Lecture 3:

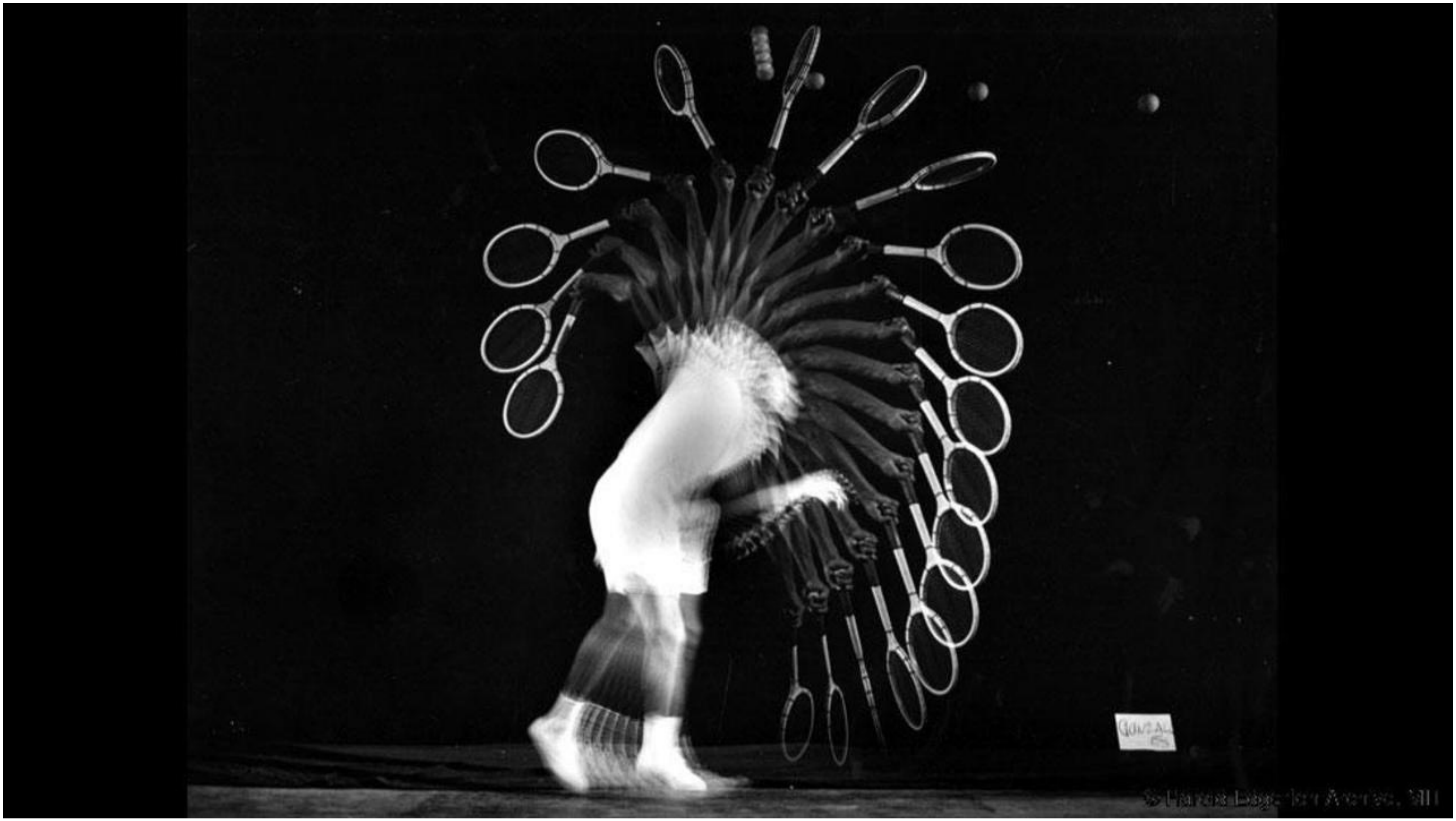
Intro to Signal Processing: Sampling, Aliasing, Antialiasing

Computer Graphics and Imaging

UC Berkeley CS184/284A

Sampling is Ubiquitous in Computer Graphics and Imaging

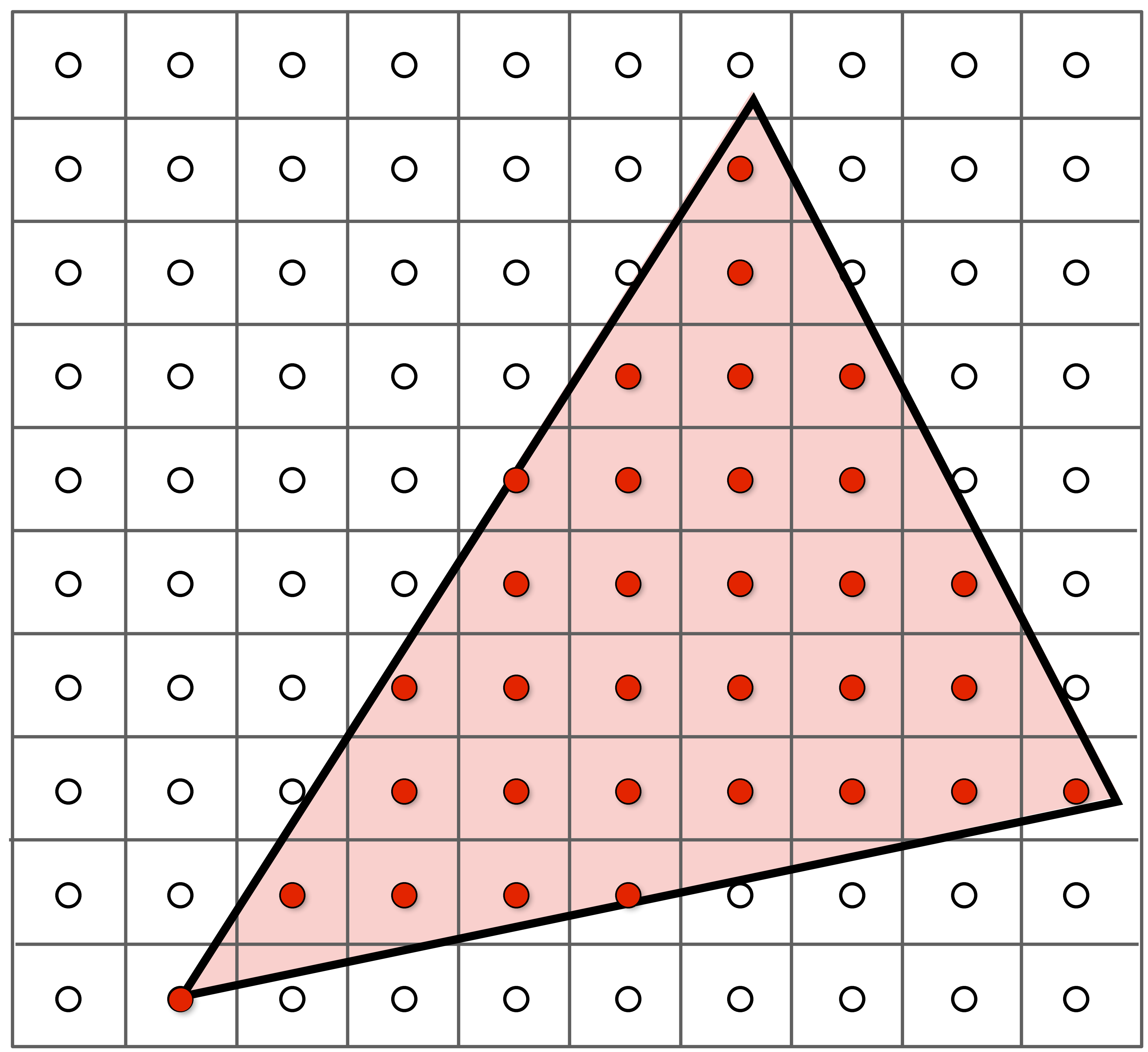
Video = Sample Time



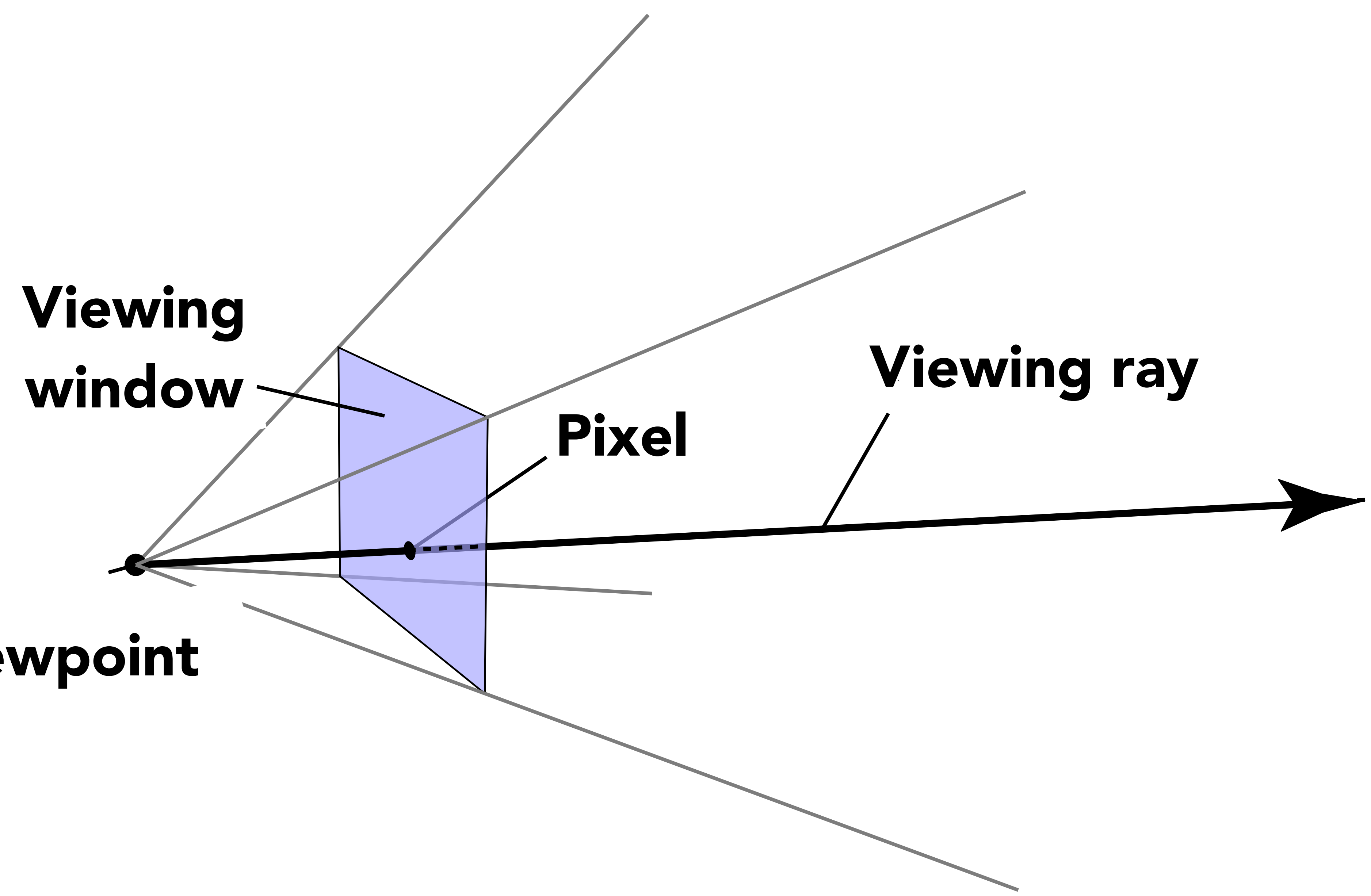
Harold Edgerton Archive, MIT

Photograph = Sample Image Sensor Plane

Rasterization = Sample 2D Positions

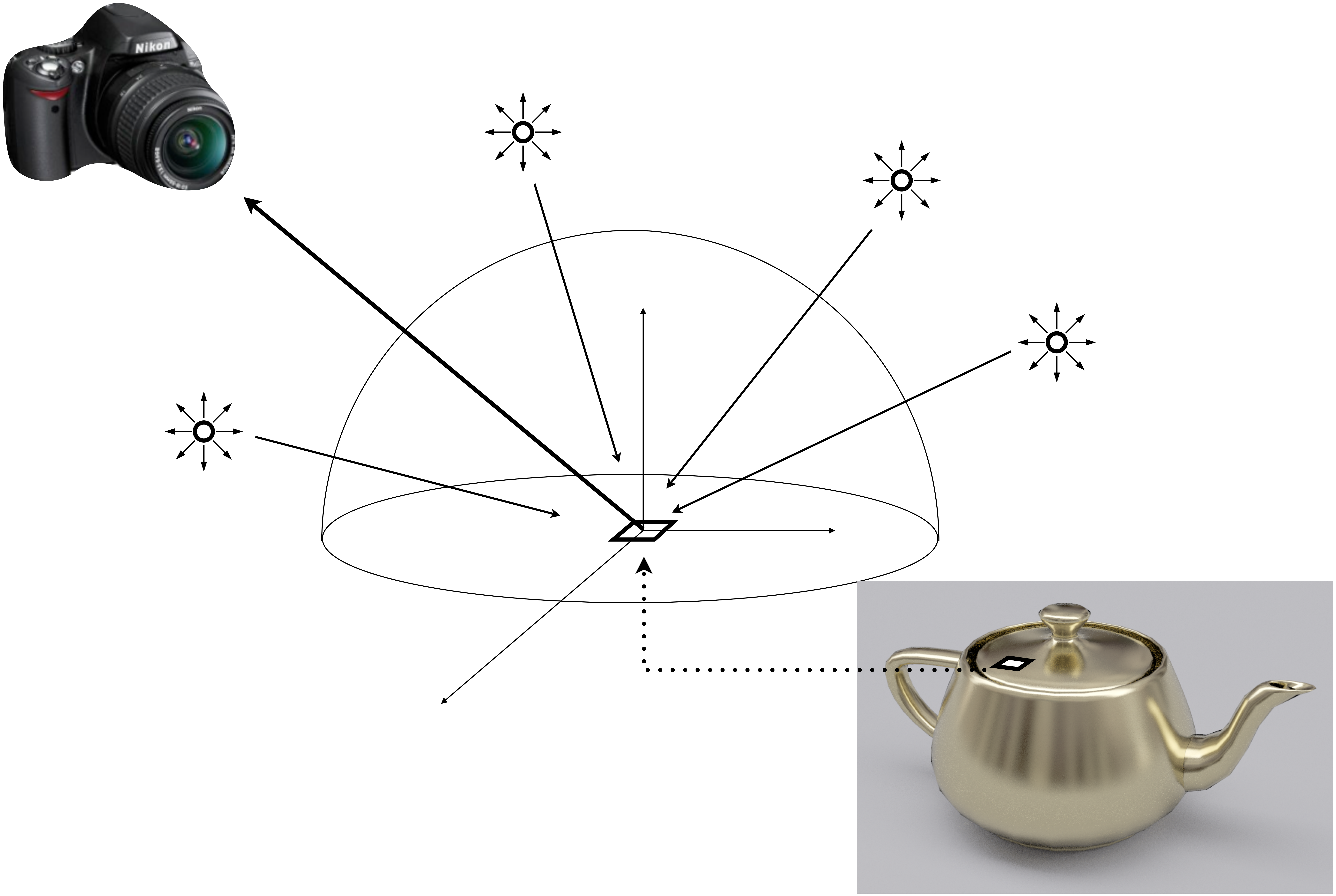


Ray Tracing = Sample Rays



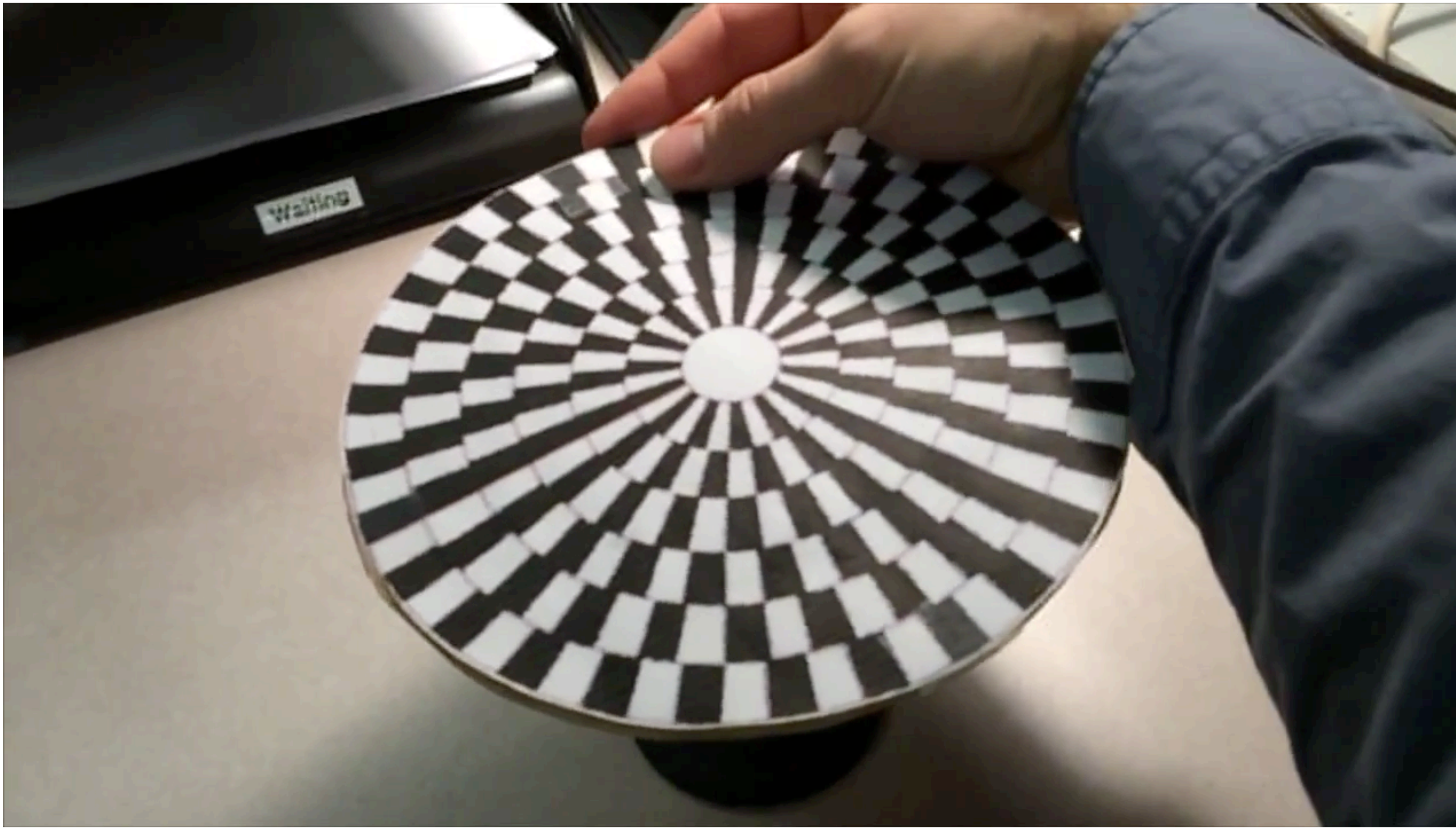
Jensen

Lighting Integrals: Sample Incident Angles



Sampling Artifacts in Graphics and Imaging

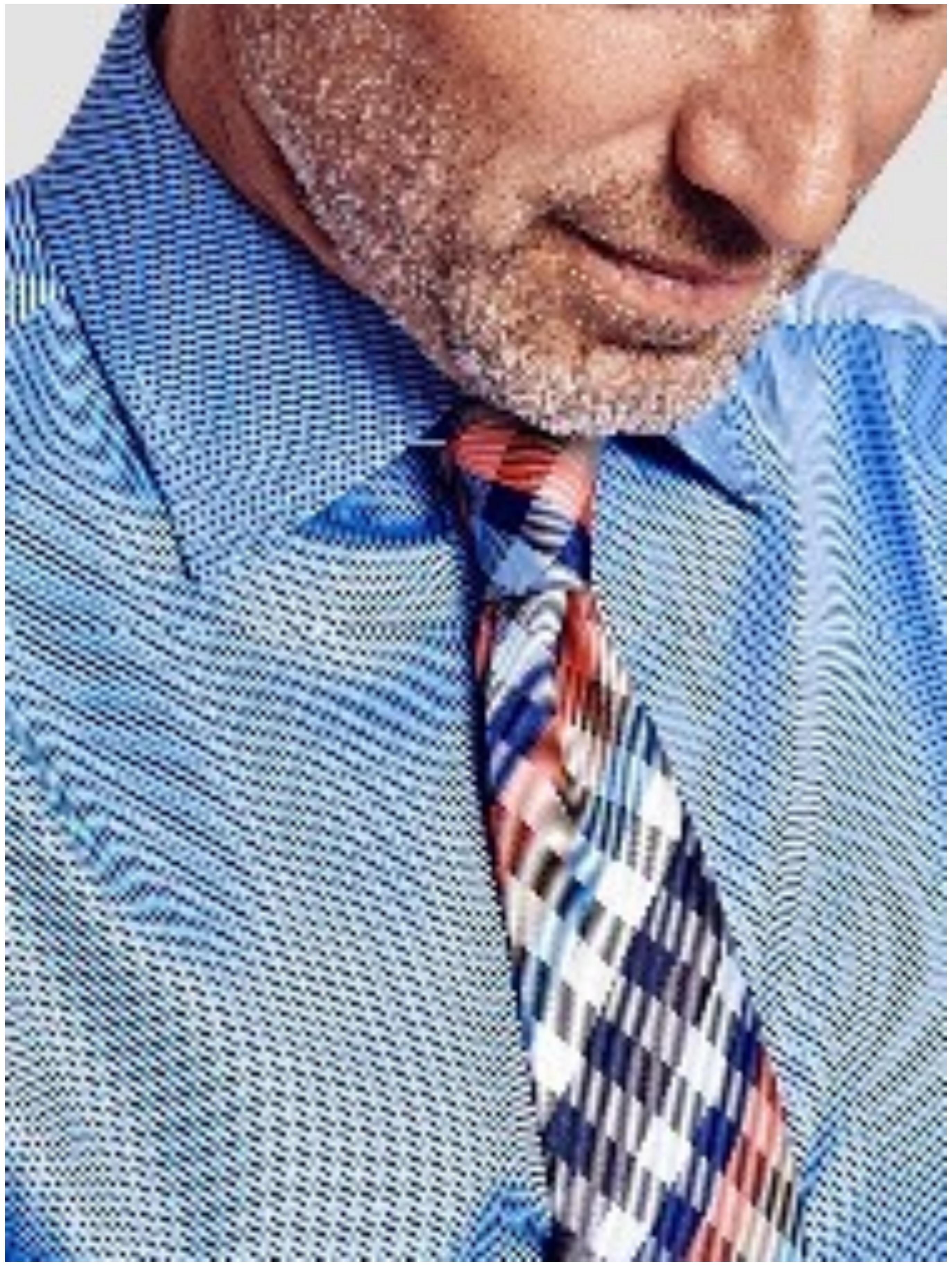
Wagon Wheel Illusion (False Motion)



