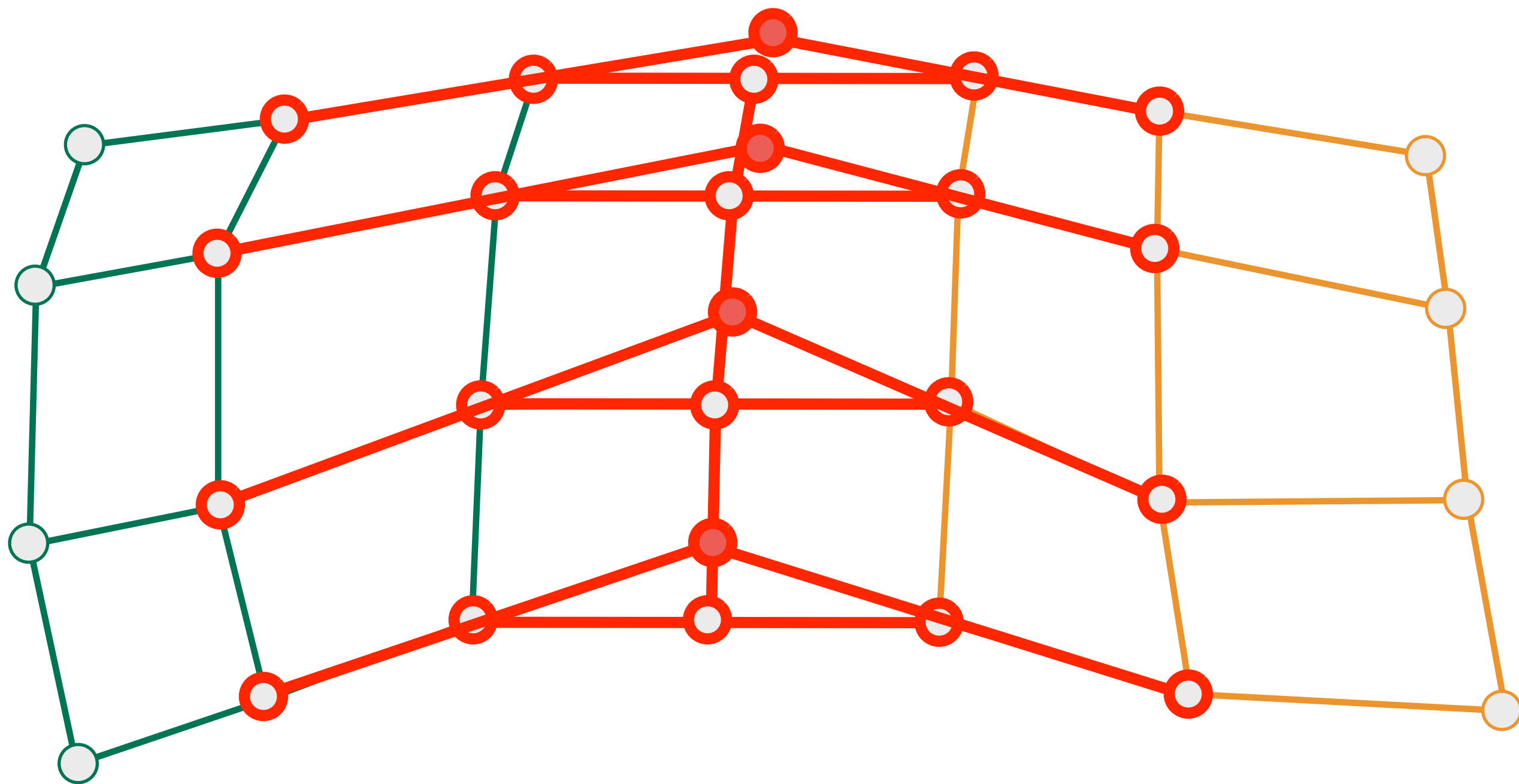
Piecewise Bézier Surfaces

C^1 continuity: Collinearity



Piecewise Bézier Surfaces

C^2 continuity: A-frames



Things to Remember

Splines

- Cubic Hermite and Catmull-Rom interpolation
- Matrix representation of cubic polynomials

Bézier curves

- Easy-to-control spline
- Recursive linear interpolation – de Casteljau algorithm
- Properties of Bézier curves
- Piecewise Bézier curve – continuity types and how to achieve

Bézier surfaces

- Bicubic Bézier patches – tensor product surface
- 2D de Casteljau algorithm

Acknowledgments

This slide set contain contributions from:

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- Pat Hanrahan
- Angjoo Kanazawa
- Steve Marschner
- Ren Ng
- James O'Brien
- Mark Pauly