Created by Jesse Mason, https://www.youtube.com/watch?v=QOwzkND_ooU

Moiré Patterns in Imaging

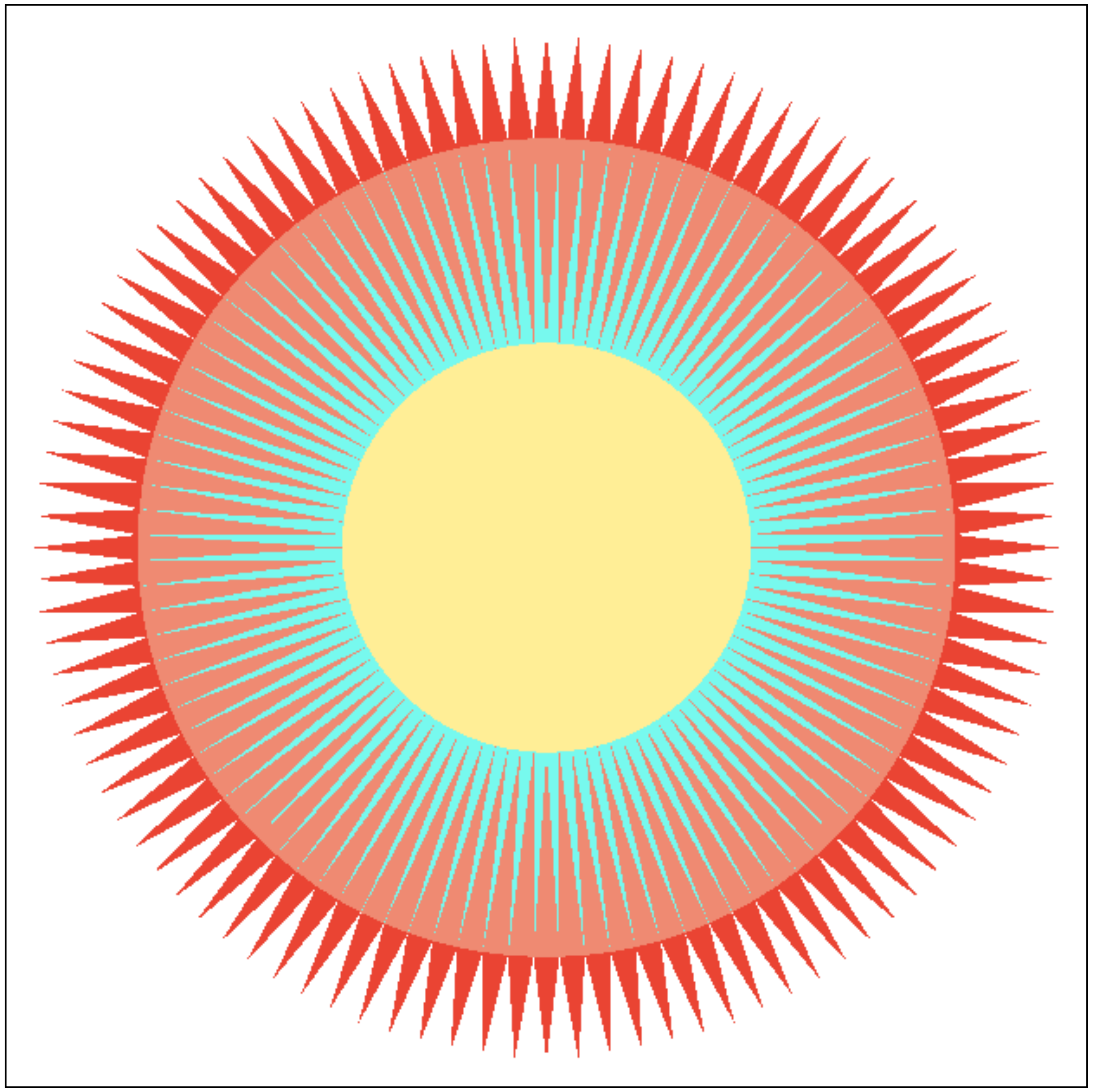
Read every sensor pixel

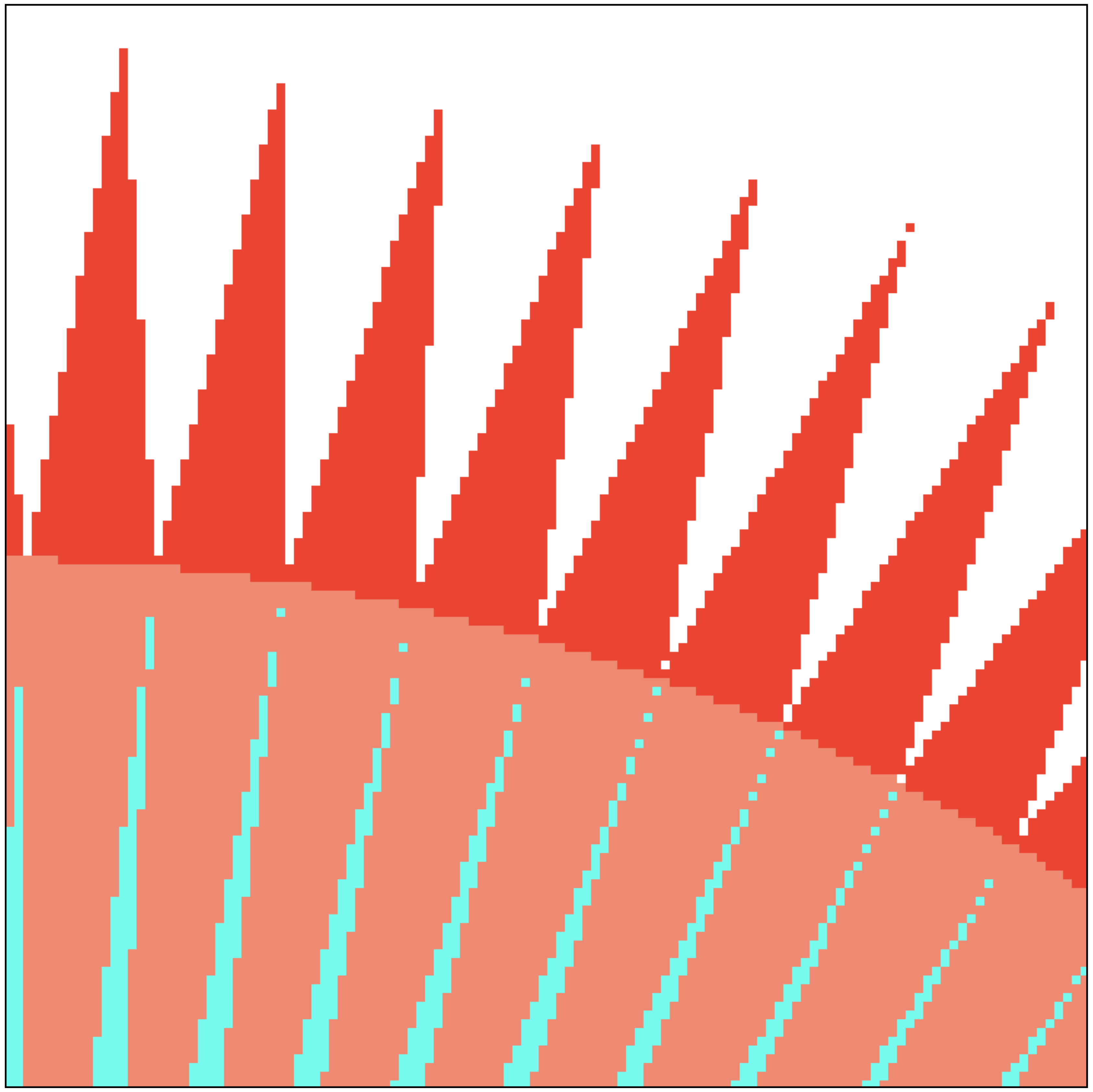


Skip odd rows and columns

lystit.com

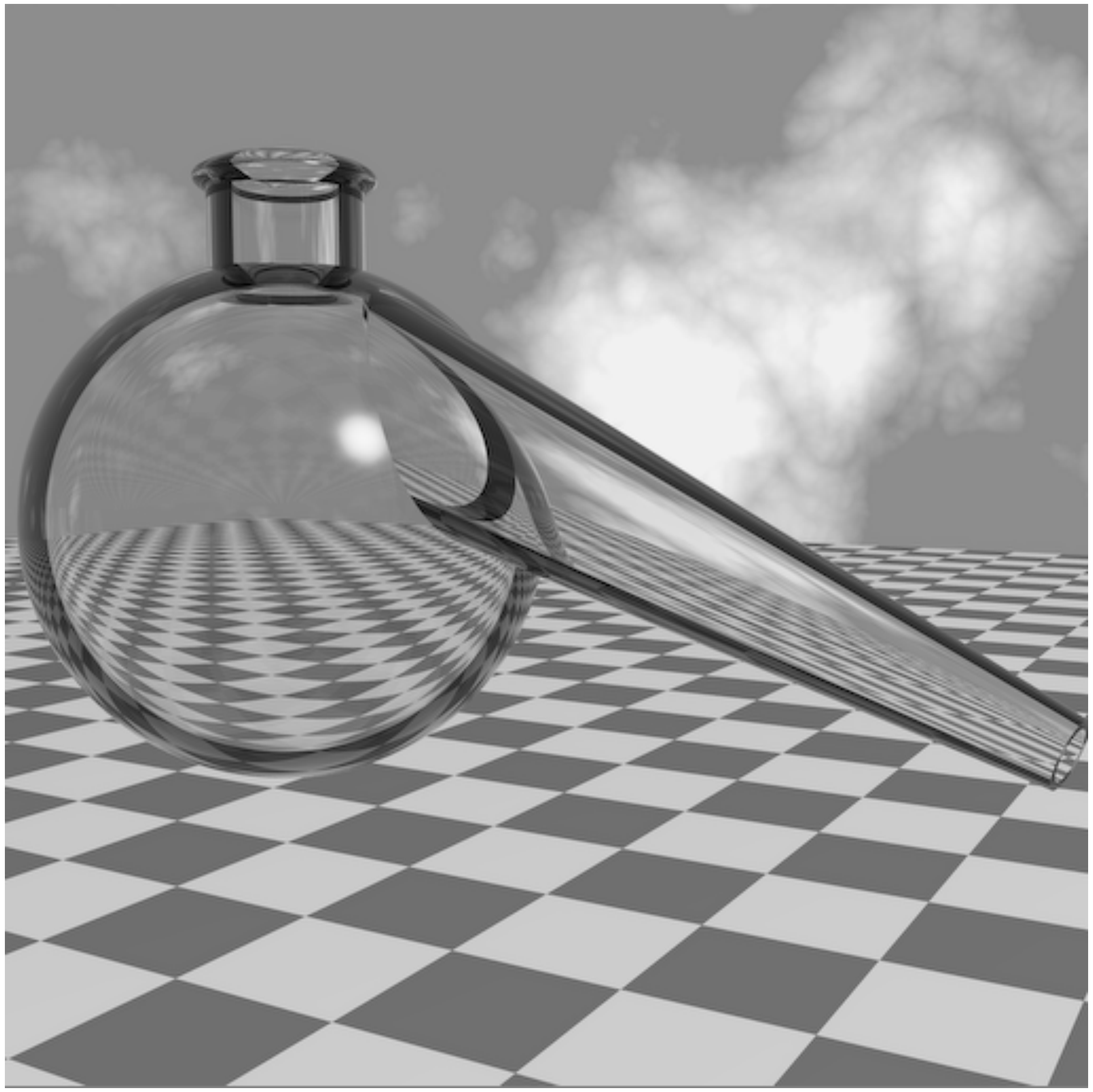
Jaggies (Staircase Pattern)

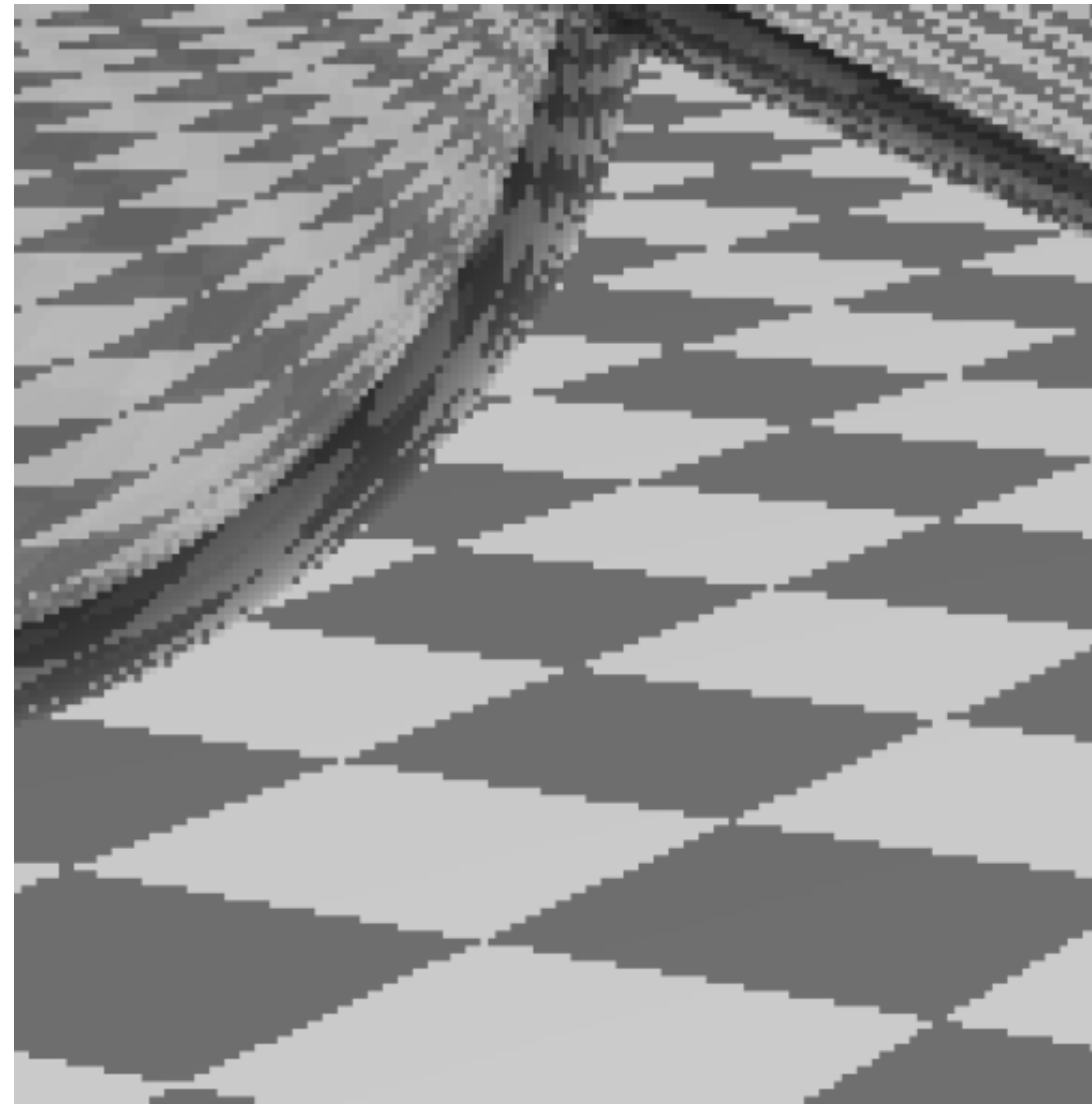




This is also an example of "aliasing" – a sampling error

Jaggies (Staircase Pattern)





Retort by Don Mitchell

Sampling Artifacts in Computer Graphics

Artifacts due to sampling - "Aliasing"

- Jaggies – sampling in space
- Wagon wheel effect – sampling in time
- Moire – undersampling images (and texture maps)
- [Many more] …

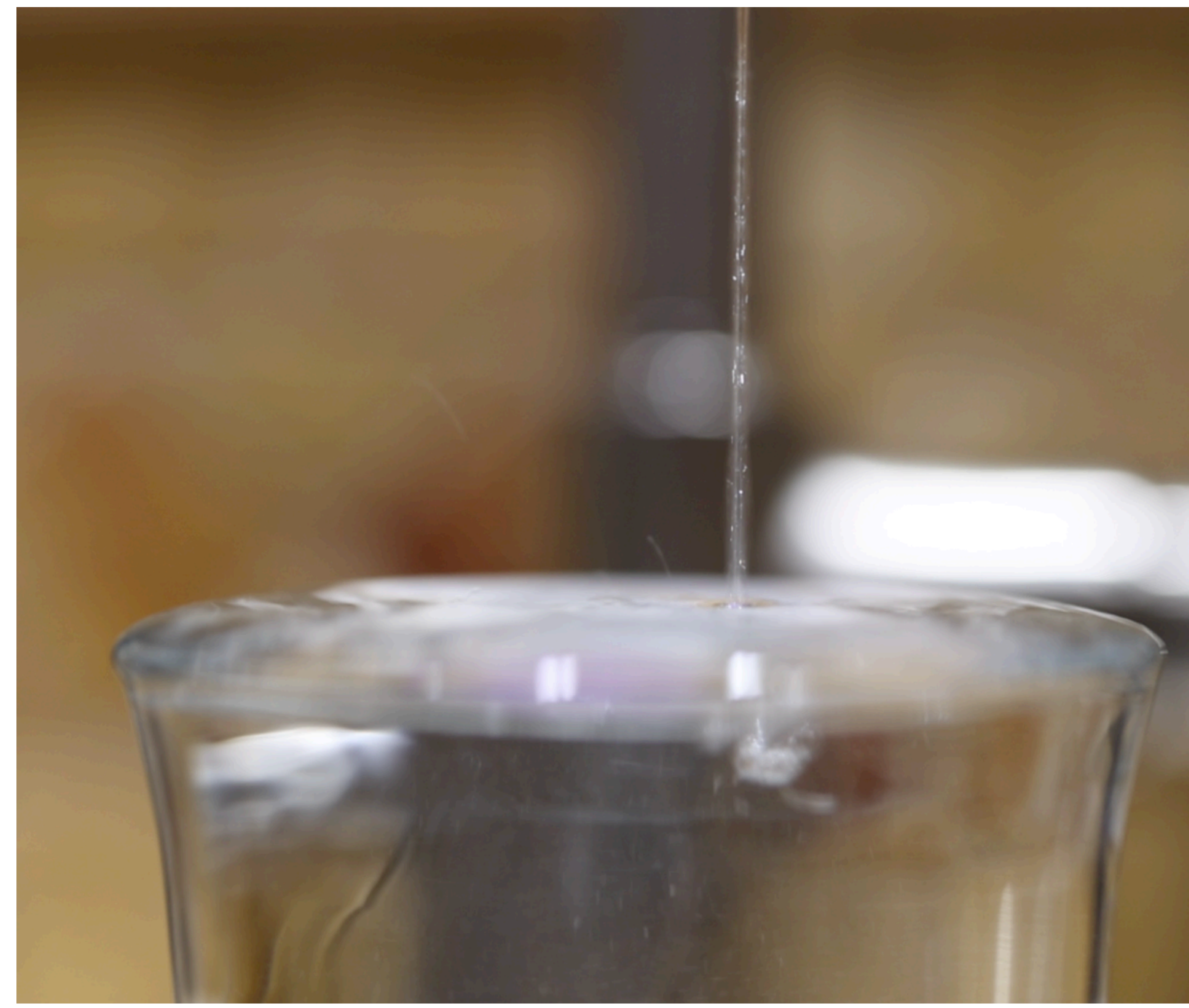
We notice this in fast-changing signals (high frequency), when we sample too slowly

Antialiasing Idea: Filter Out High Frequencies Before Sampling

Video: Point Sampling vs Antialiased Sampling in Time

Thin stream of water from kitchen tap

Point in Time
1/4000 sec exposure

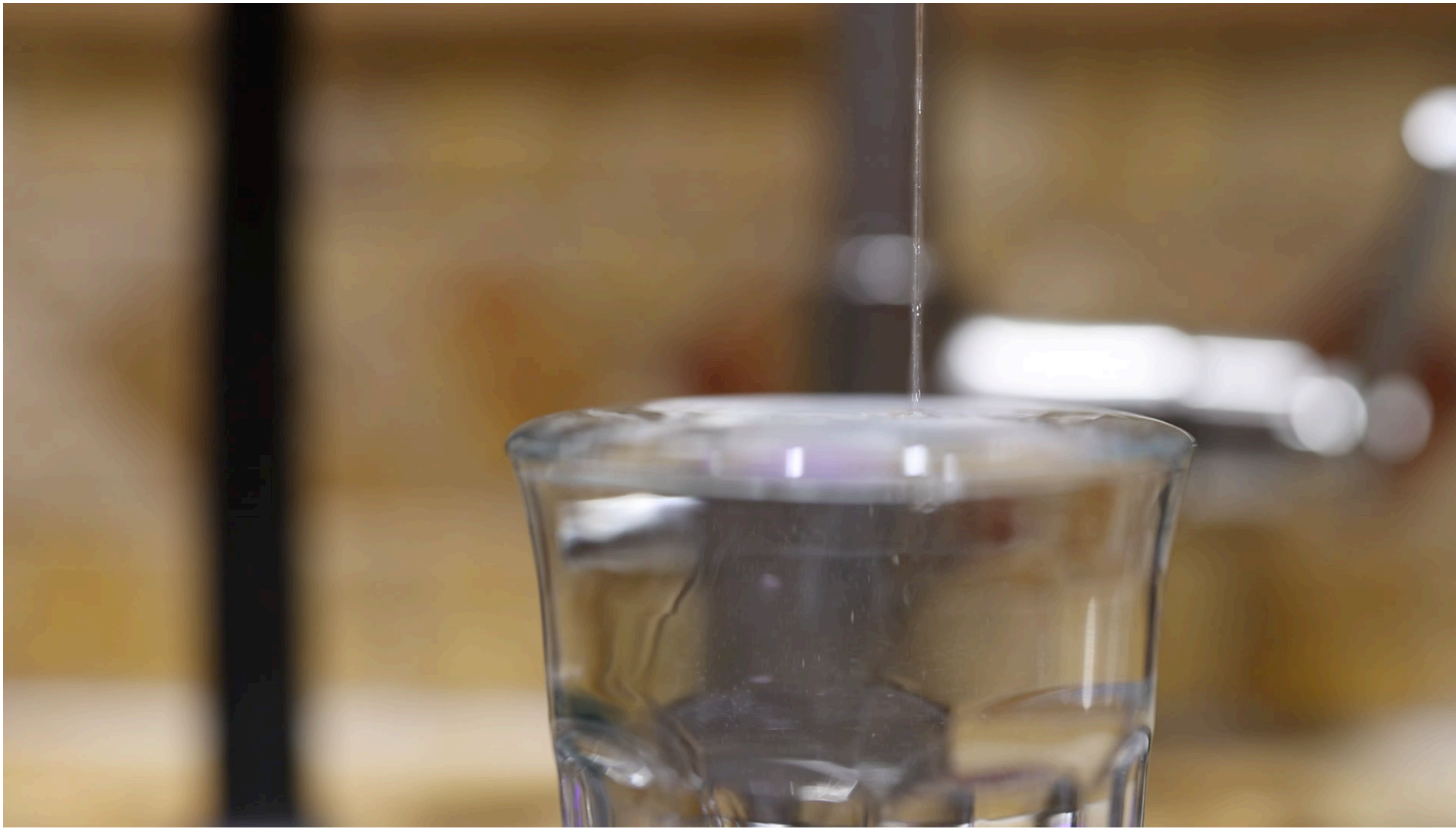


Motion Blurred
1/60 sec exposure

Video: Point Sampling in Time

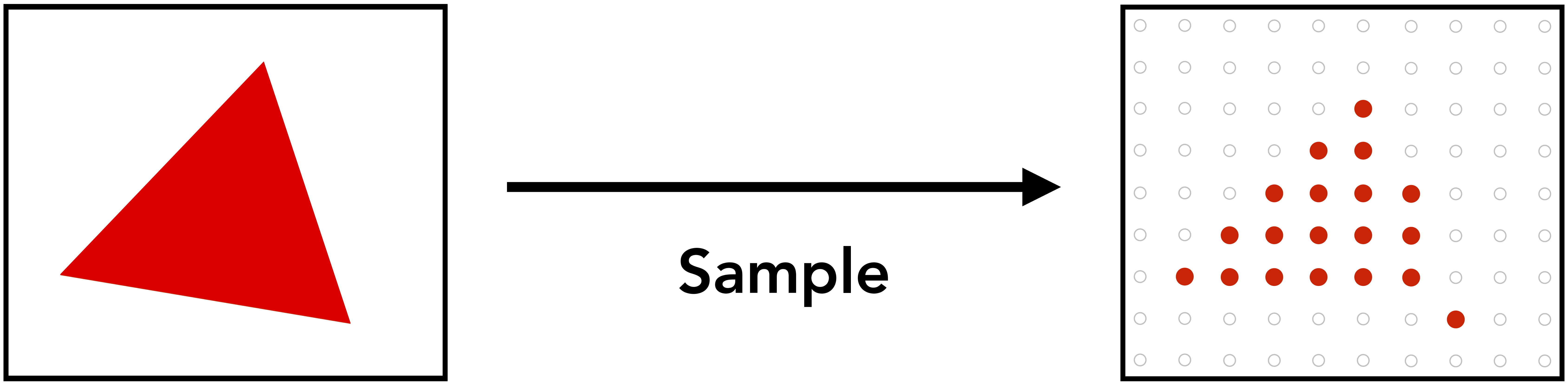
60 fps video. 1/4000 second exposure is sharp in time, causes time aliasing.

Video: Motion-Blurred (Antialiased) Sampling in Time



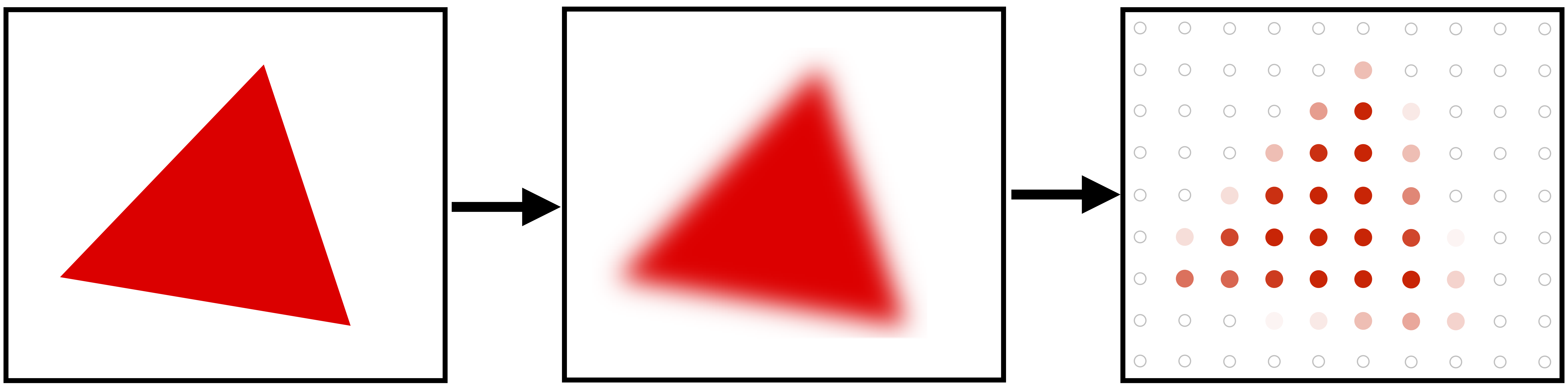
60 fps video. 1/60 second exposure is motion-blurred in time, no aliasing.

Rasterization: Point Sampling in Space



Note jaggies in rasterized triangle
where pixel values are pure red or white

Rasterization: Antialiased Sampling



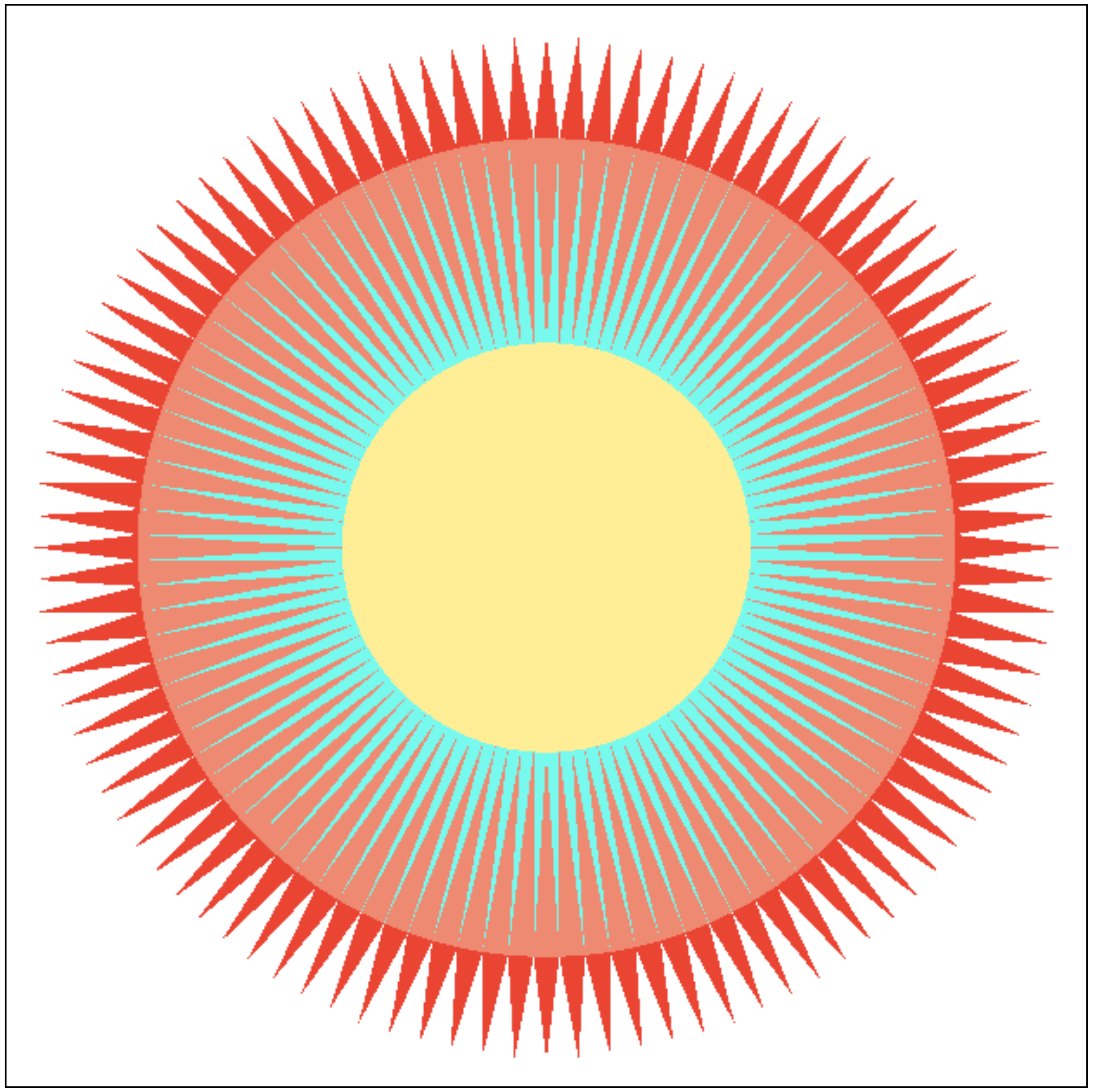
Pre-Filter

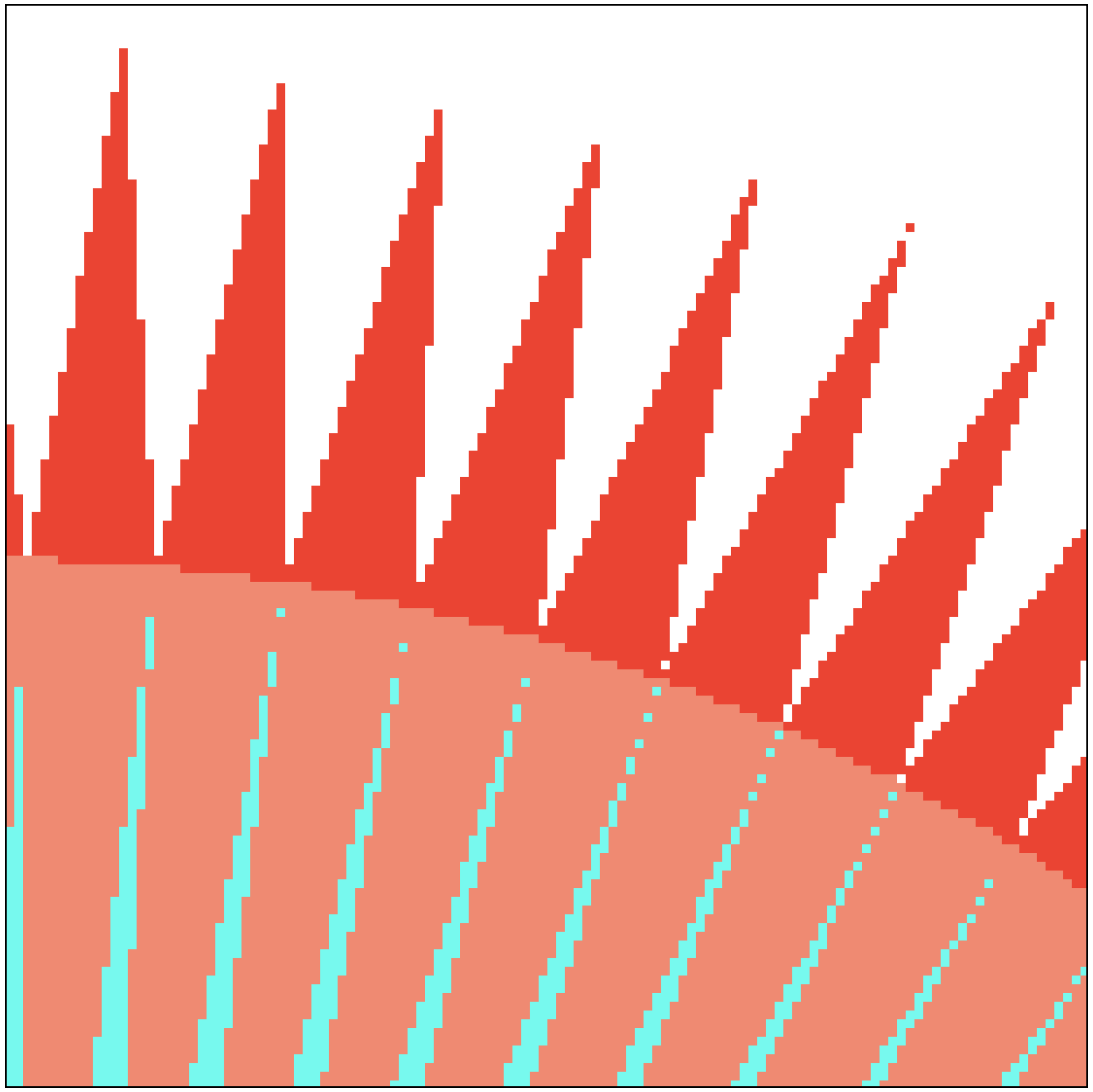
(remove frequencies above Nyquist)

Sample

Note antialiased edges in rasterized triangle where pixel values take intermediate values

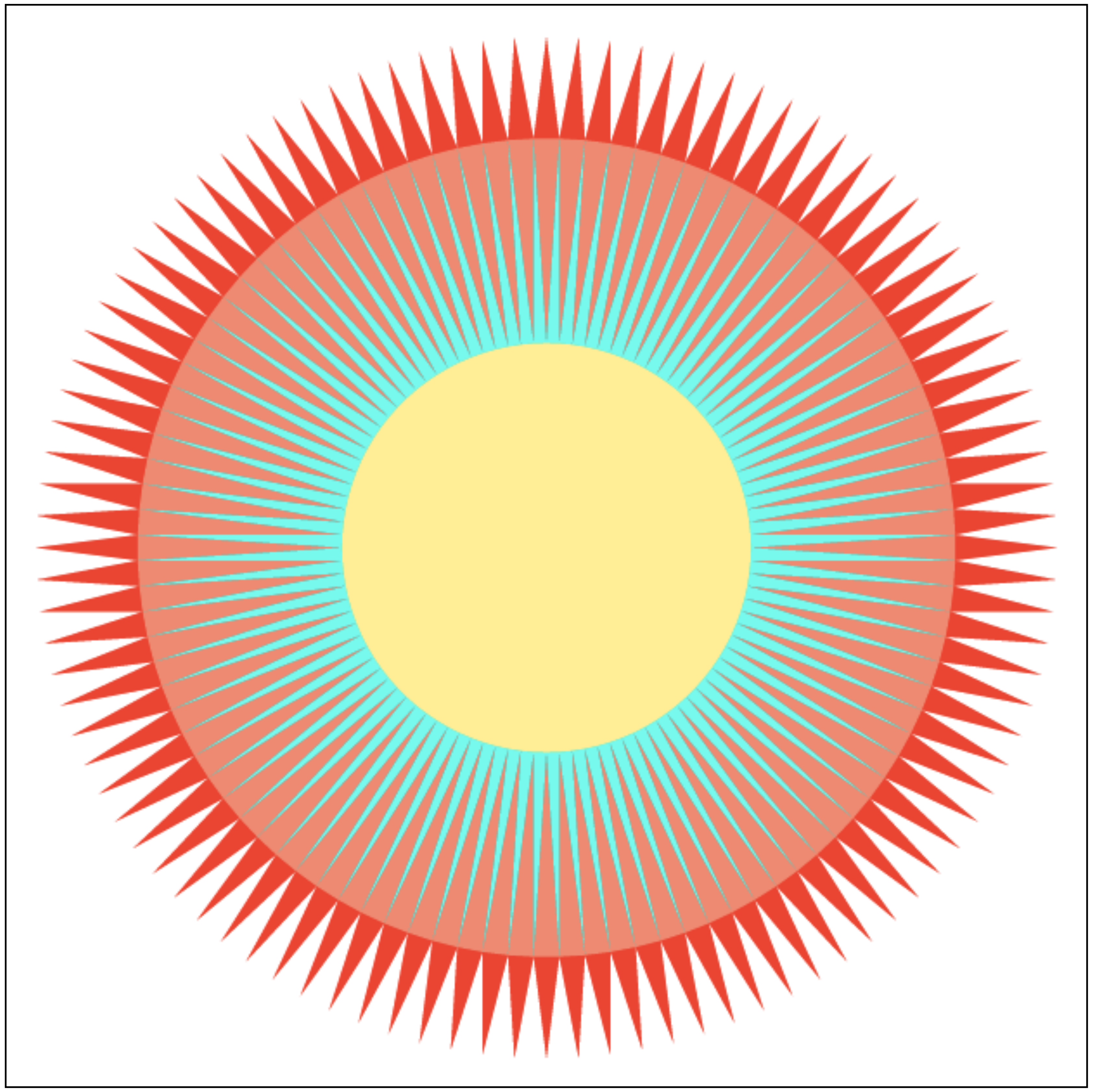
Point Sampling

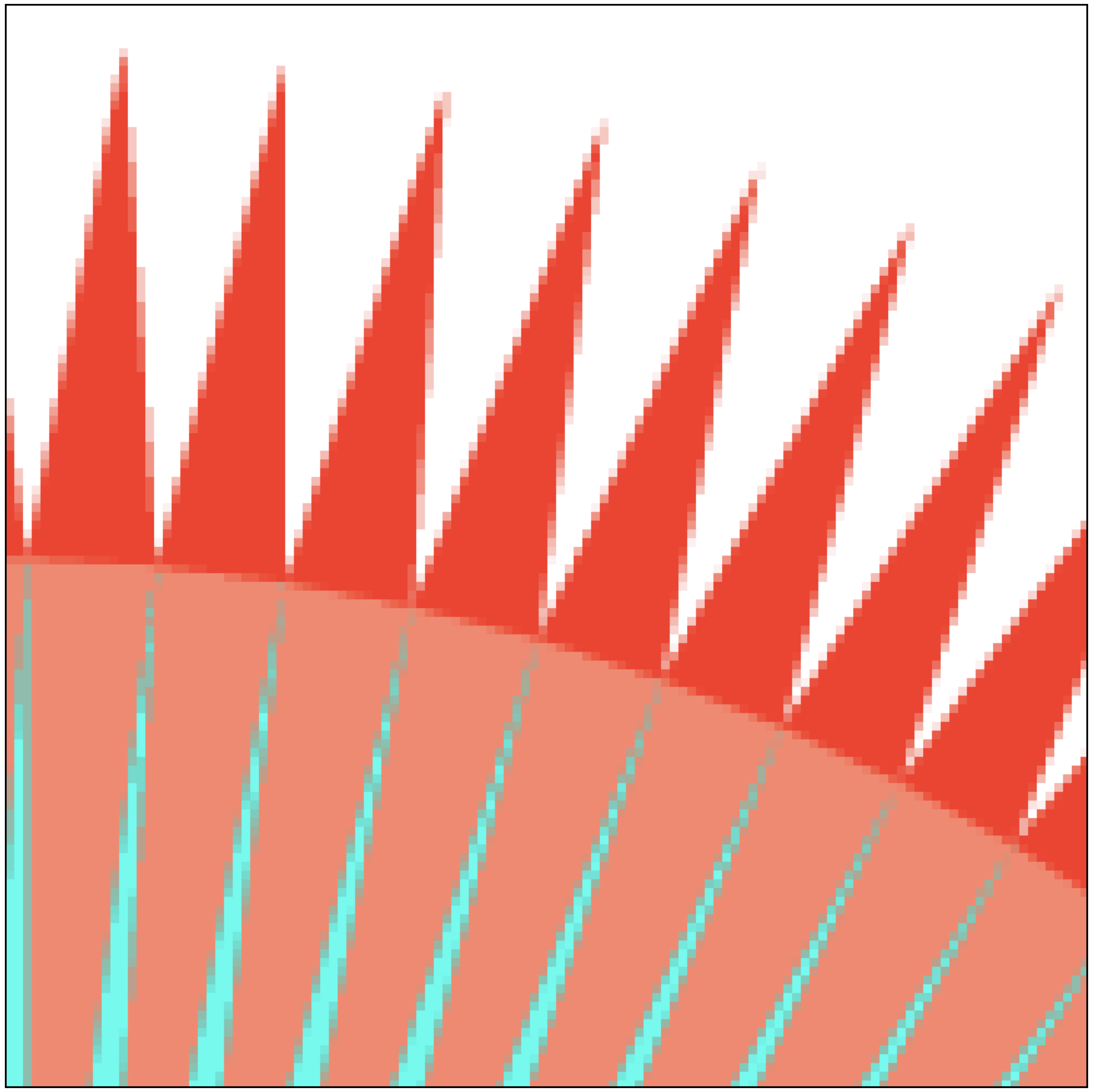




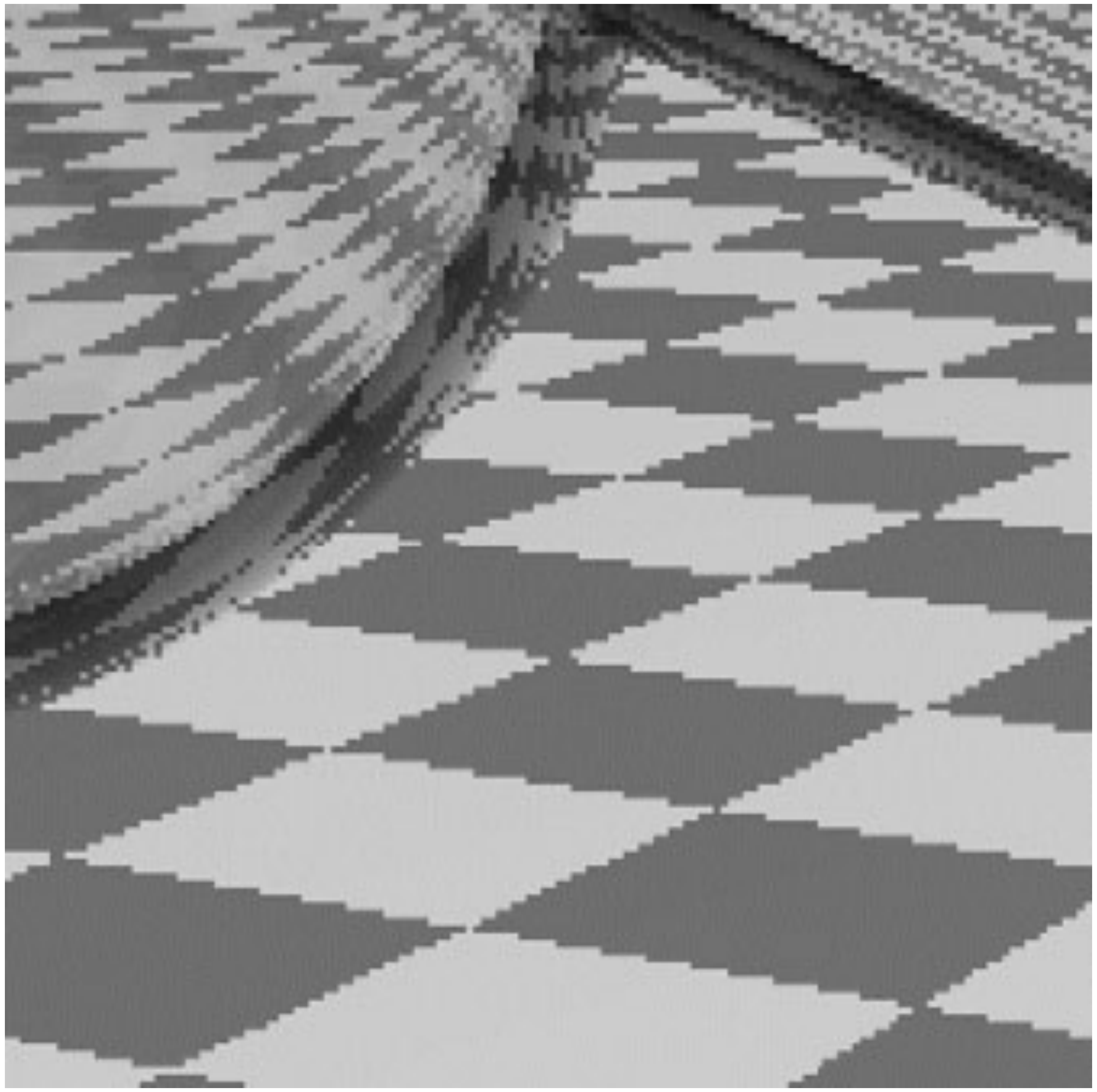
One sample per pixel

Antialiasing

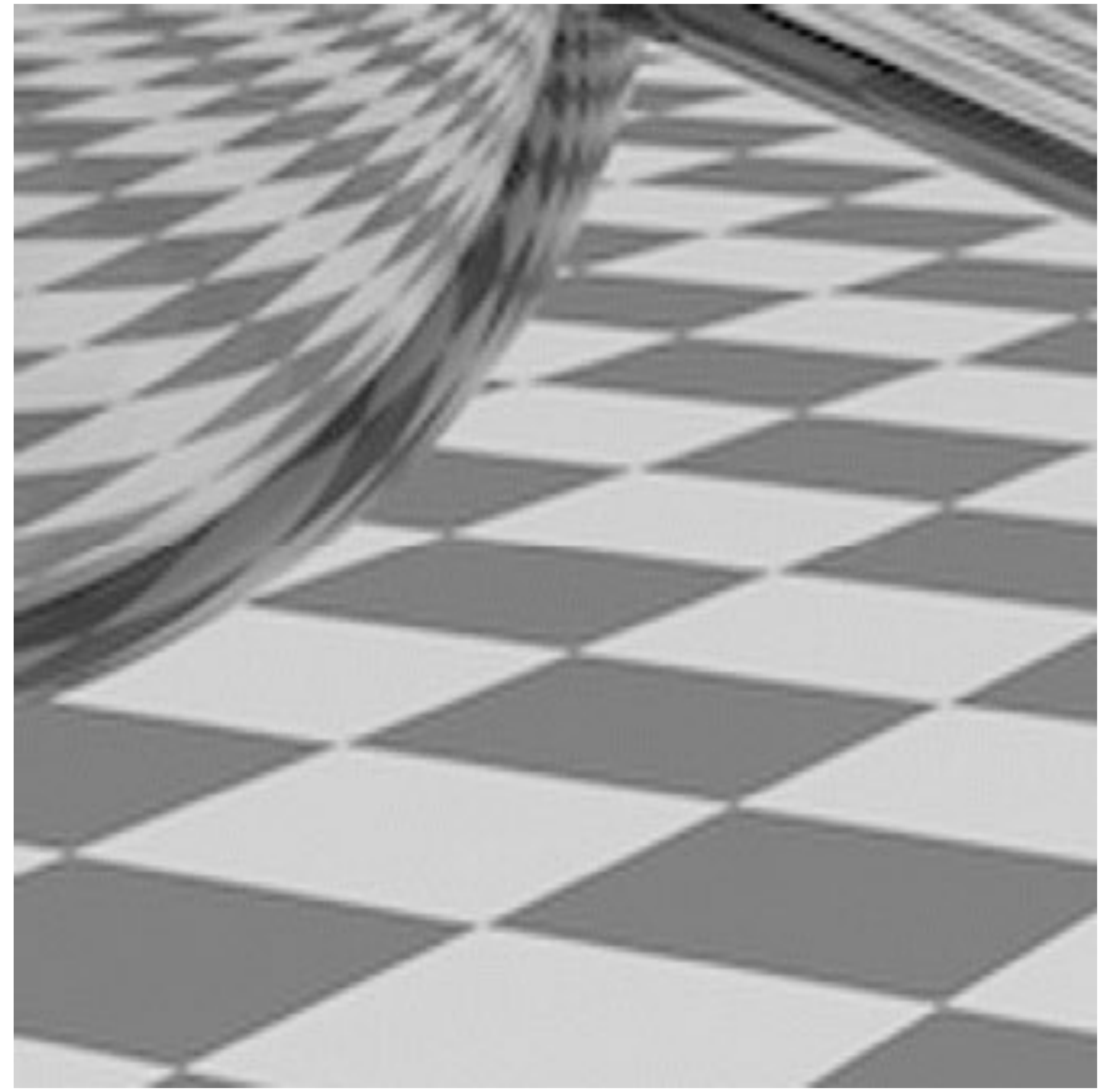




Point Sampling vs Antialiasing

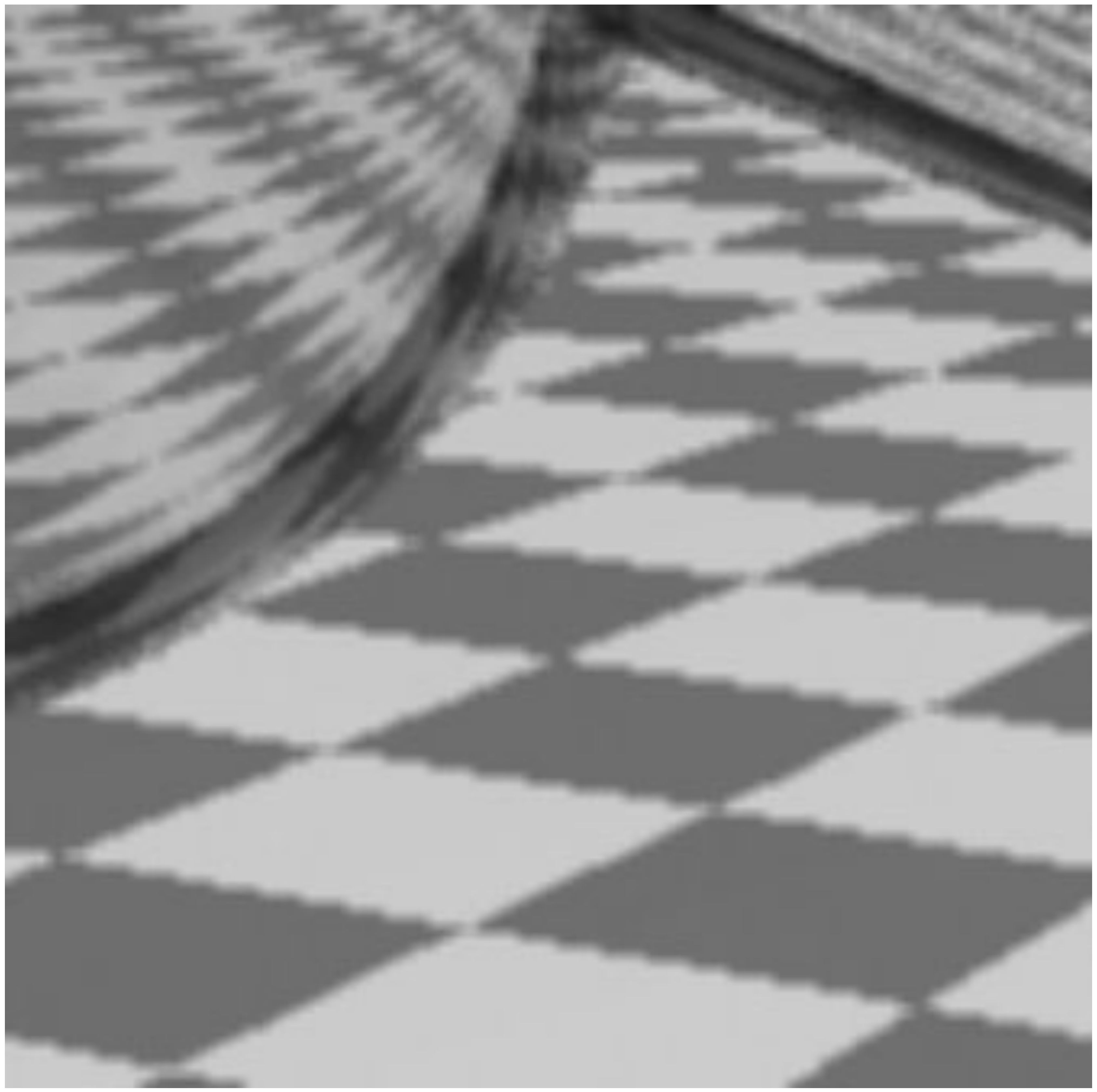


Jaggies

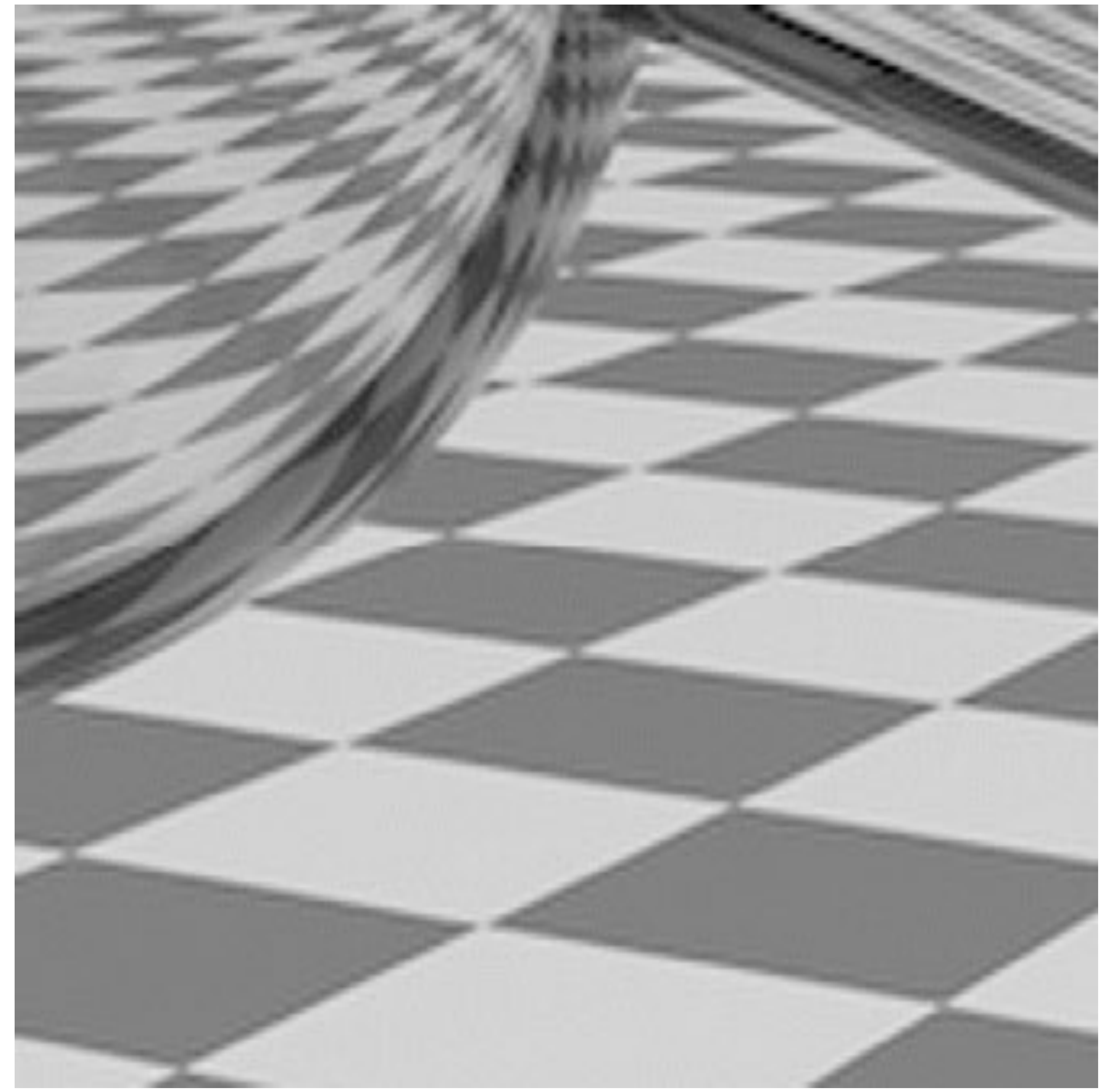


Pre-Filtered

Antialiasing vs Blurred Aliasing



Blurred Jaggies
(Sample then filter)



Pre-Filtered
(Filter then sample)

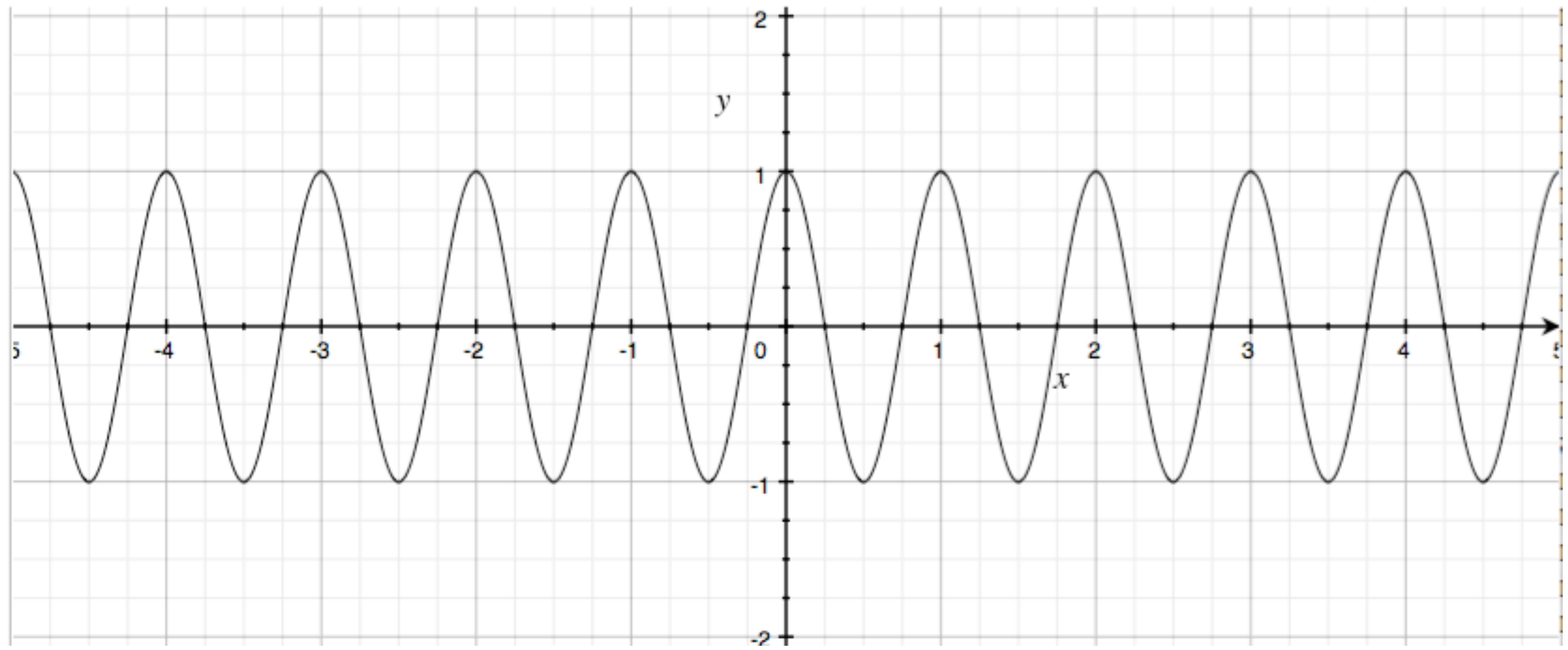
This Lecture

Let's dig into the fundamental reasons why this works

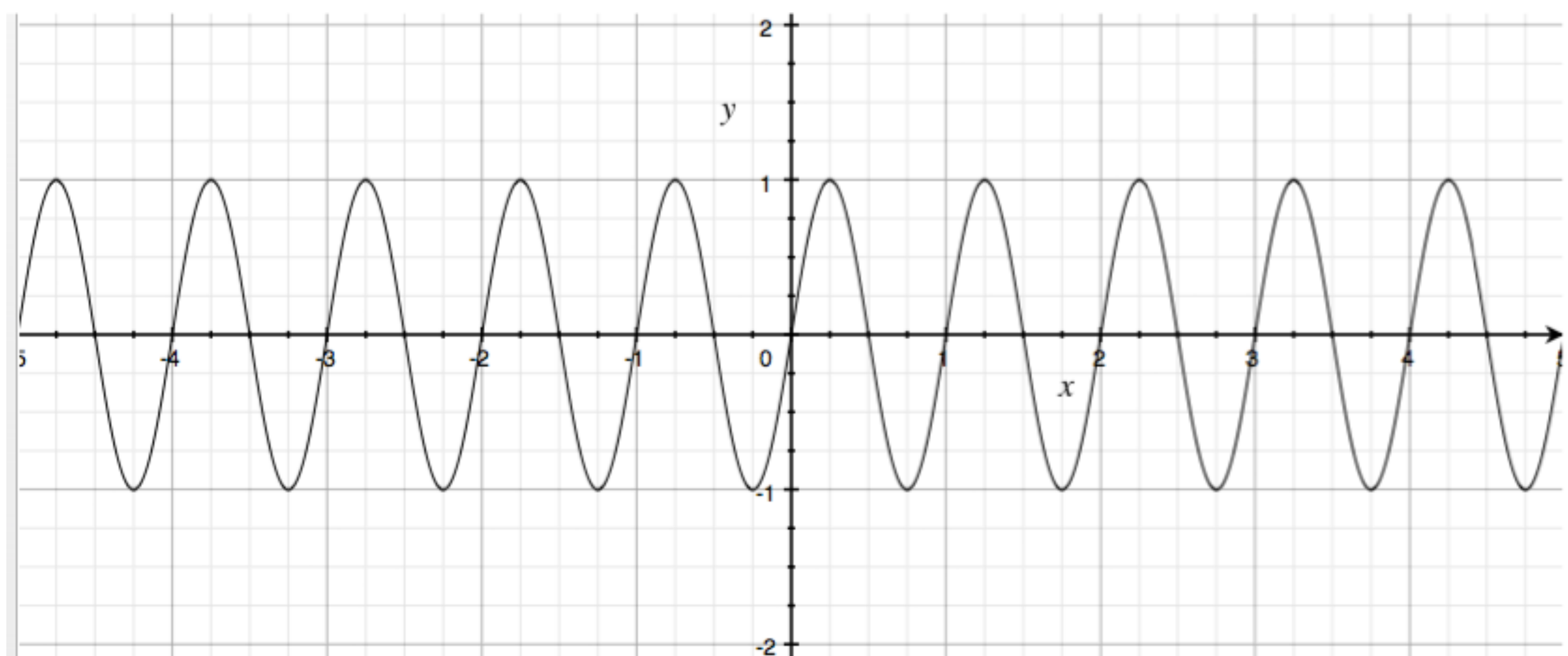
And look at how to implement antialiased rasterization

Frequency Space

Sines and Cosines



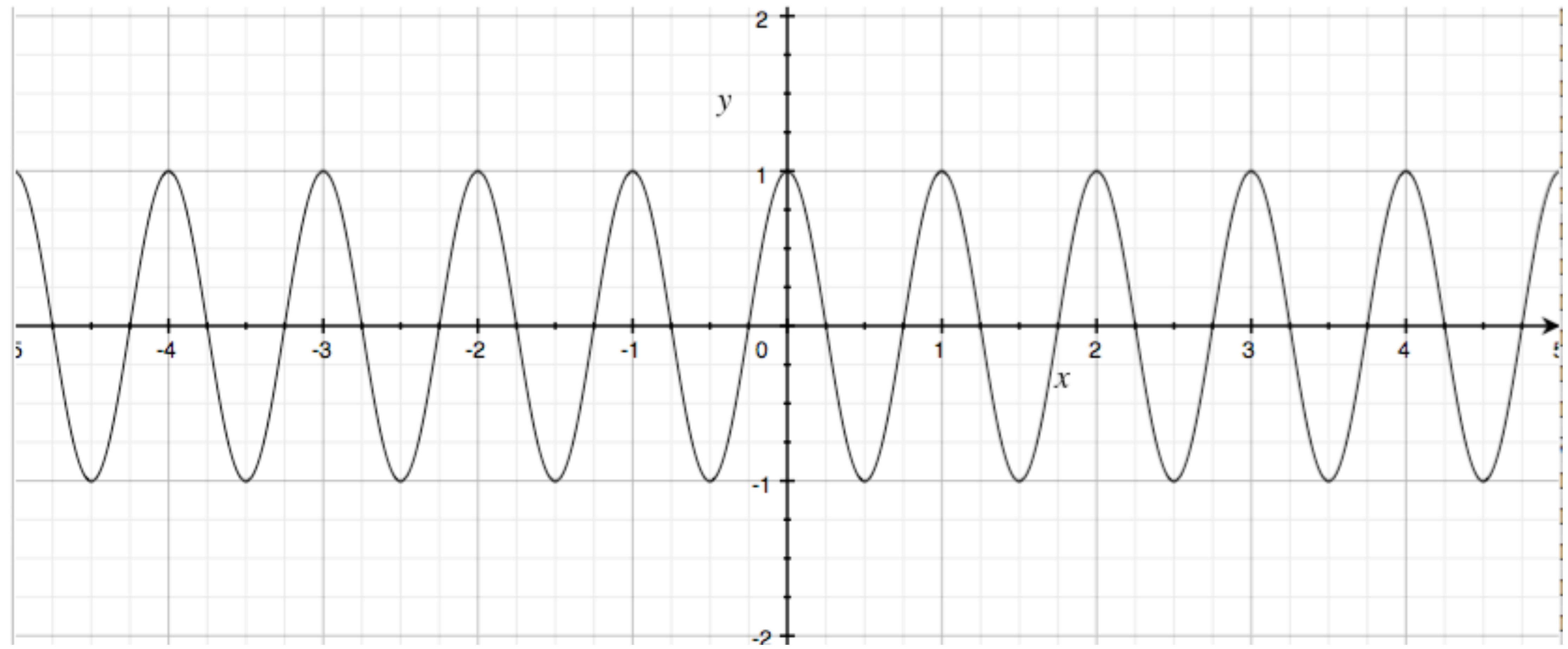
$$\cos 2\pi x$$



$$\sin 2\pi x$$

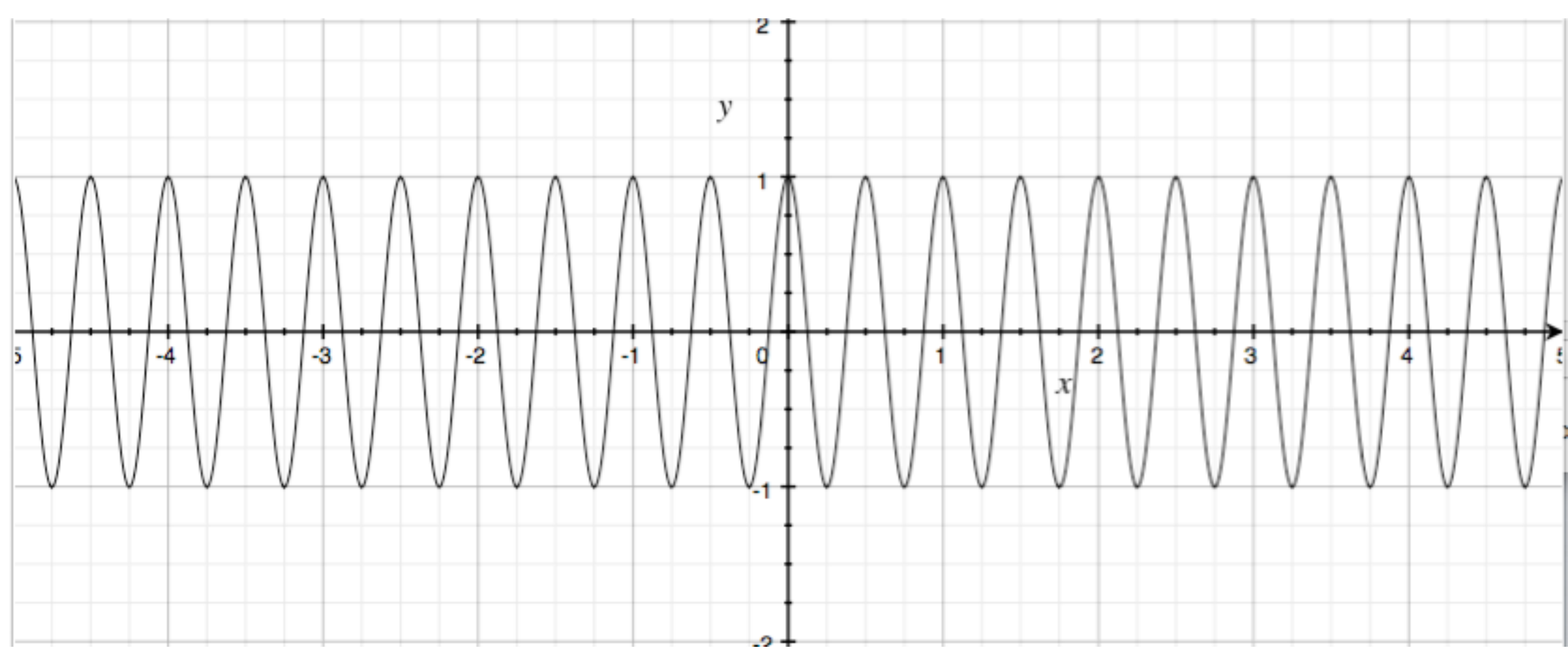
Frequencies $\cos 2\pi f x$

$$f = \frac{1}{T}$$



$f = 1$

$\cos 2\pi x$



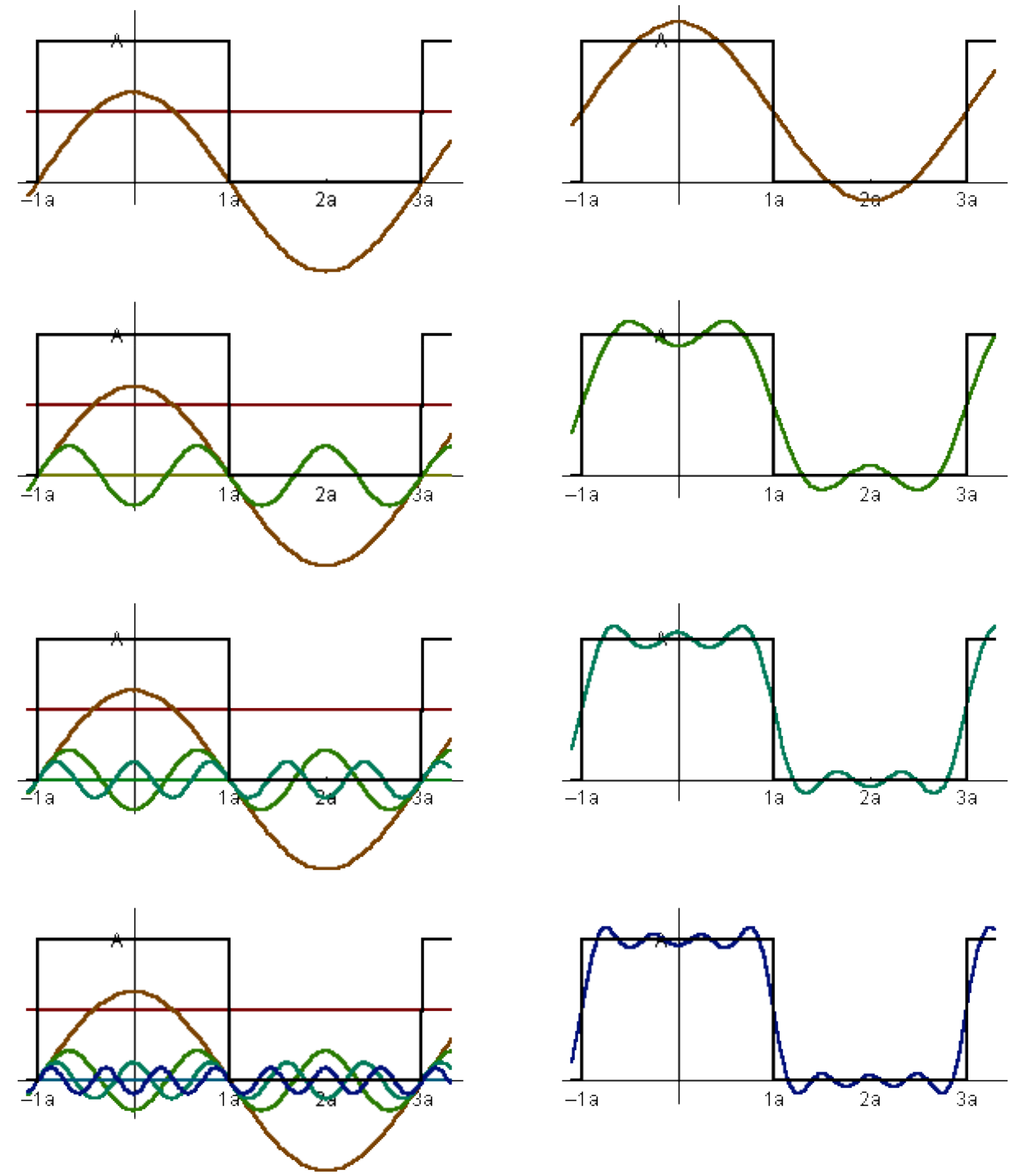
$f = 2$

$\cos 4\pi x$

Fourier Transform

Represent a function as a weighted sum of sines and cosines

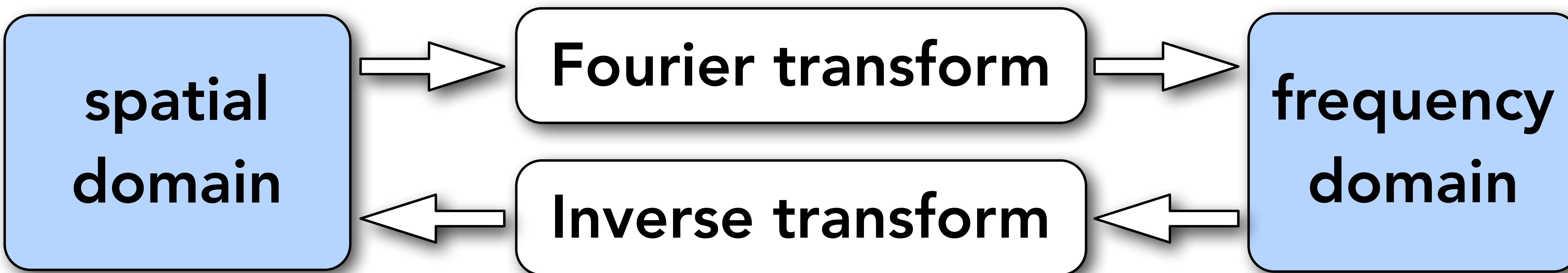
Joseph Fourier 1768 - 1830



$$f(x) = \frac{A}{2} + \frac{2A\cos(t\omega)}{\pi} - \frac{2A\cos(3t\omega)}{3\pi} + \frac{2A\cos(5t\omega)}{5\pi} - \frac{2A\cos(7t\omega)}{7\pi} + \cdots$$

Fourier Transform Decomposes A Signal Into Frequencies

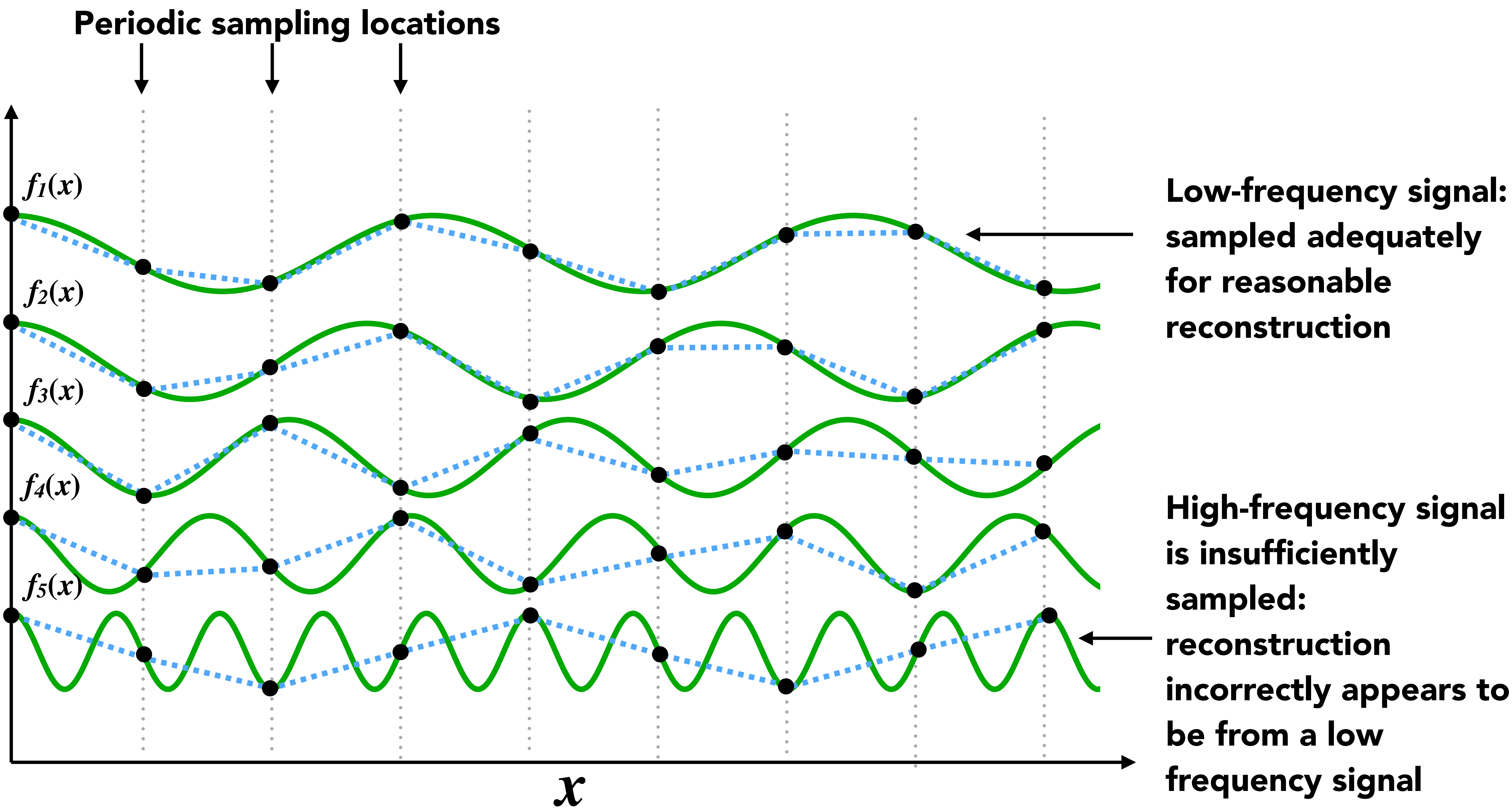
$$f(x) \qquad F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx \qquad F(\omega)$$



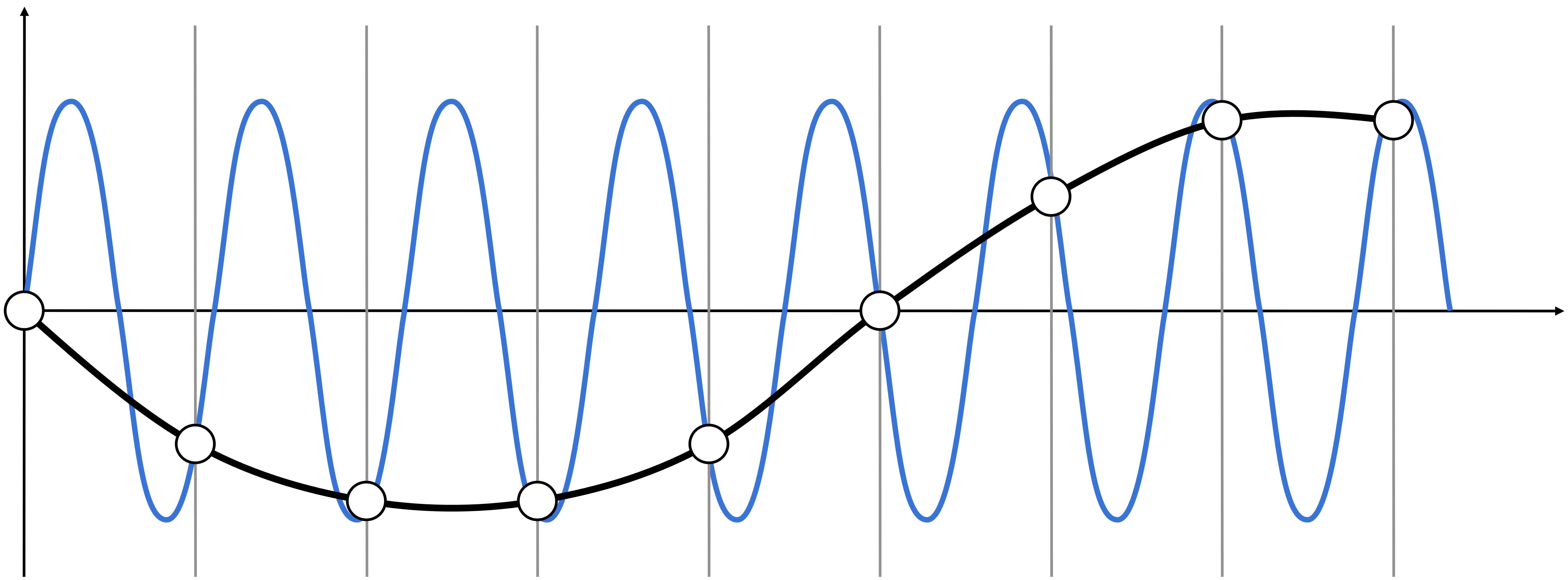
$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega x} d\omega$$

Recall $e^{ix} = \cos x + i \sin x$

Higher Frequencies Need Faster Sampling



Undersampling Creates Frequency Aliases



High-frequency signal is insufficiently sampled: samples erroneously appear to be from a low-frequency signal

Two frequencies that are indistinguishable at a given sampling rate are called "aliases"

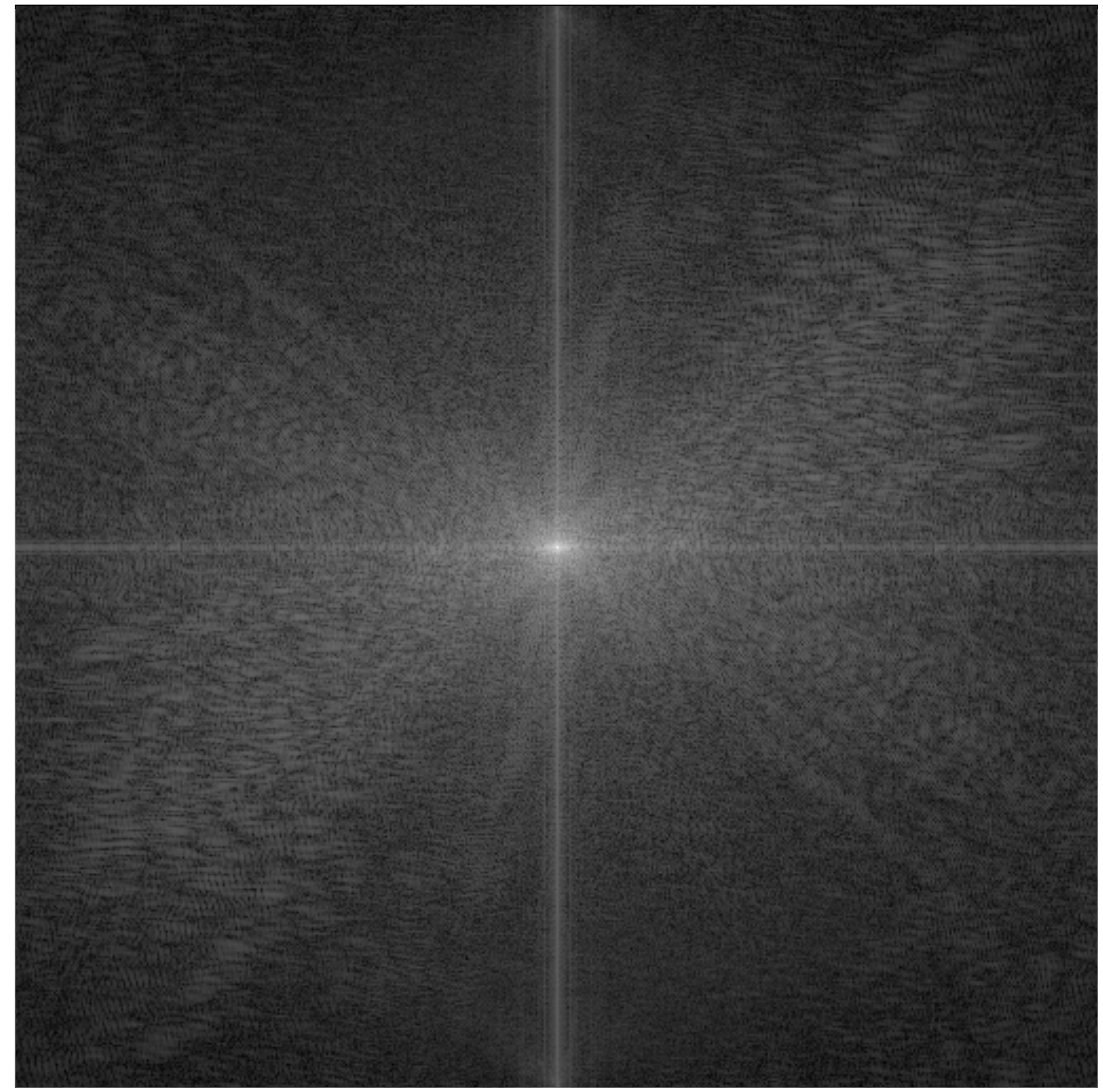
“Alias” = False Identity

“Batman” = Bruce Wayne’s alias to hide his true identity

Visualization of Frequency Space

2D Frequency Space

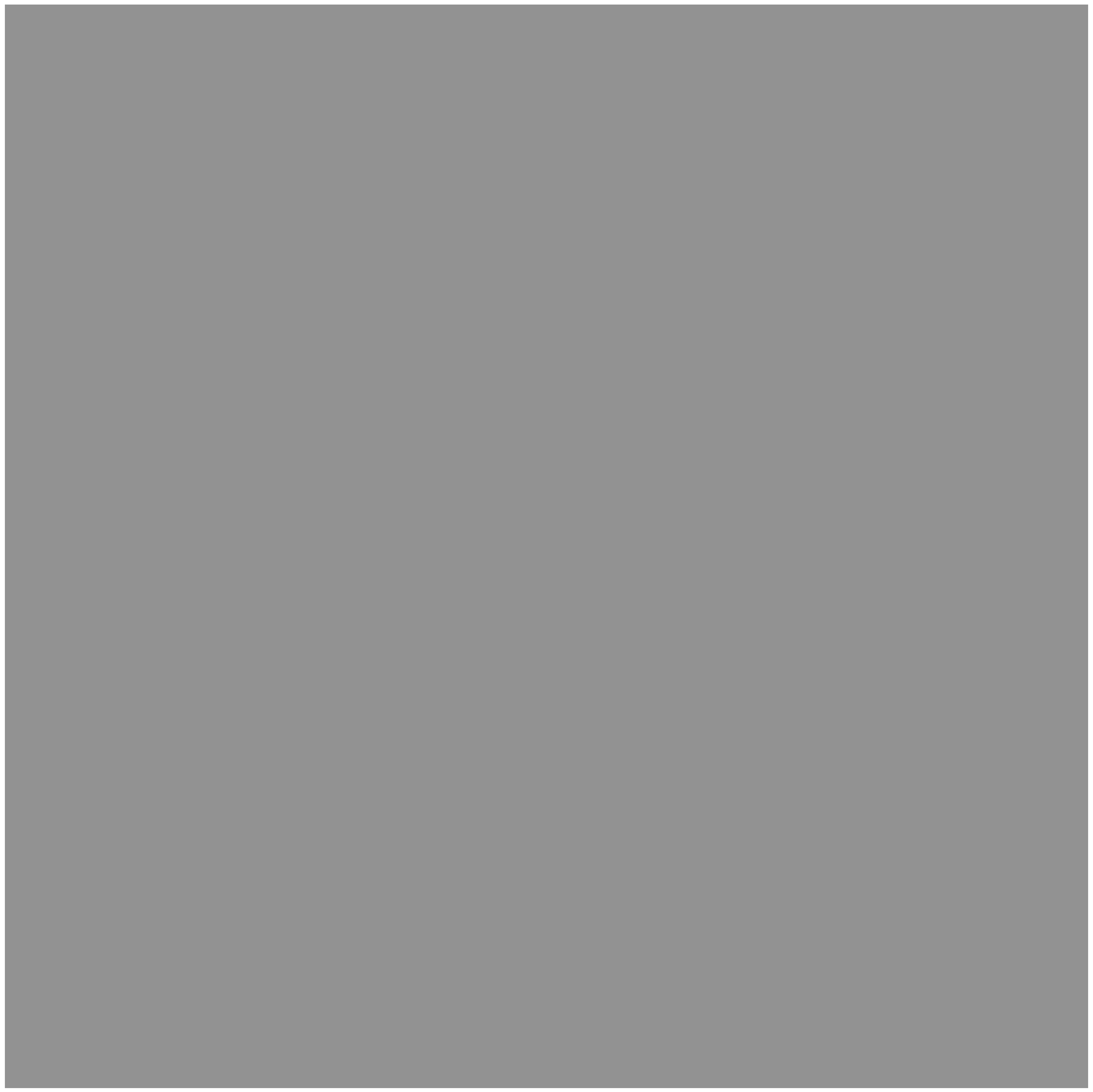
Spatial Domain



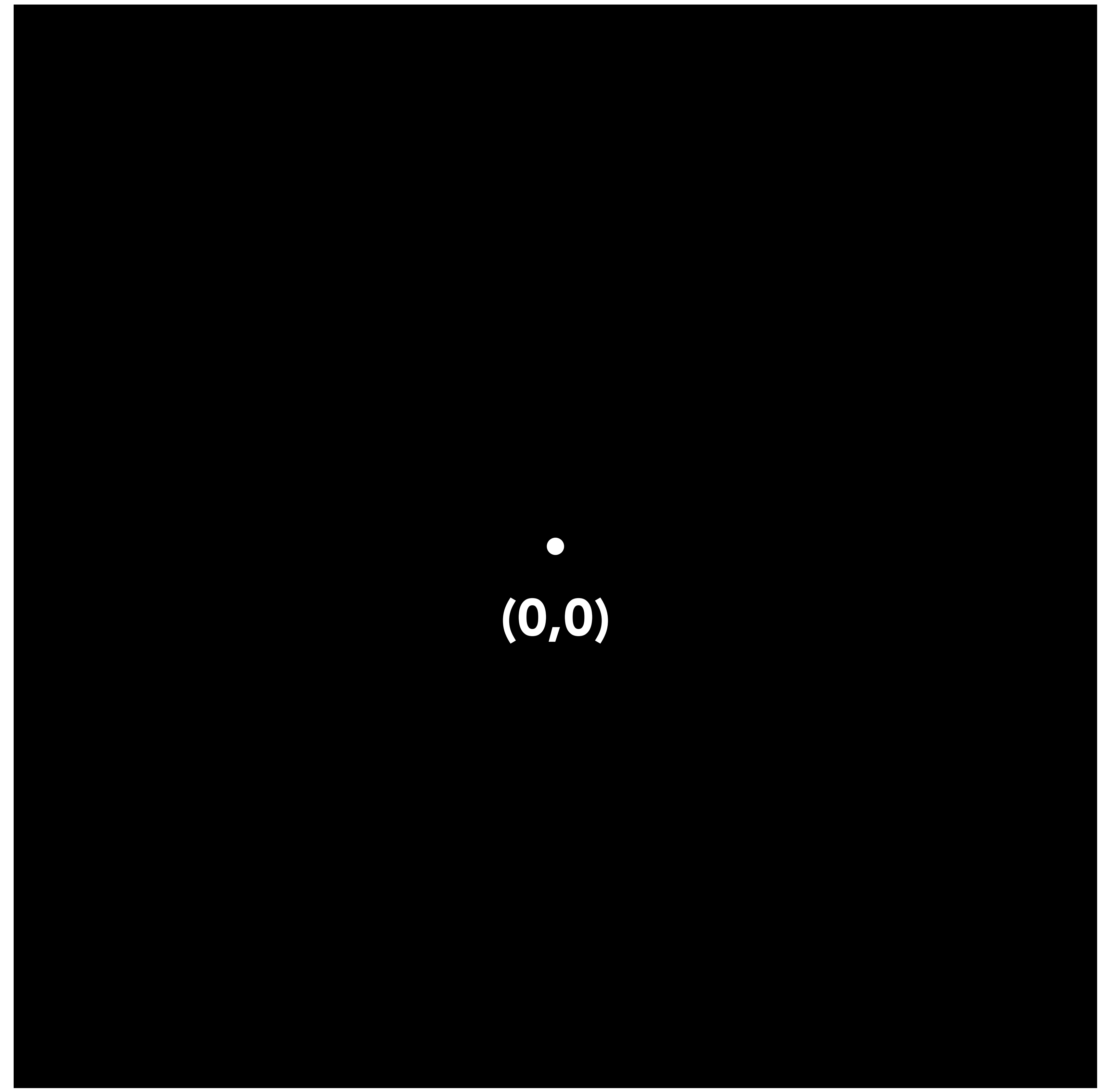
Frequency Domain

Note: Frequency domain also known as frequency space, Fourier domain, spectrum, ...

Constant

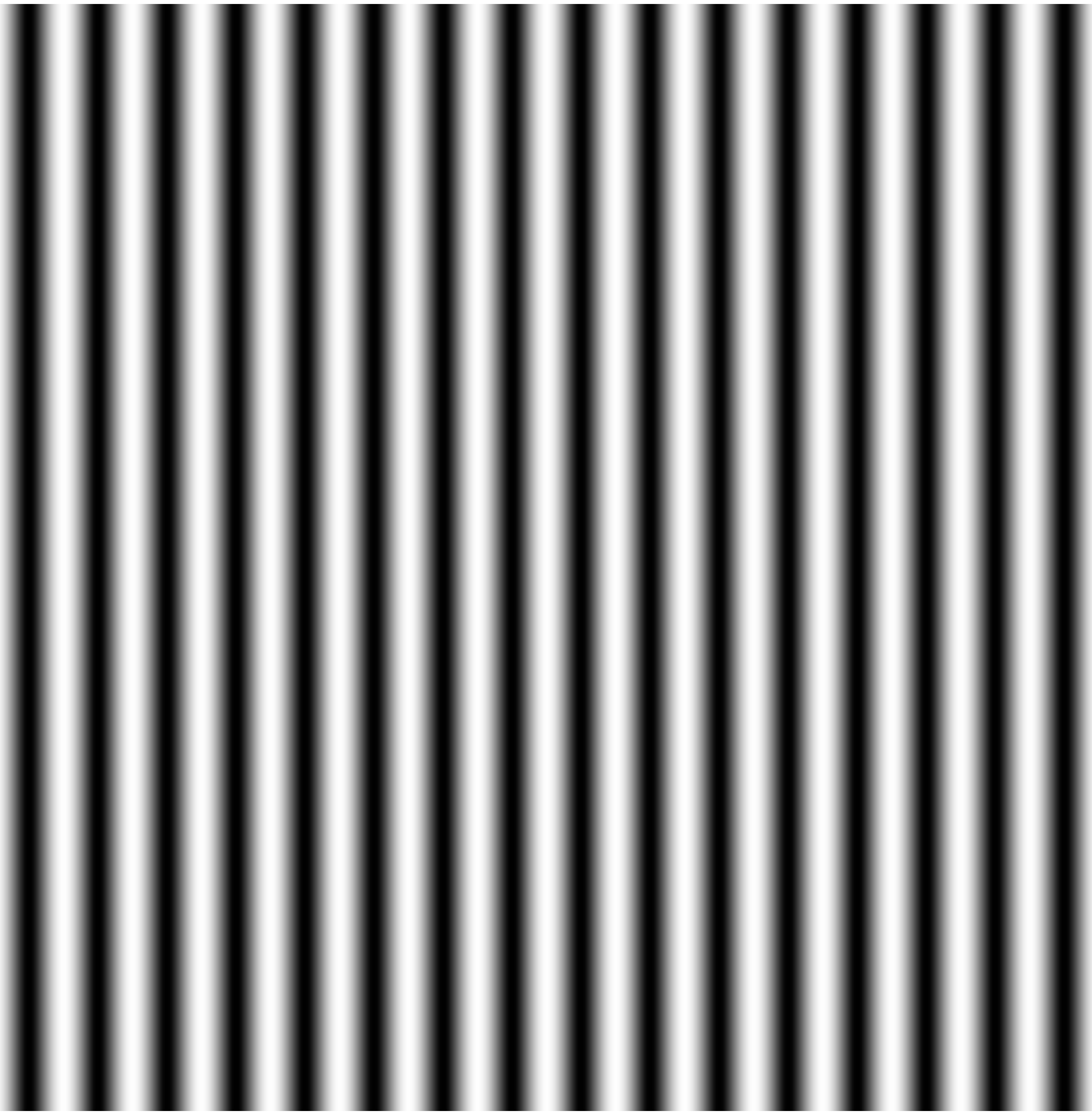


Spatial Domain

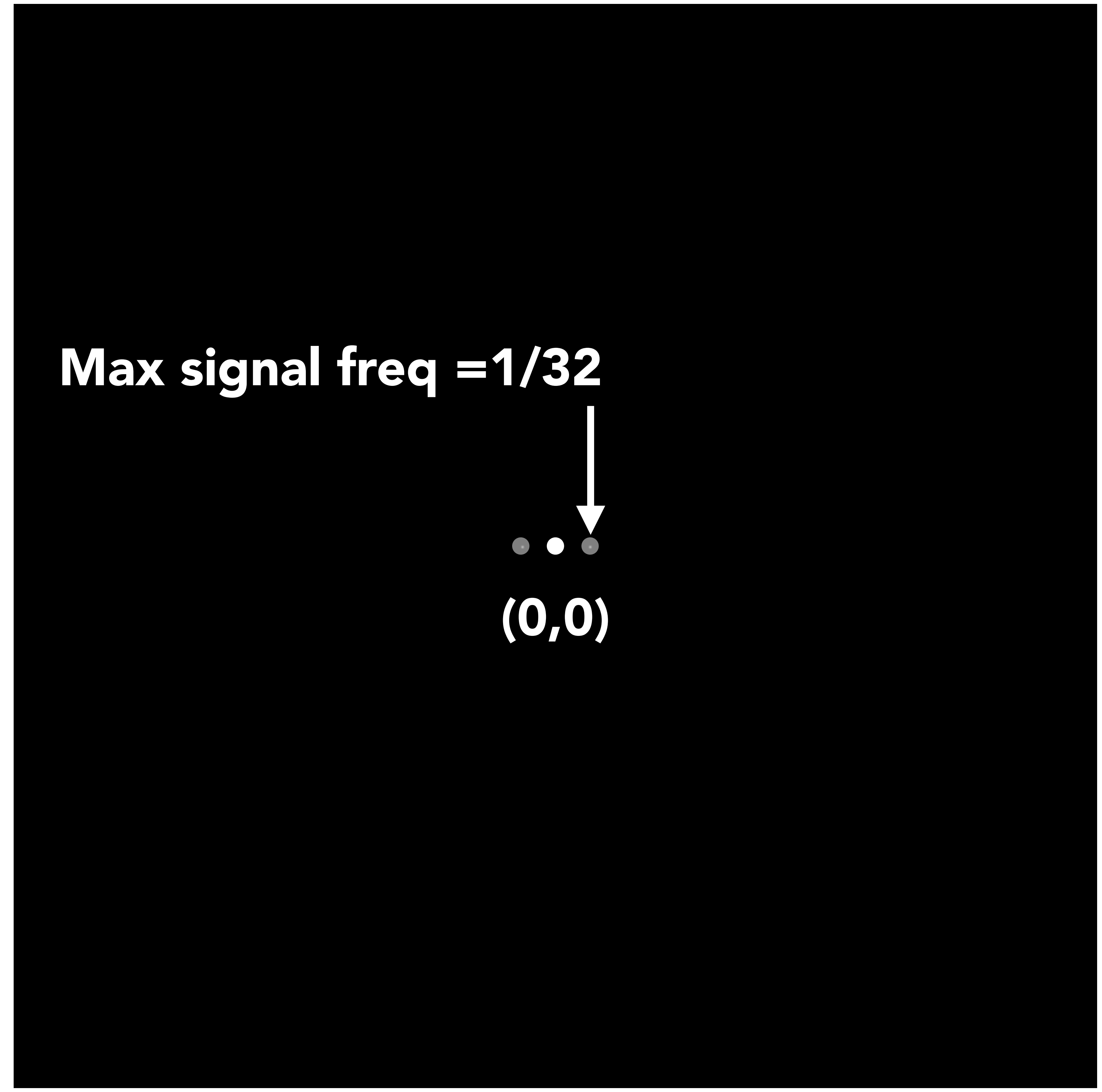


Frequency Domain

$\sin(2\pi/32)x$ — frequency 1/32; 32 pixels per cycle

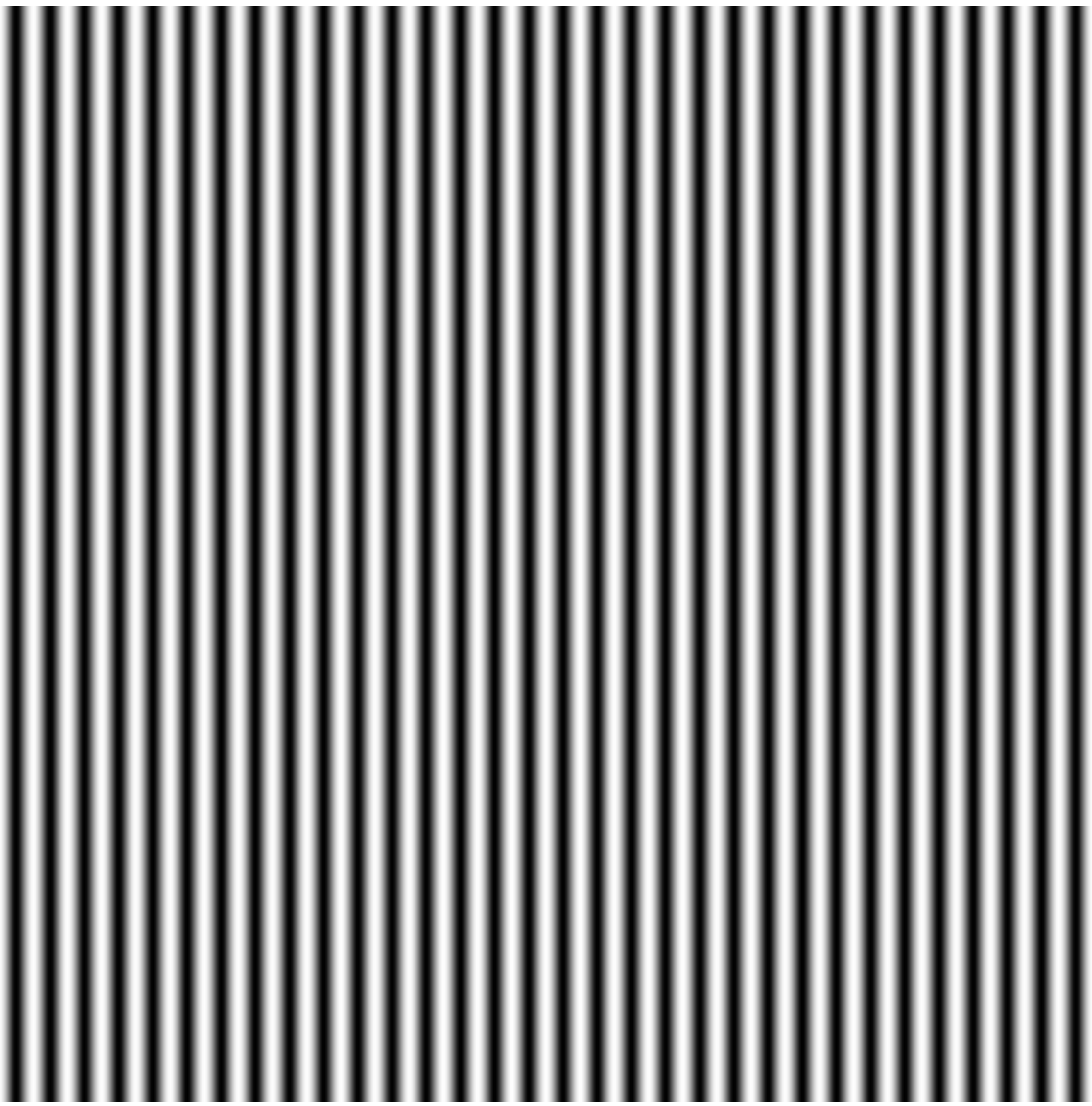


Spatial Domain

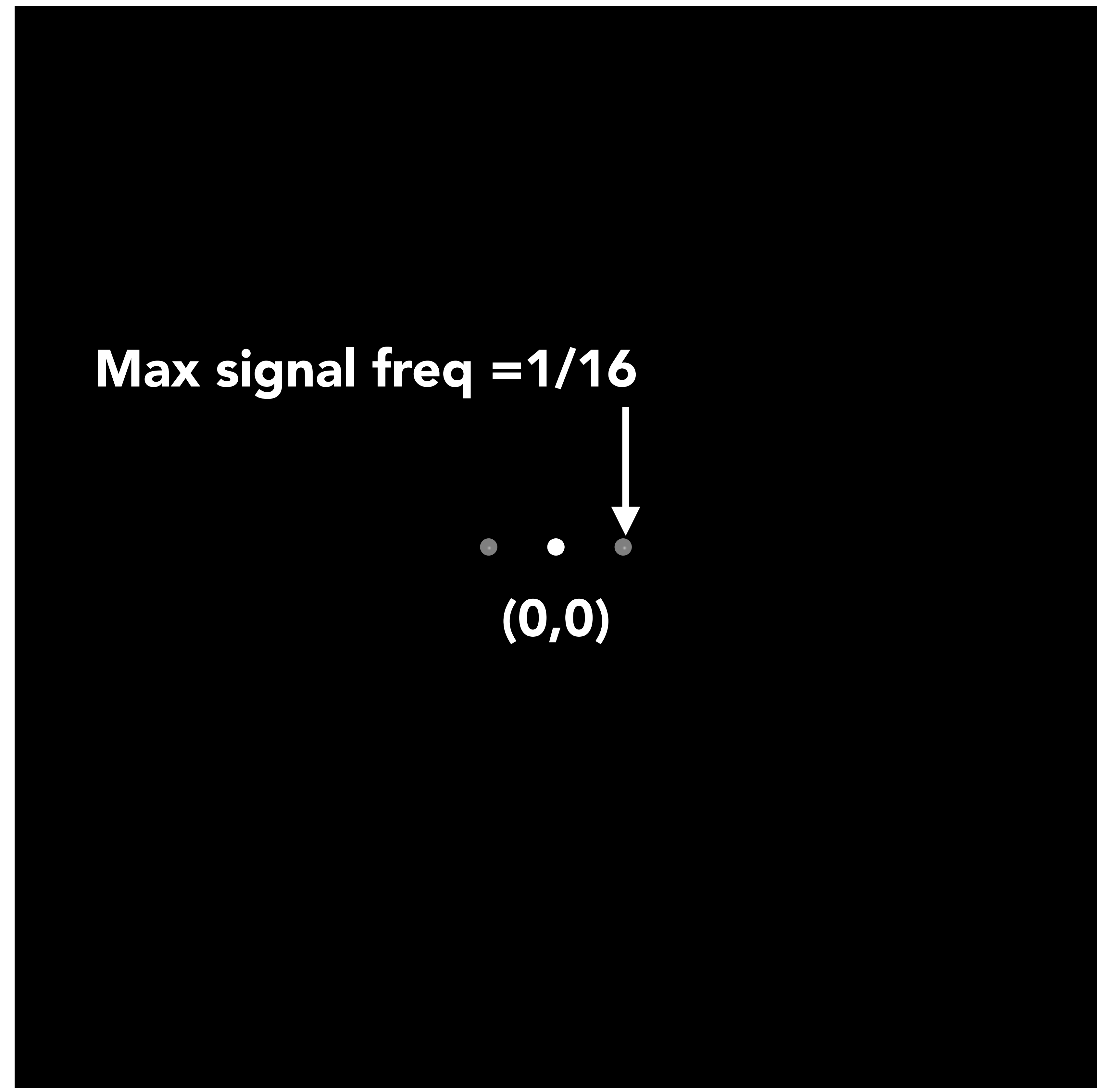


Frequency Domain

$\sin(2\pi/16)x$ — frequency 1/16; 16 pixels per cycle

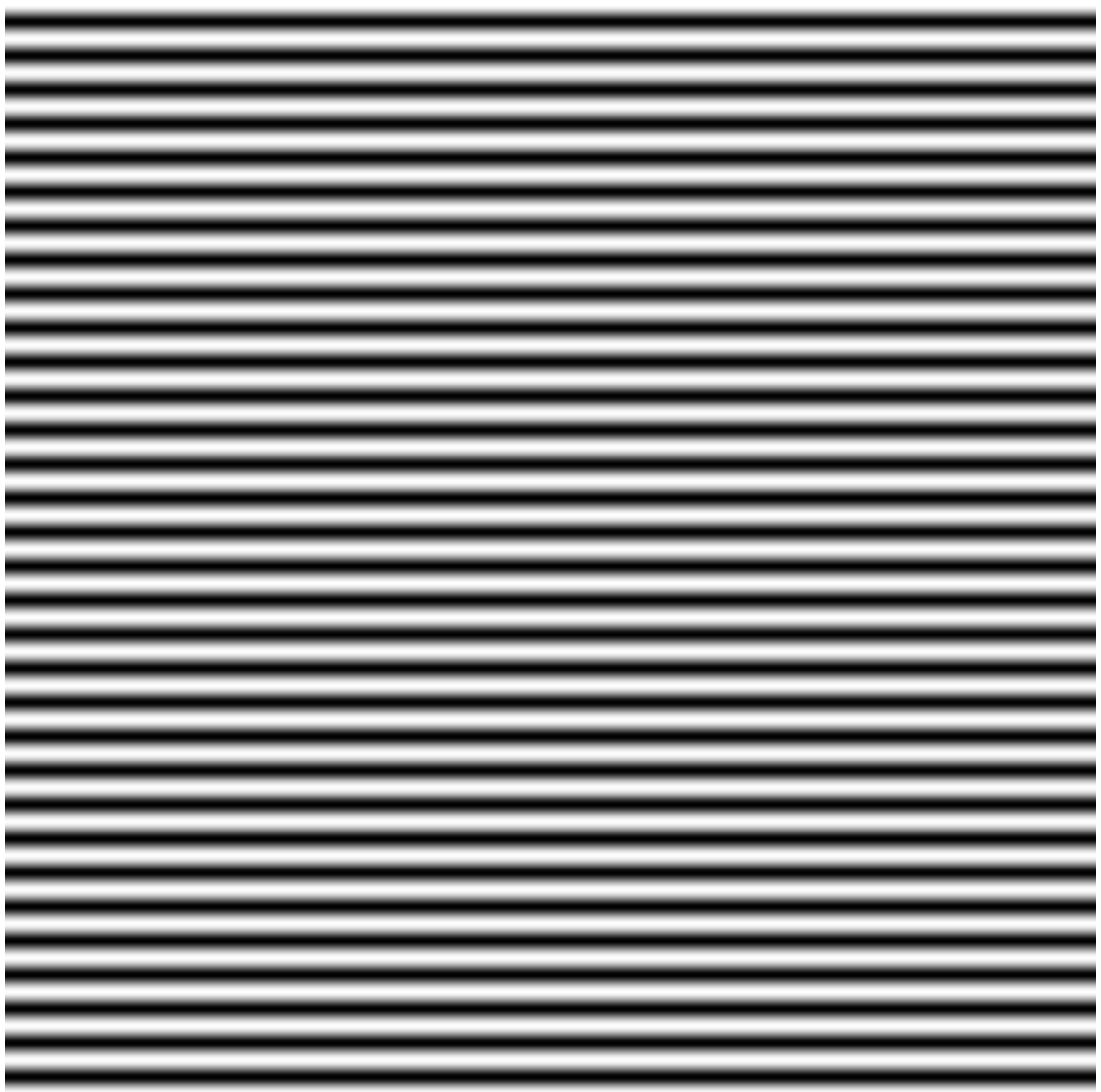


Spatial Domain

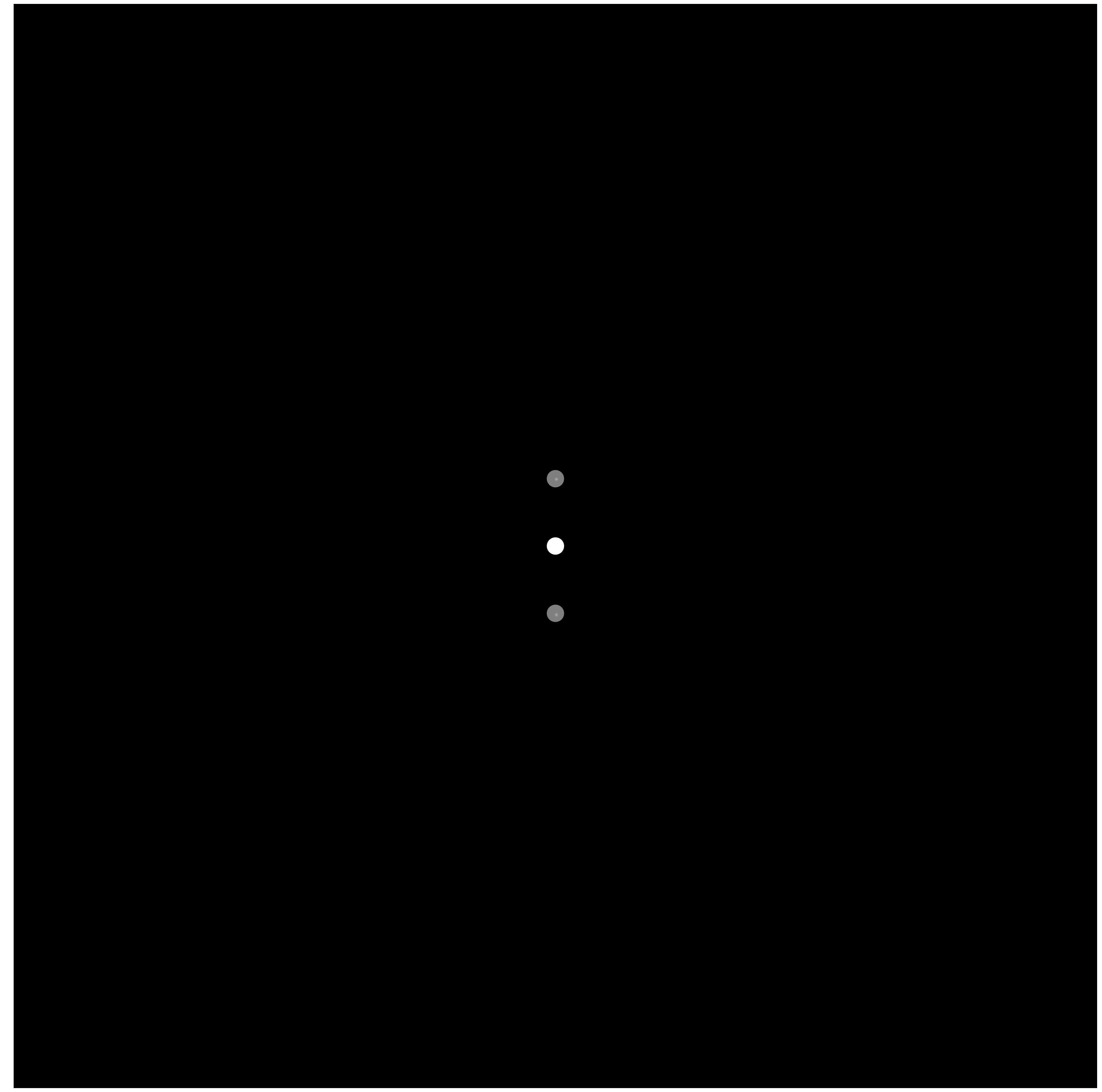


Frequency Domain

$$\sin(2\pi/16)y$$

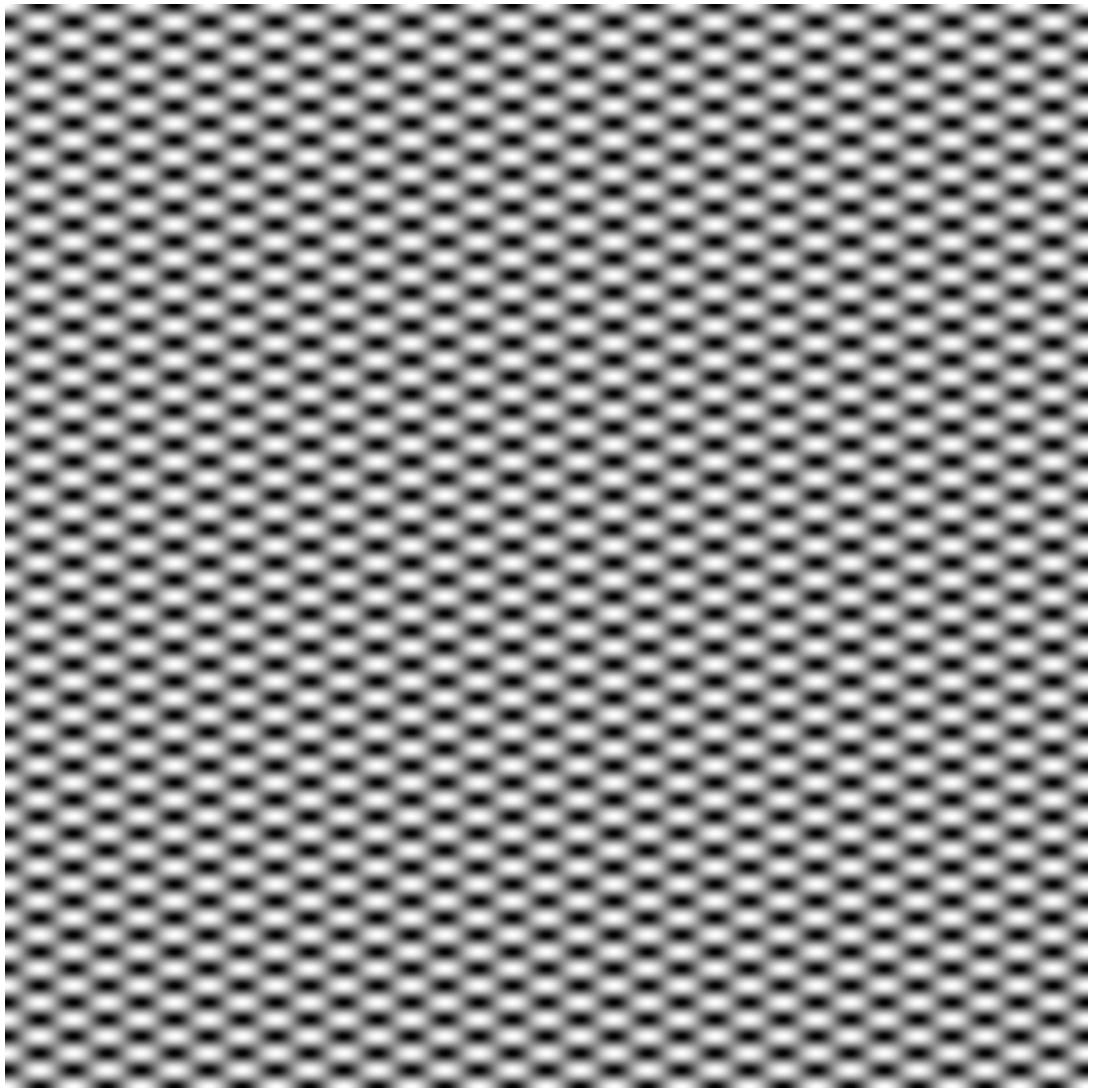


Spatial Domain

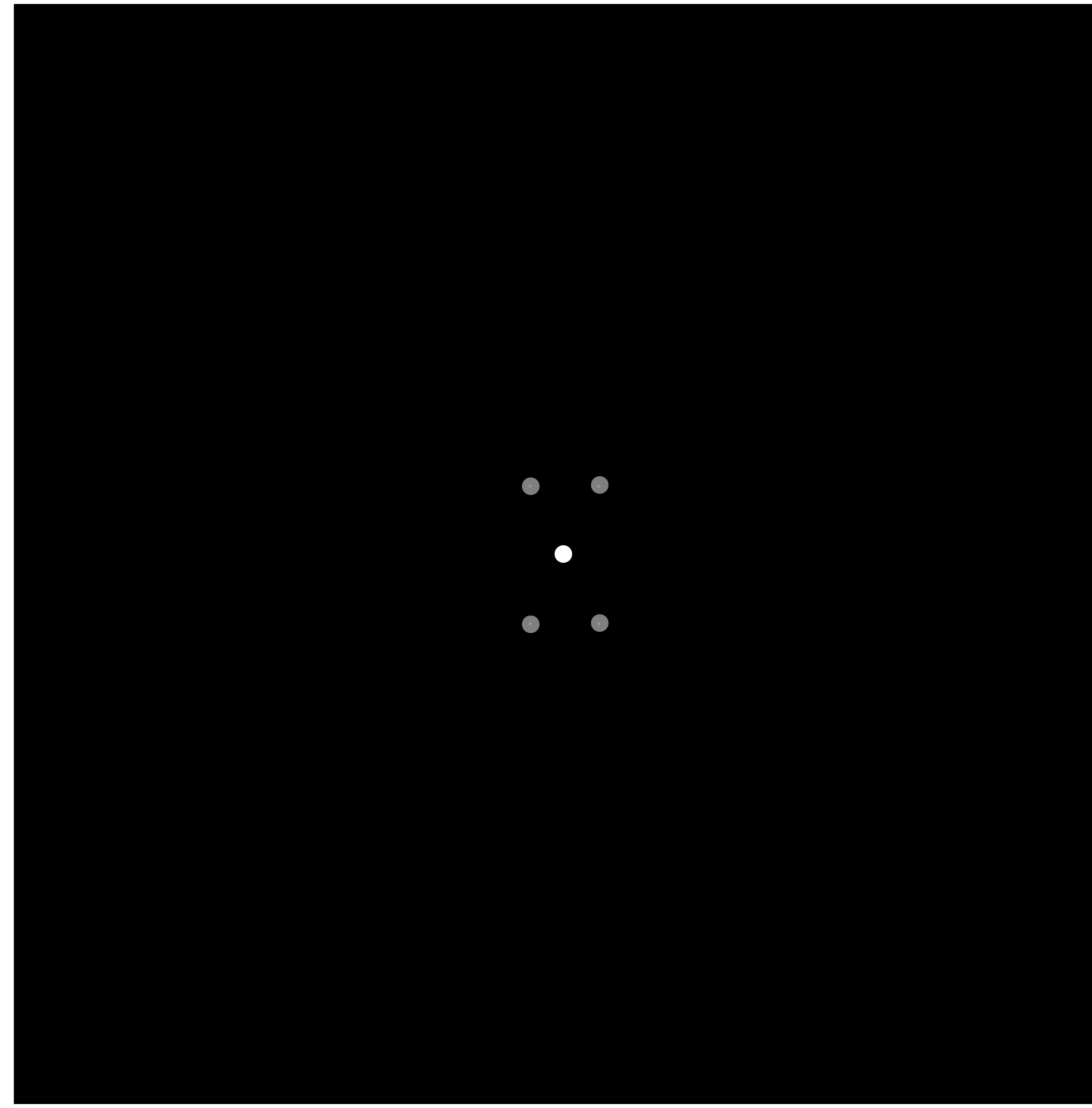


Frequency Domain

$$\sin(2\pi/32)x \times \sin(2\pi/16)y$$

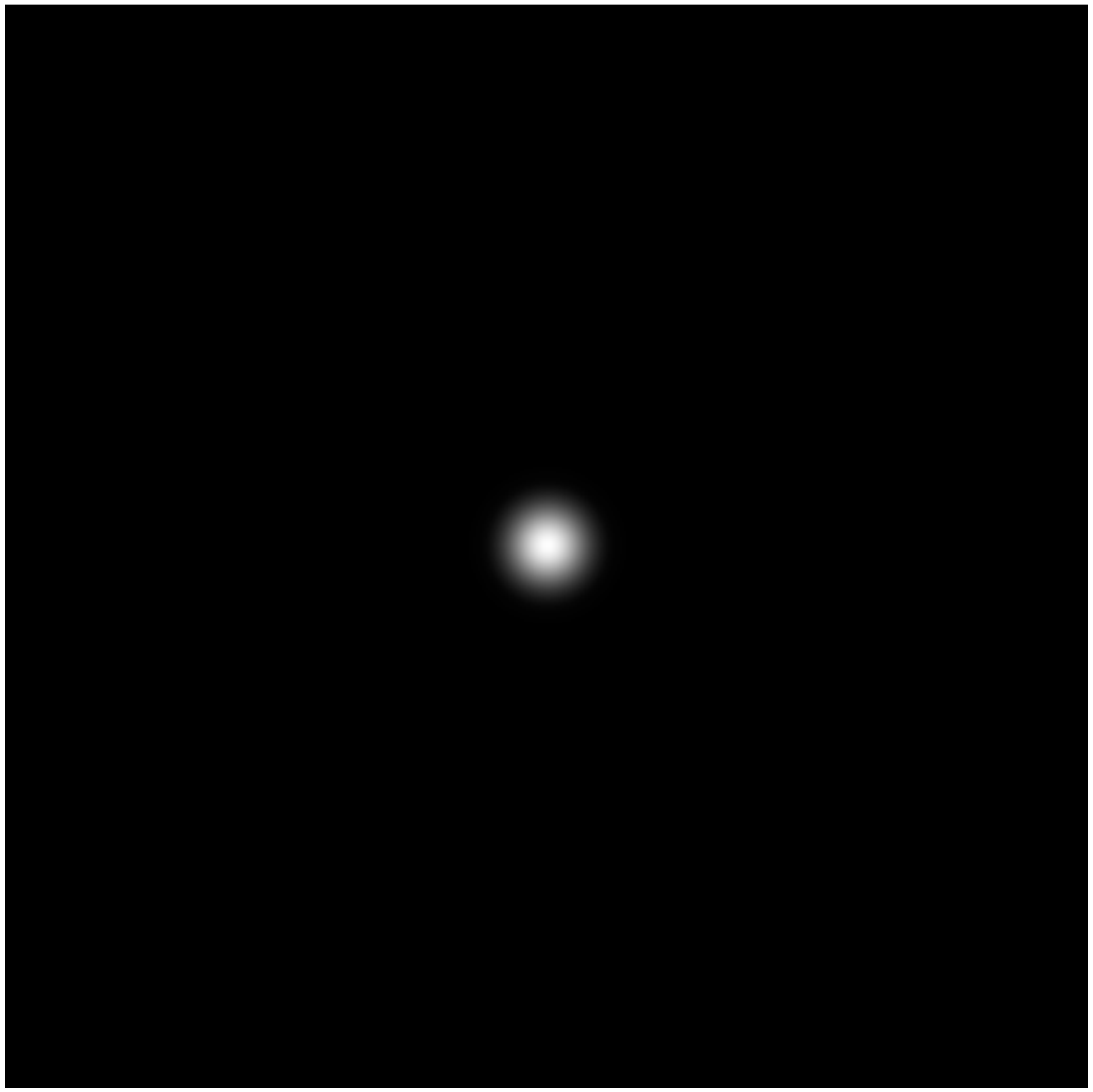


Spatial Domain

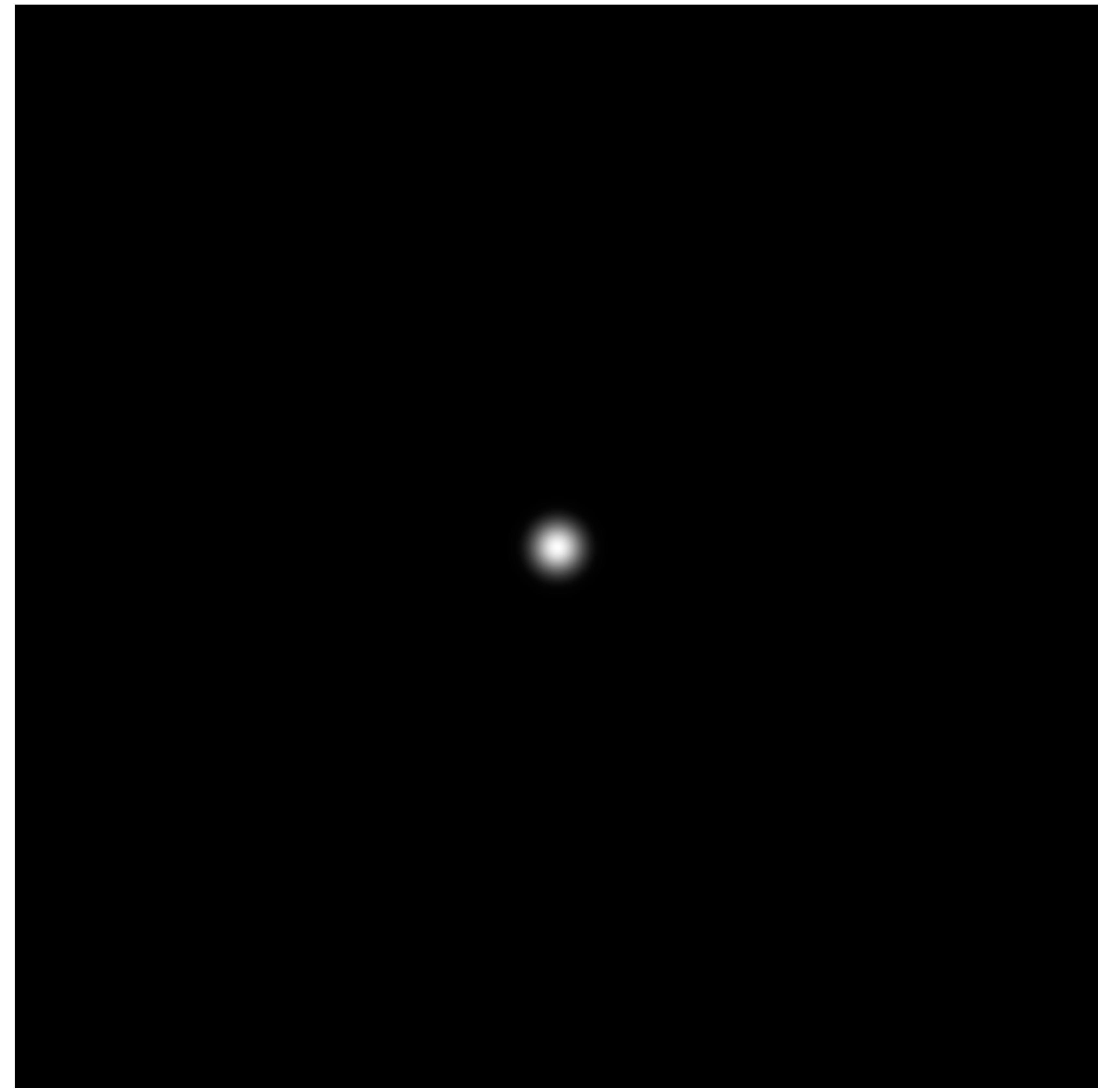


Frequency Domain

$$\exp(-r^2/16^2)$$

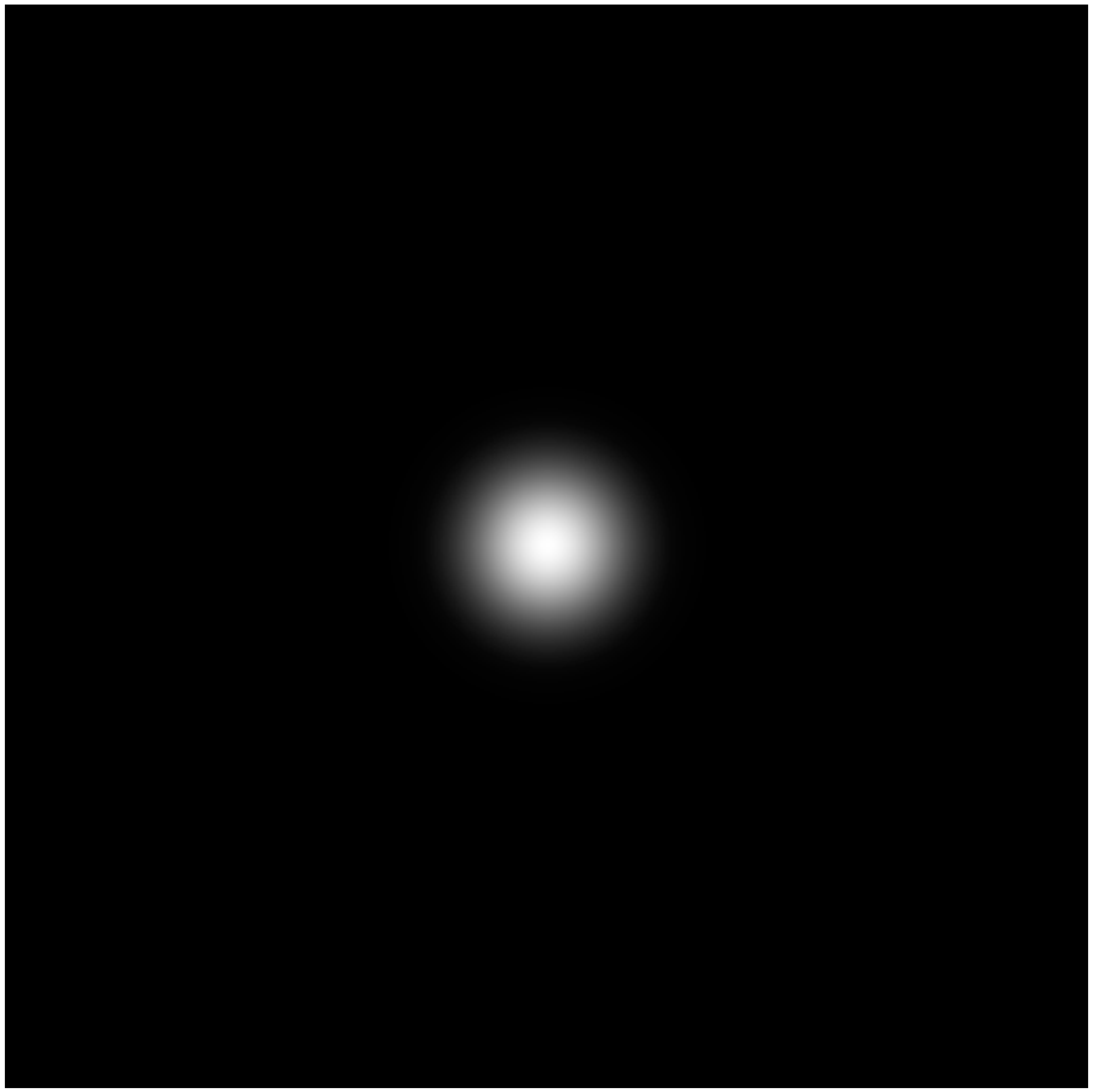


Spatial Domain

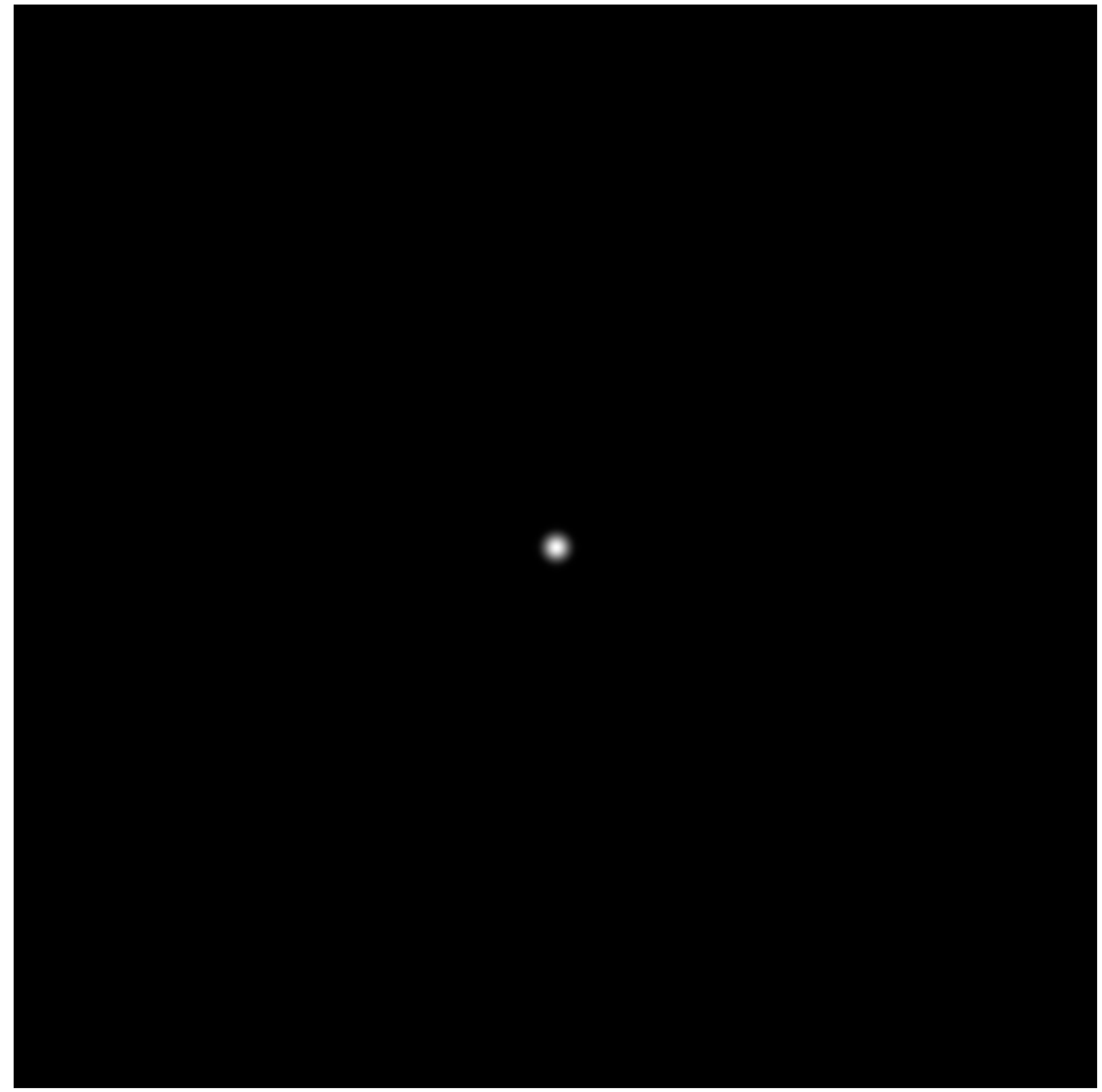


Frequency Domain

$$\exp(-r^2/32^2)$$

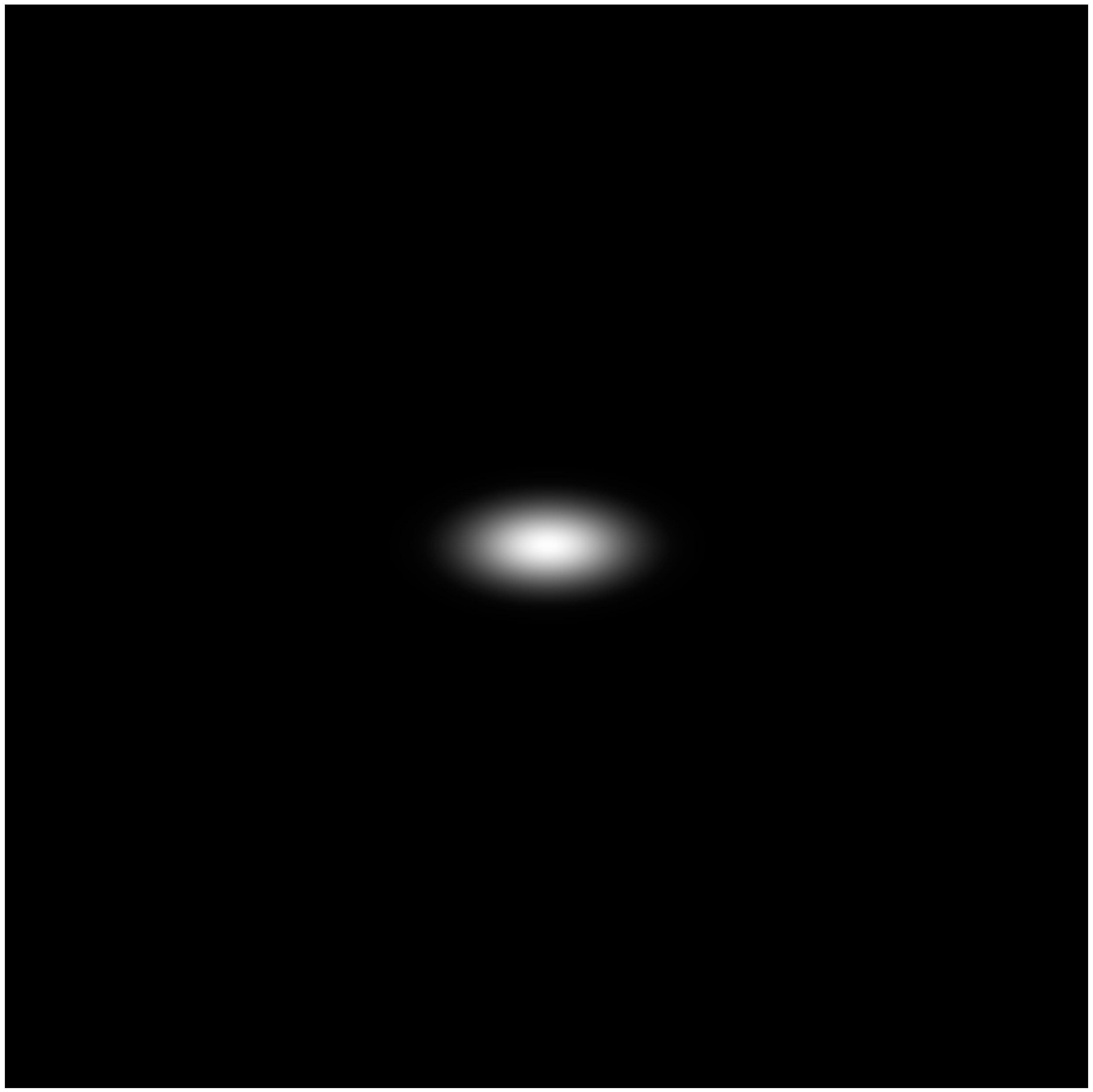


Spatial Domain

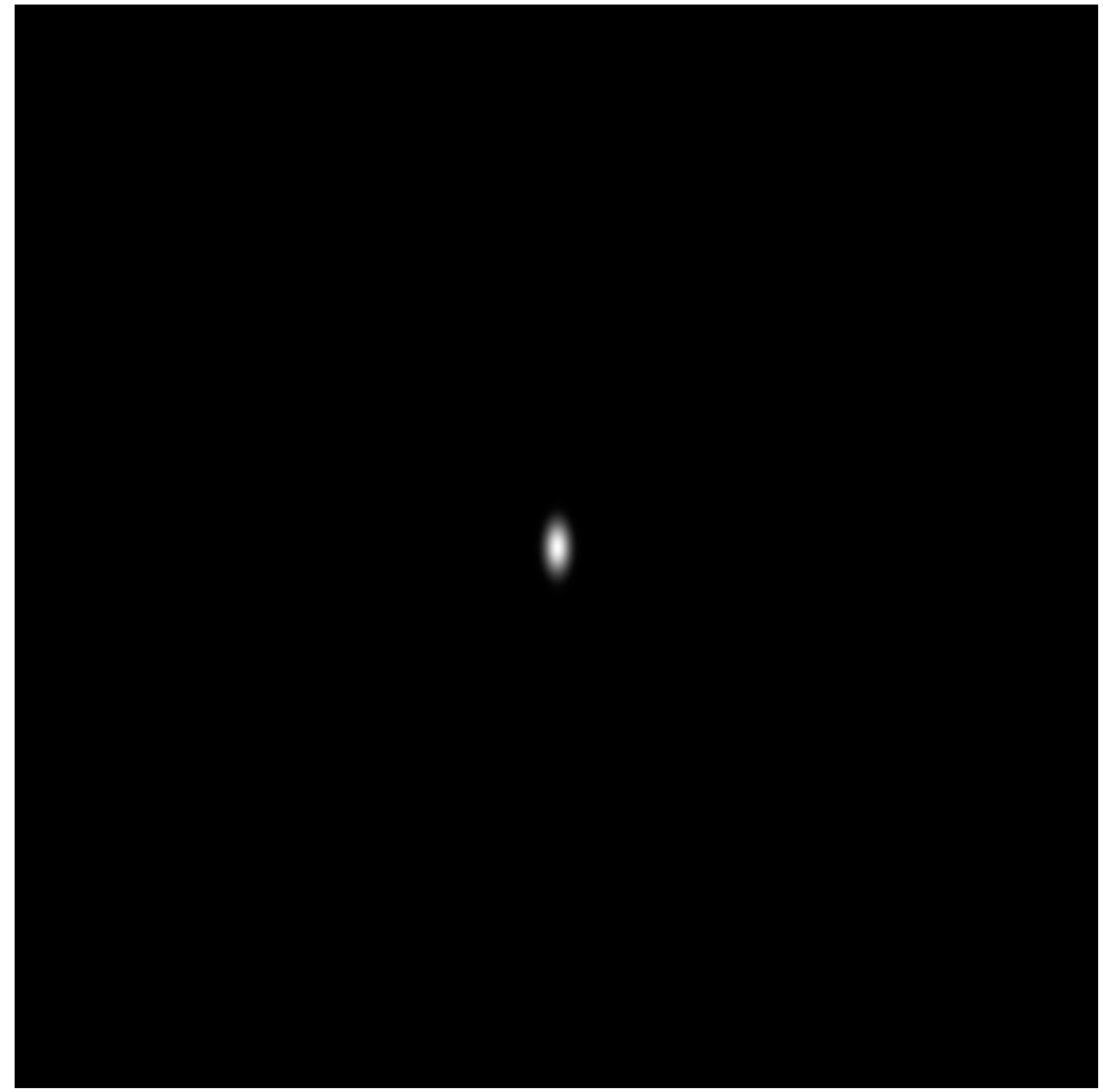


Frequency Domain

$$\exp(-x^2/32^2) \times \exp(-y^2/16^2)$$

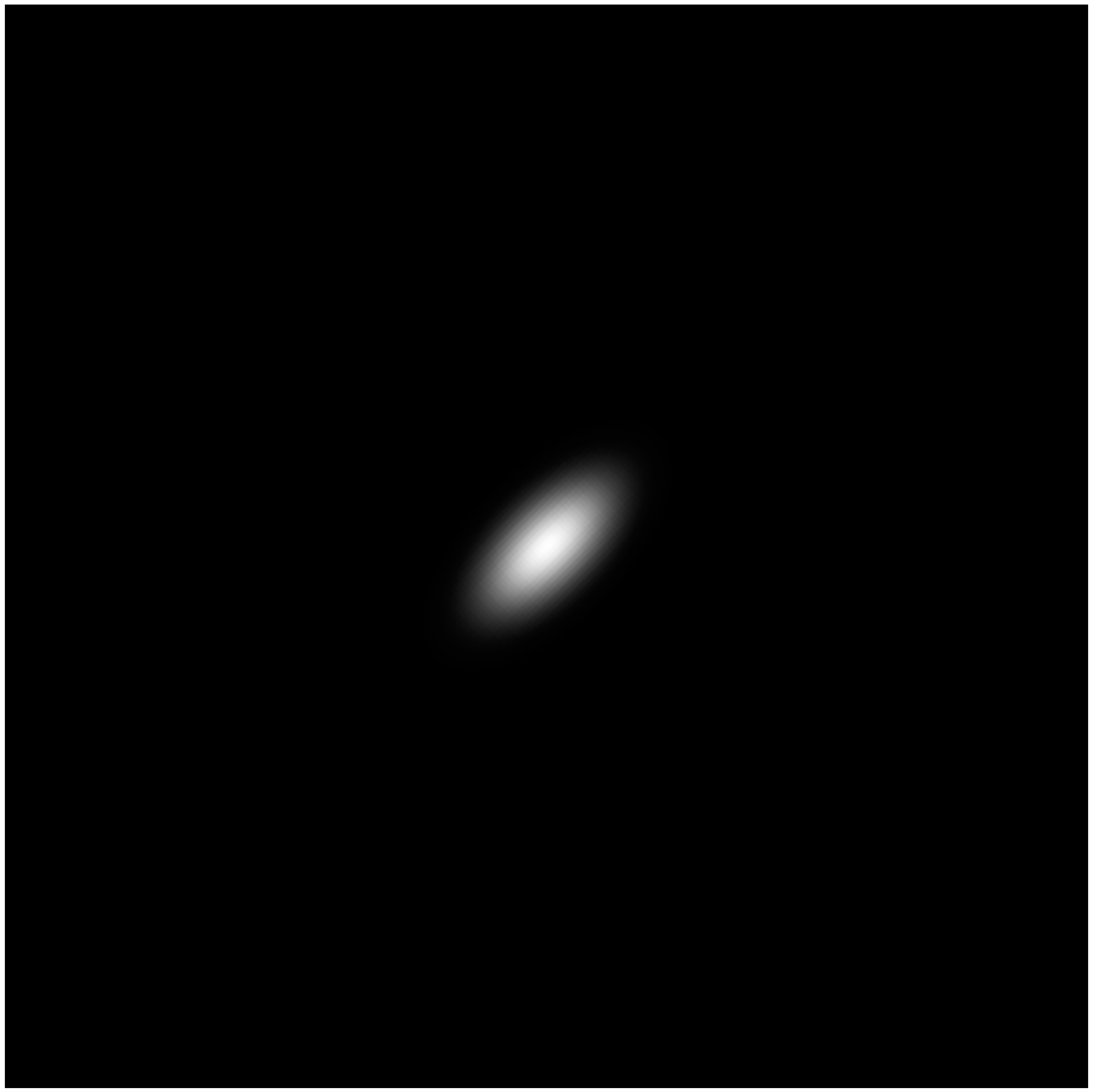


Spatial Domain

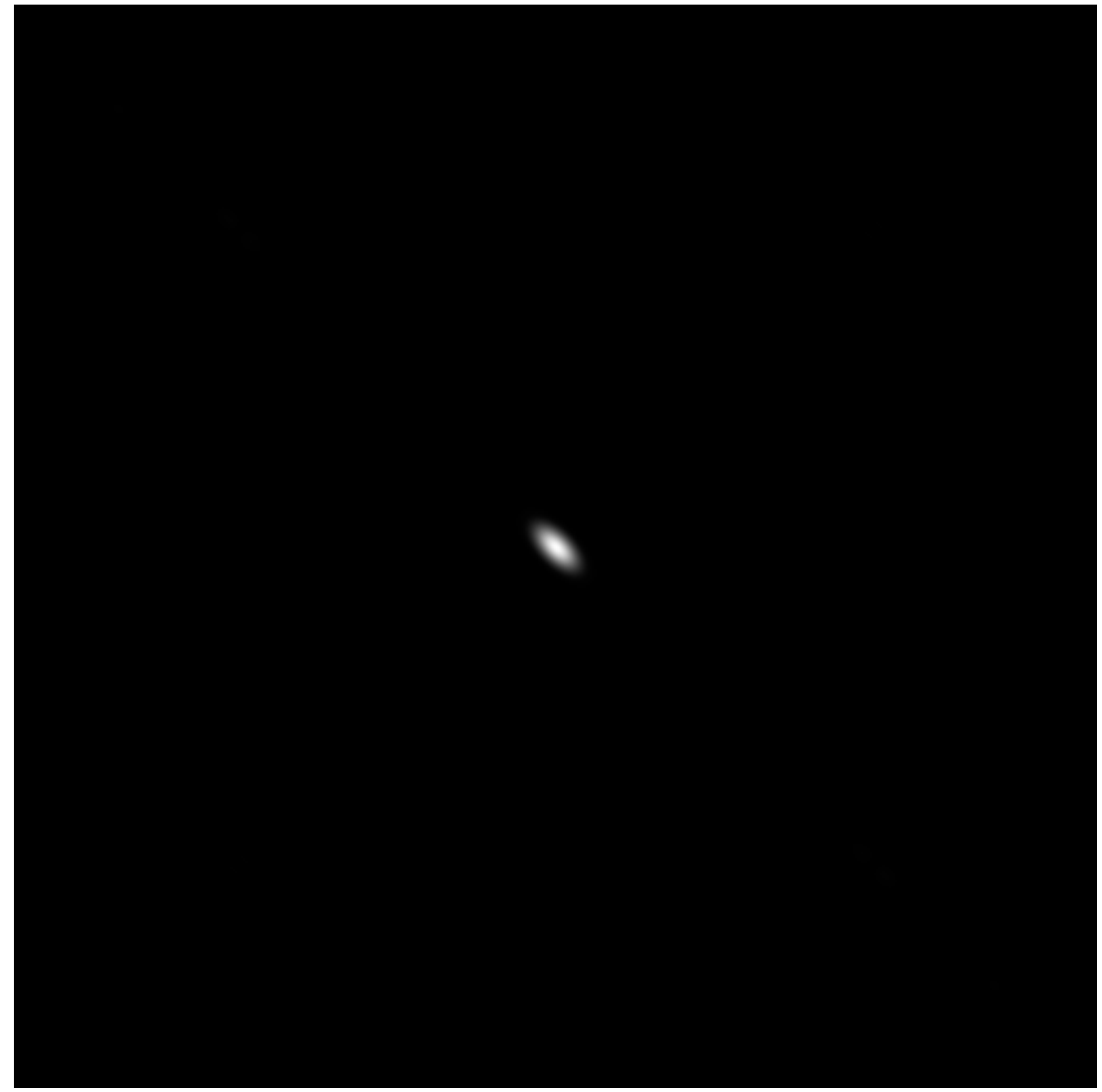


Frequency Domain

Rotate 45 $\exp(-x^2/32^2) \times \exp(-y^2/16^2)$



Spatial Domain

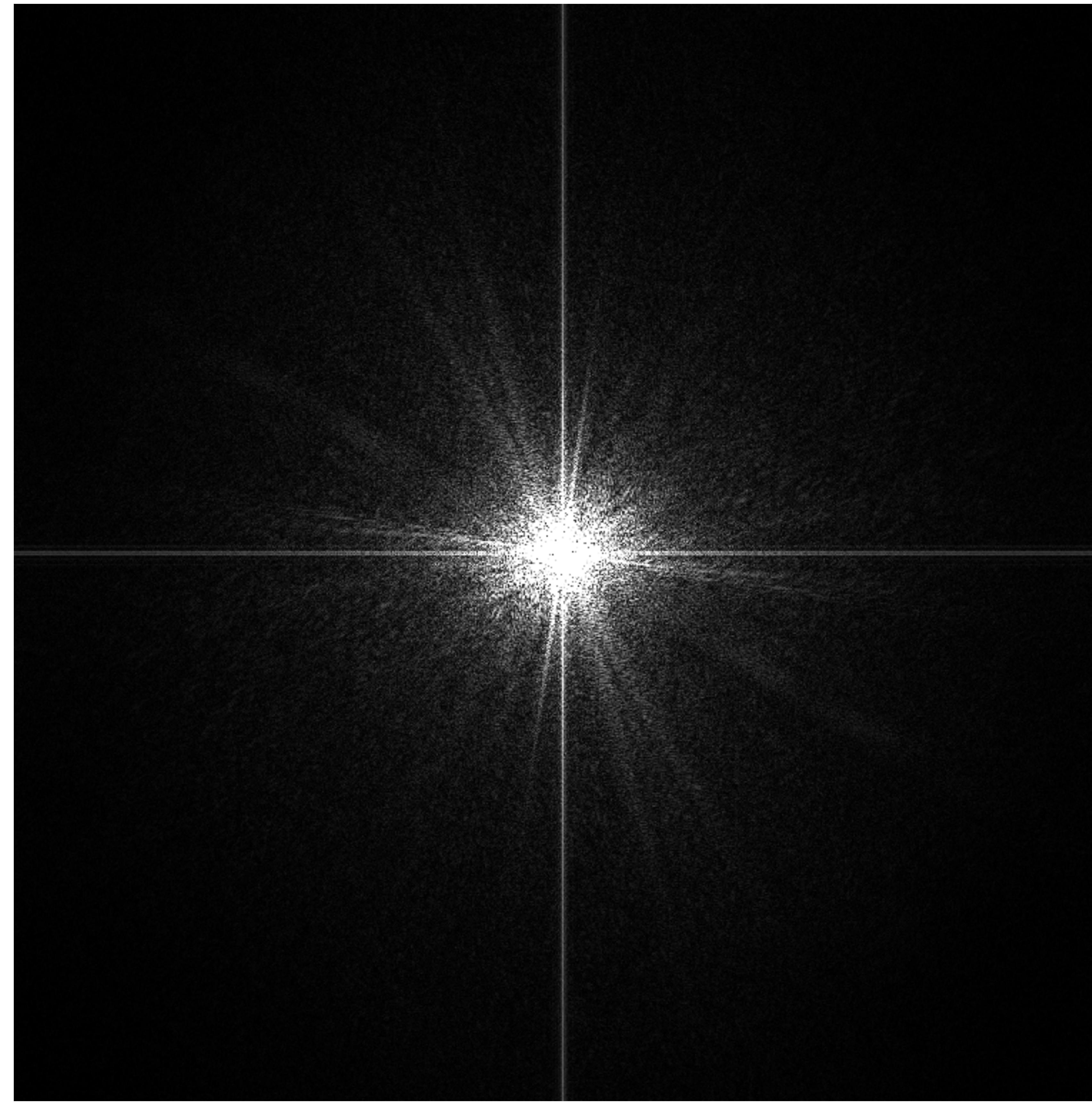


Frequency Domain

Filtering

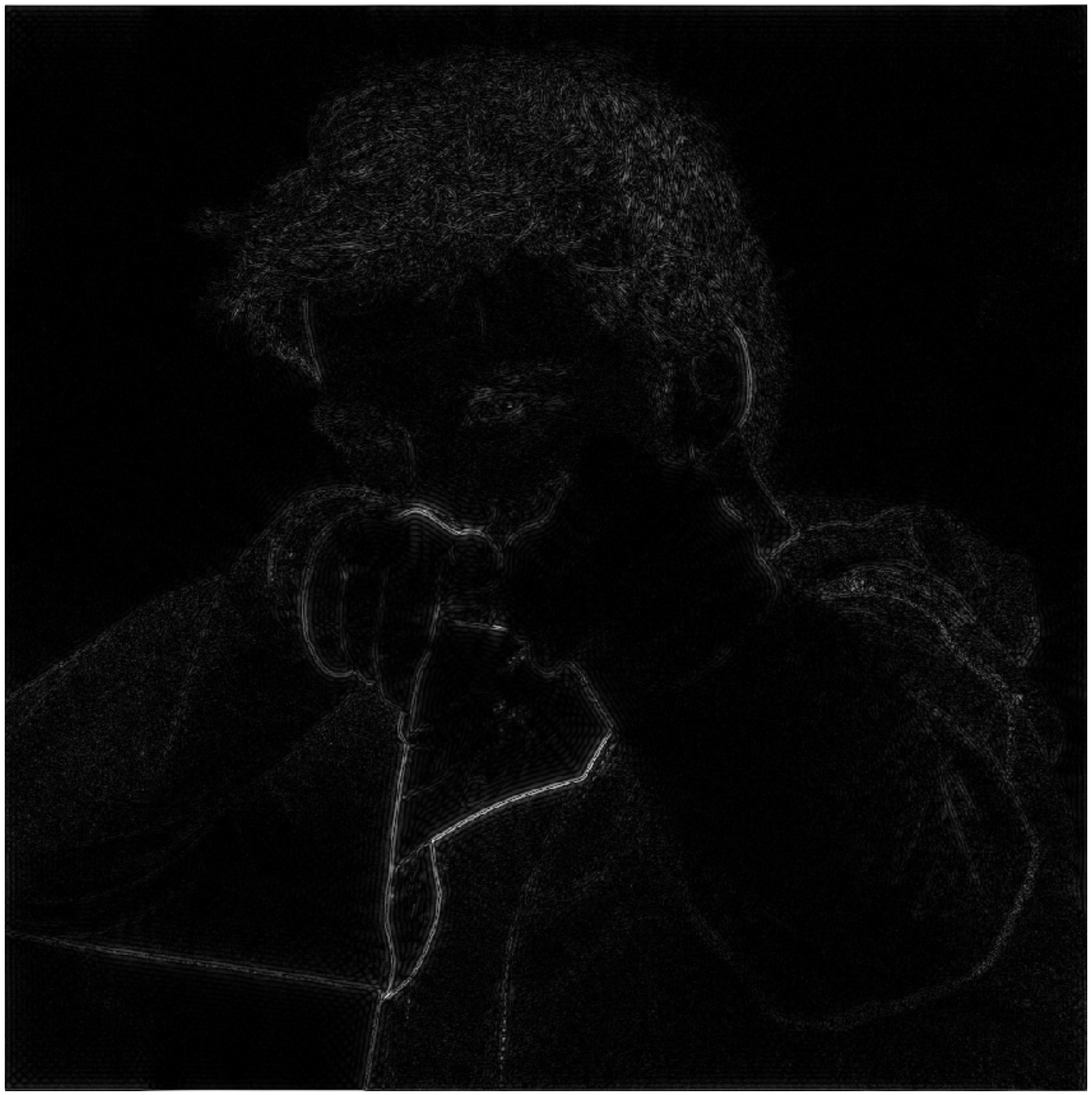
Visualizing Image Frequency Content

Spatial Domain

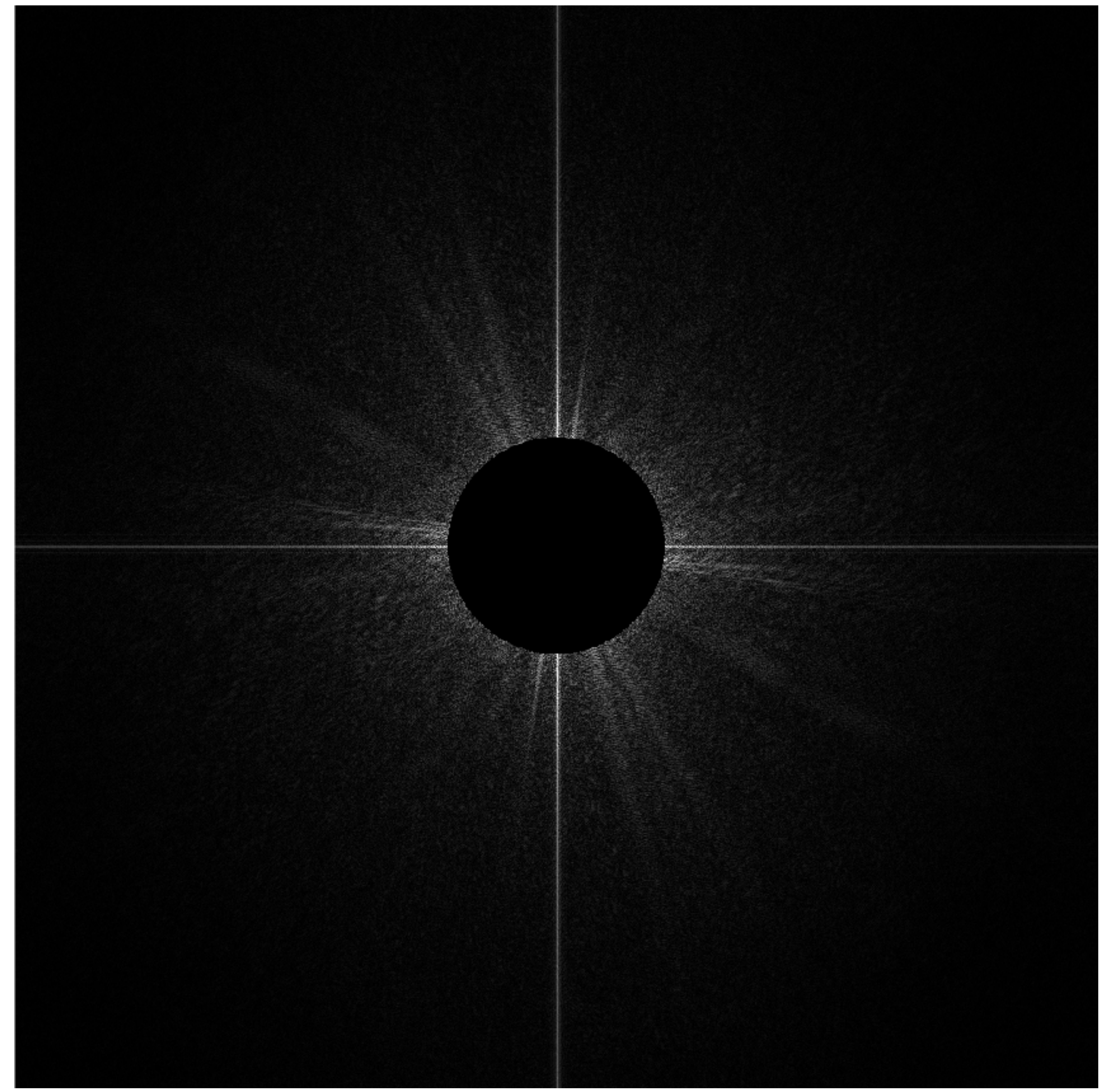


Frequency Domain

Filter Out Low Frequencies Only (Edges)



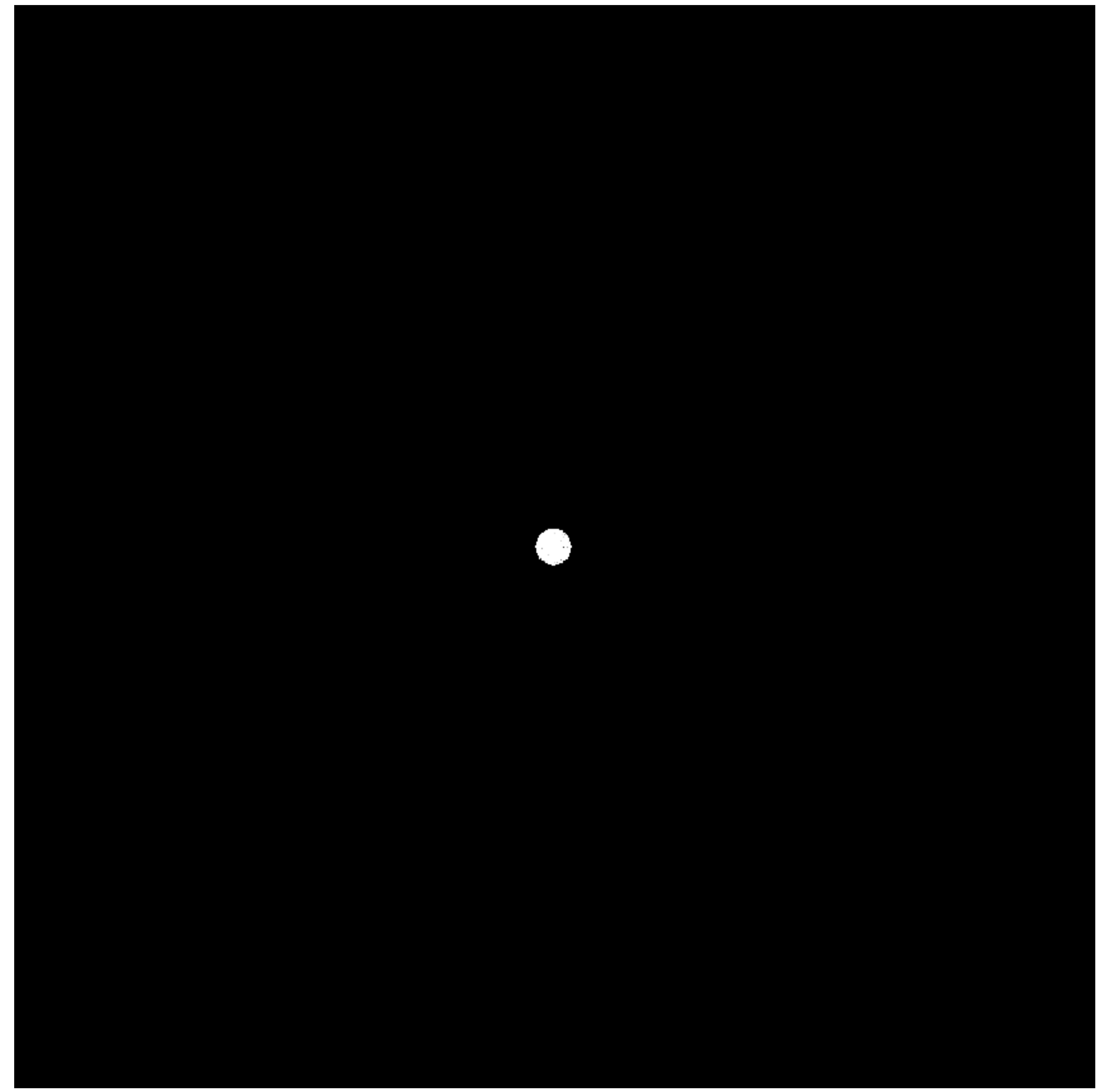
Spatial Domain



Frequency Domain

Filter Out High Frequencies (Blur)

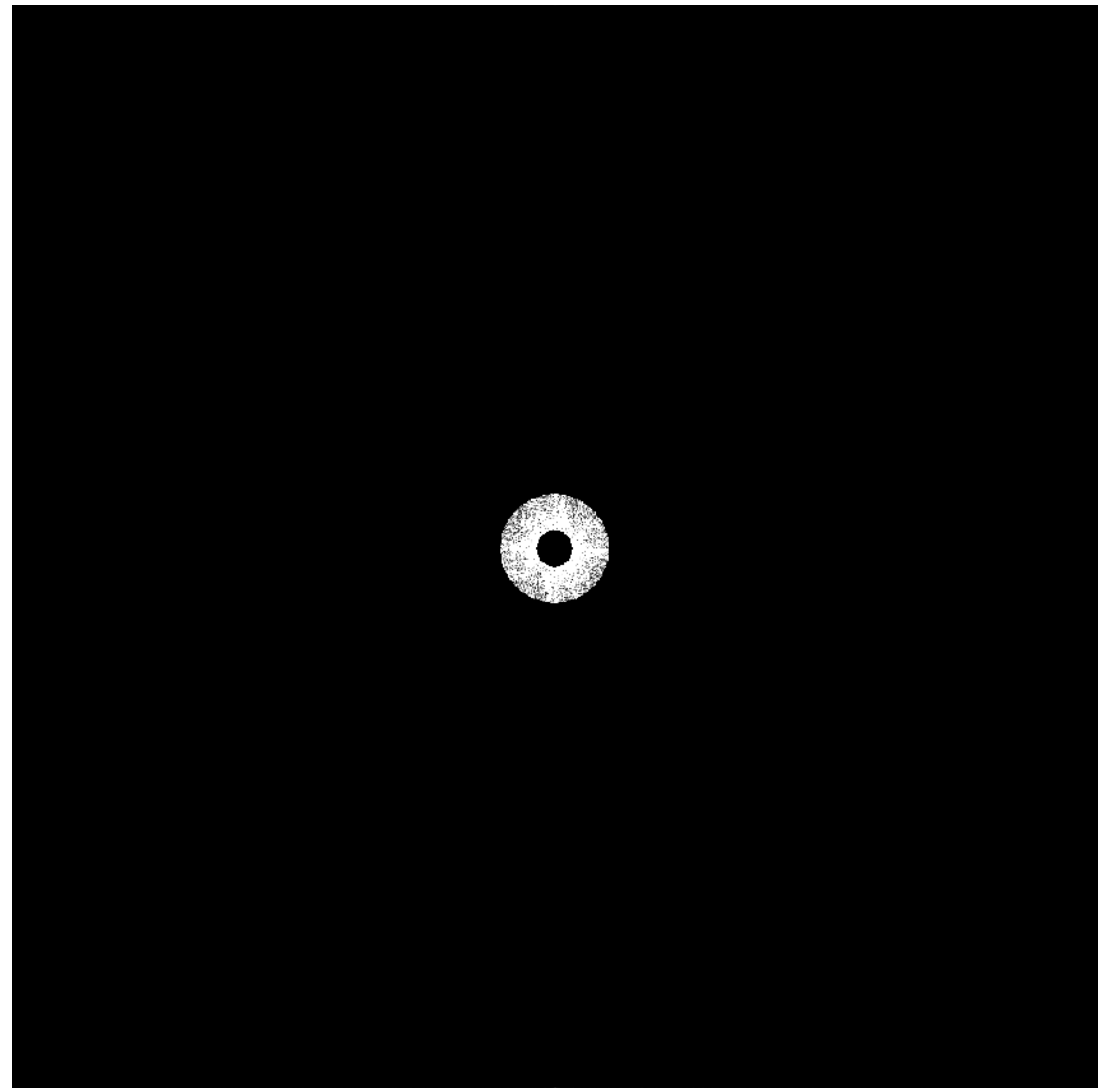
Spatial Domain



Frequency Domain

Filter Out Low and High Frequencies

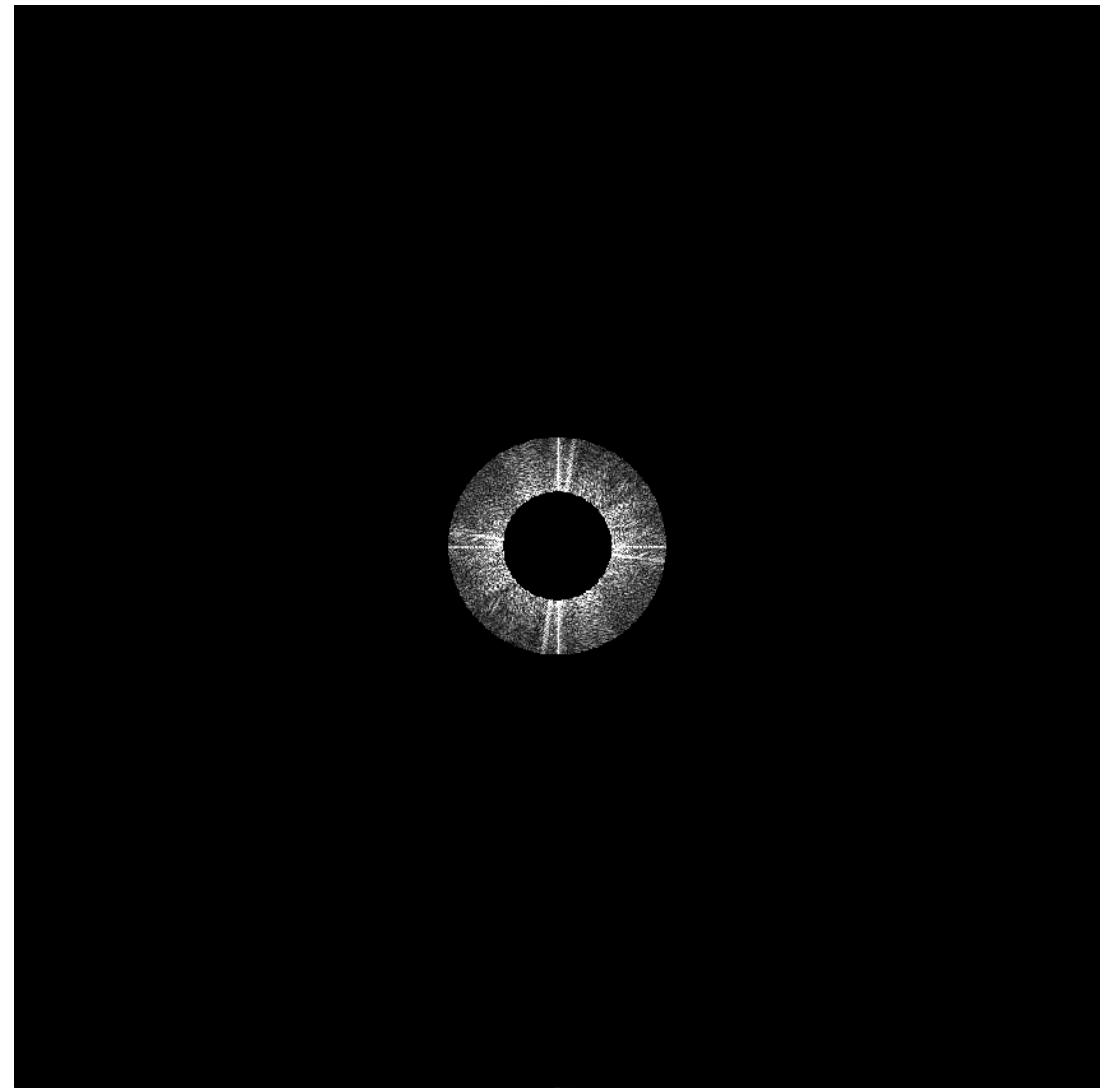
Spatial Domain



Frequency Domain

Filter Out Low and High Frequencies

Spatial Domain



Frequency Domain

Ren Ng

Filtering = Convolution

Convolution

Signal	1	3	5	3	7	1	3	8	6	4

Filter	1	2	1

Convolution

Signal	1	3	5	3	7	1	3	8	6	4

Filter	1	2	1

1x1 + 3x2 + 5x1 = 12

Result	12									

Convolution

Signal										
	1	3	5	3	7	1	3	8	6	4

Filter			
	1	2	1

3x1 + 5x2 + 3x1 = 16

Result										
	12	16								

Convolution

Signal

1	3	5	3	7	1	3	8	6	4

Filter

1	2	1

5x1 + 3x2 + 7x1 = 18

Result

12	16	18							

Convolution Theorem

Convolution in the spatial domain is equal to multiplication in the frequency domain, and vice versa

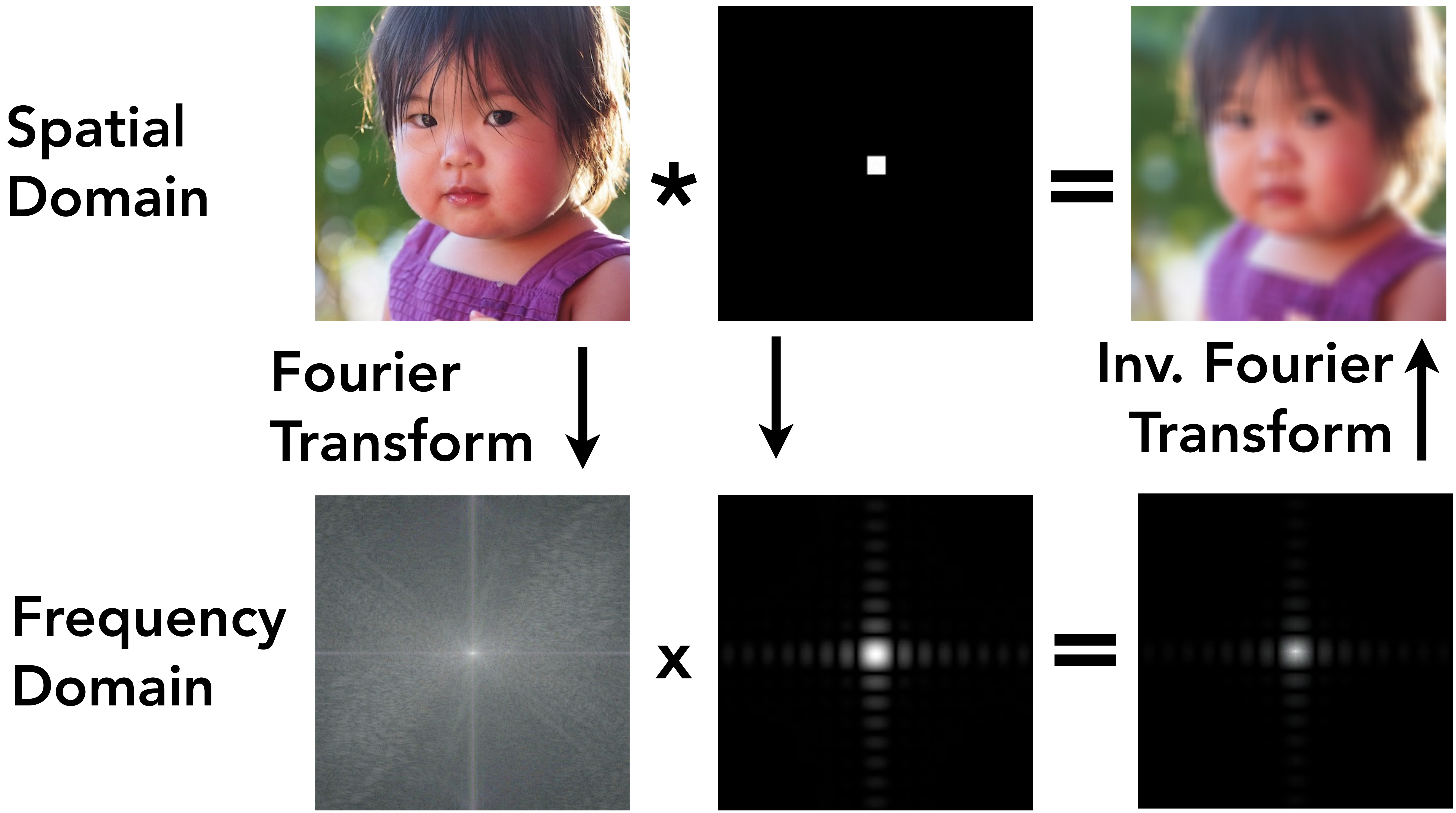
Option 1:

- Filter by convolution in the spatial domain

Option 2:

- Transform to frequency domain (Fourier transform)
- Multiply by Fourier transform of convolution kernel
- Transform back to spatial domain (inverse Fourier)

Convolution Theorem

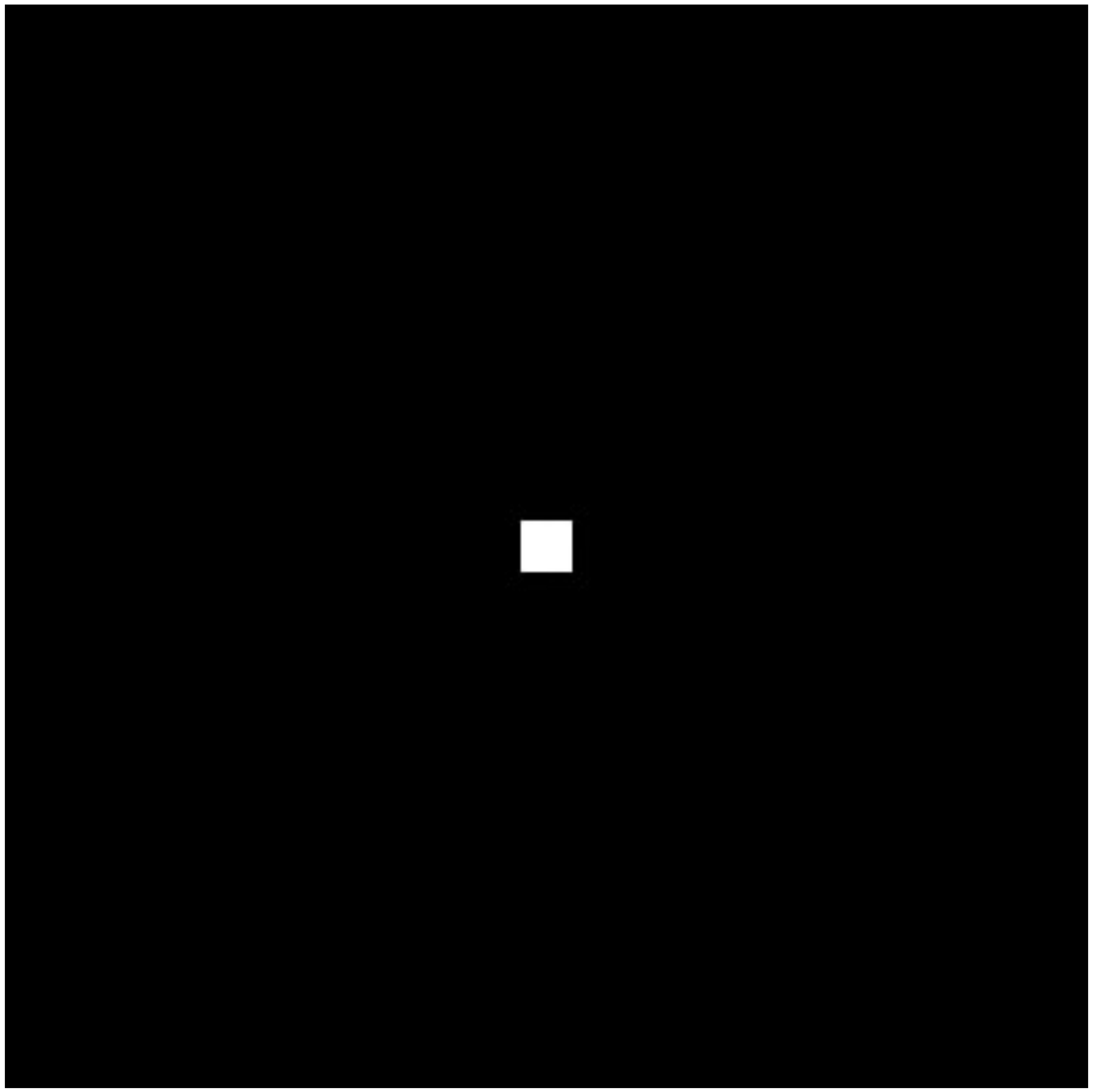


Box Filter

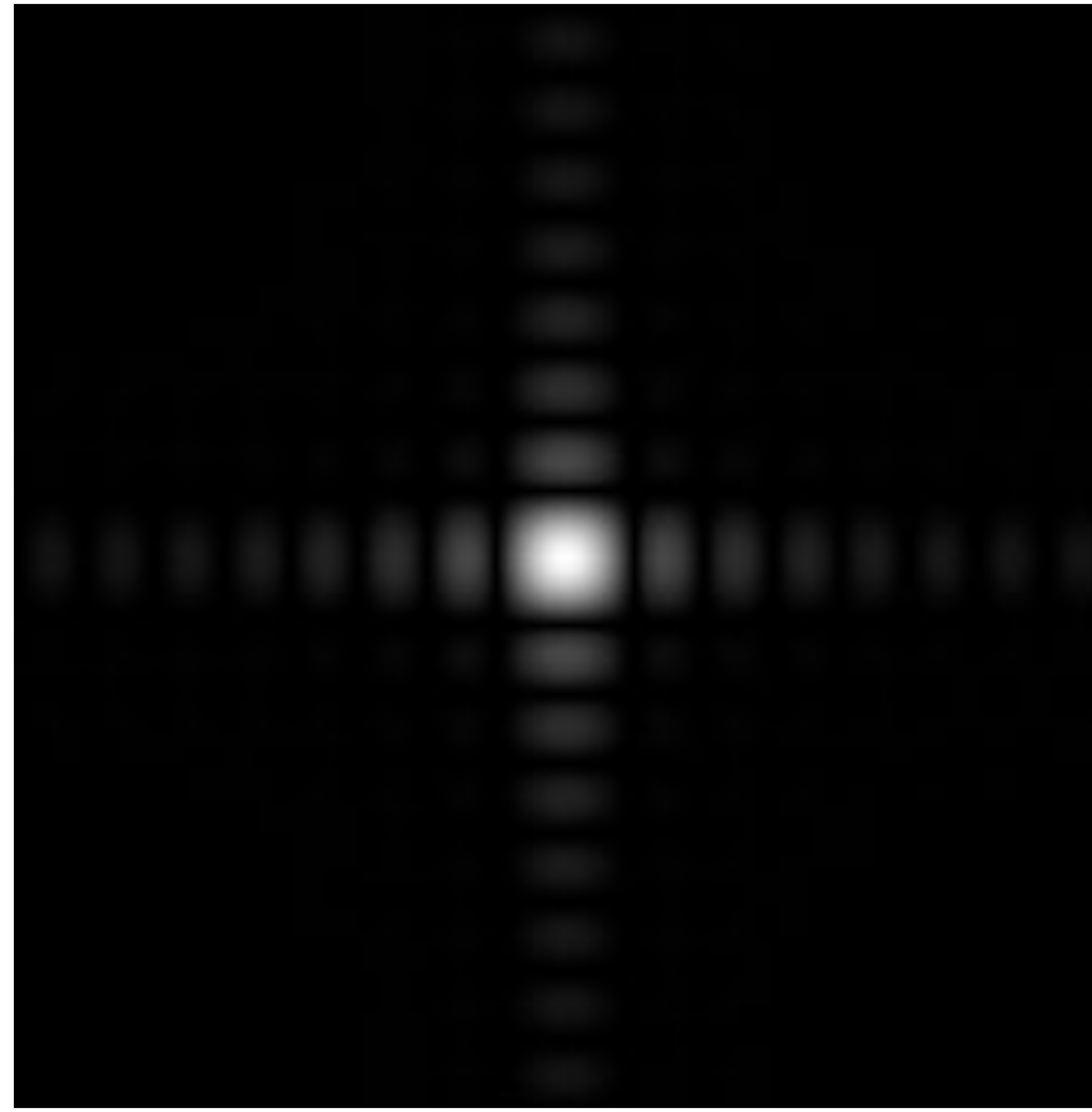
$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example: 3x3 box filter

Box Function = “Low Pass” Filter



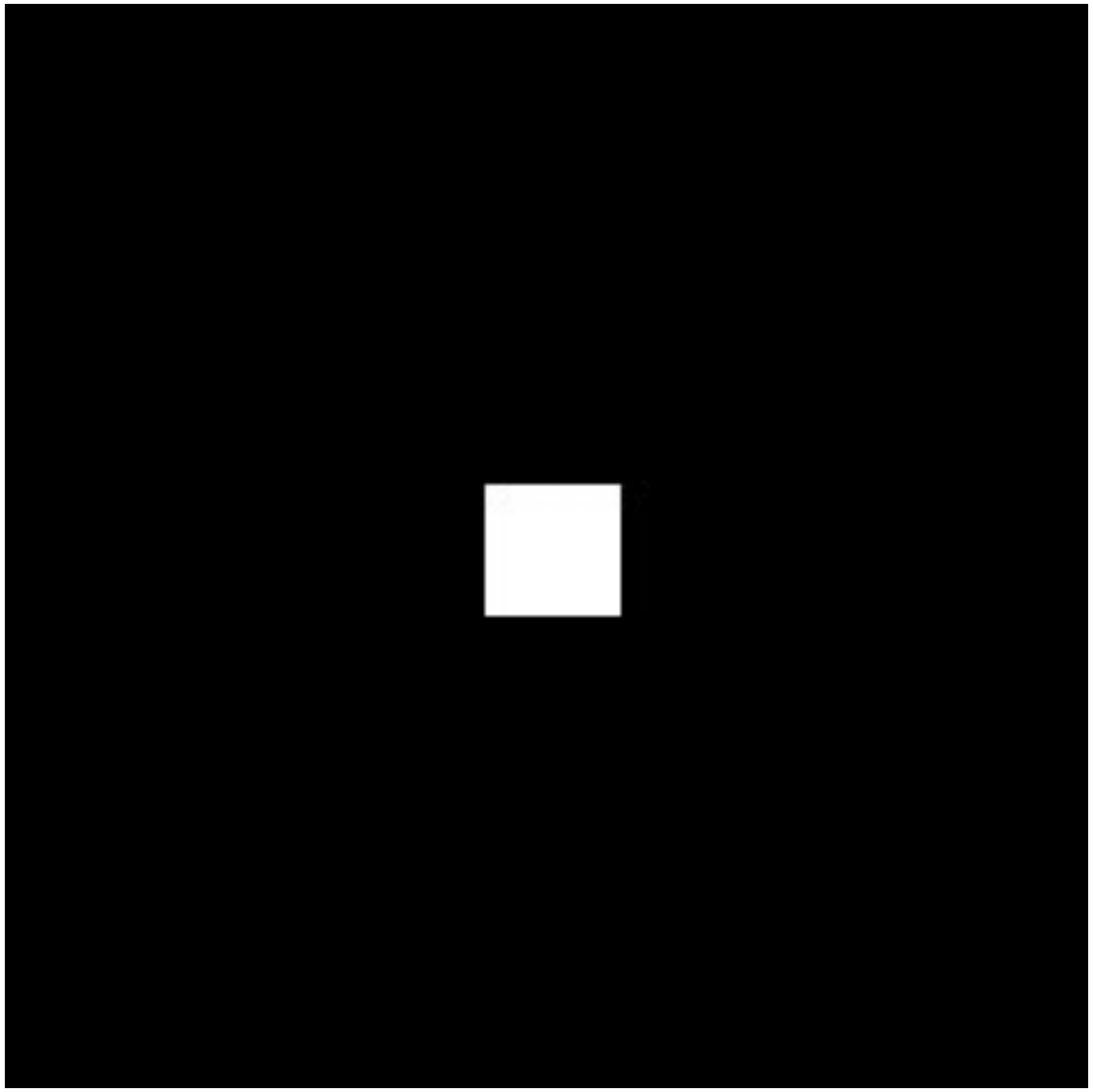
Spatial Domain



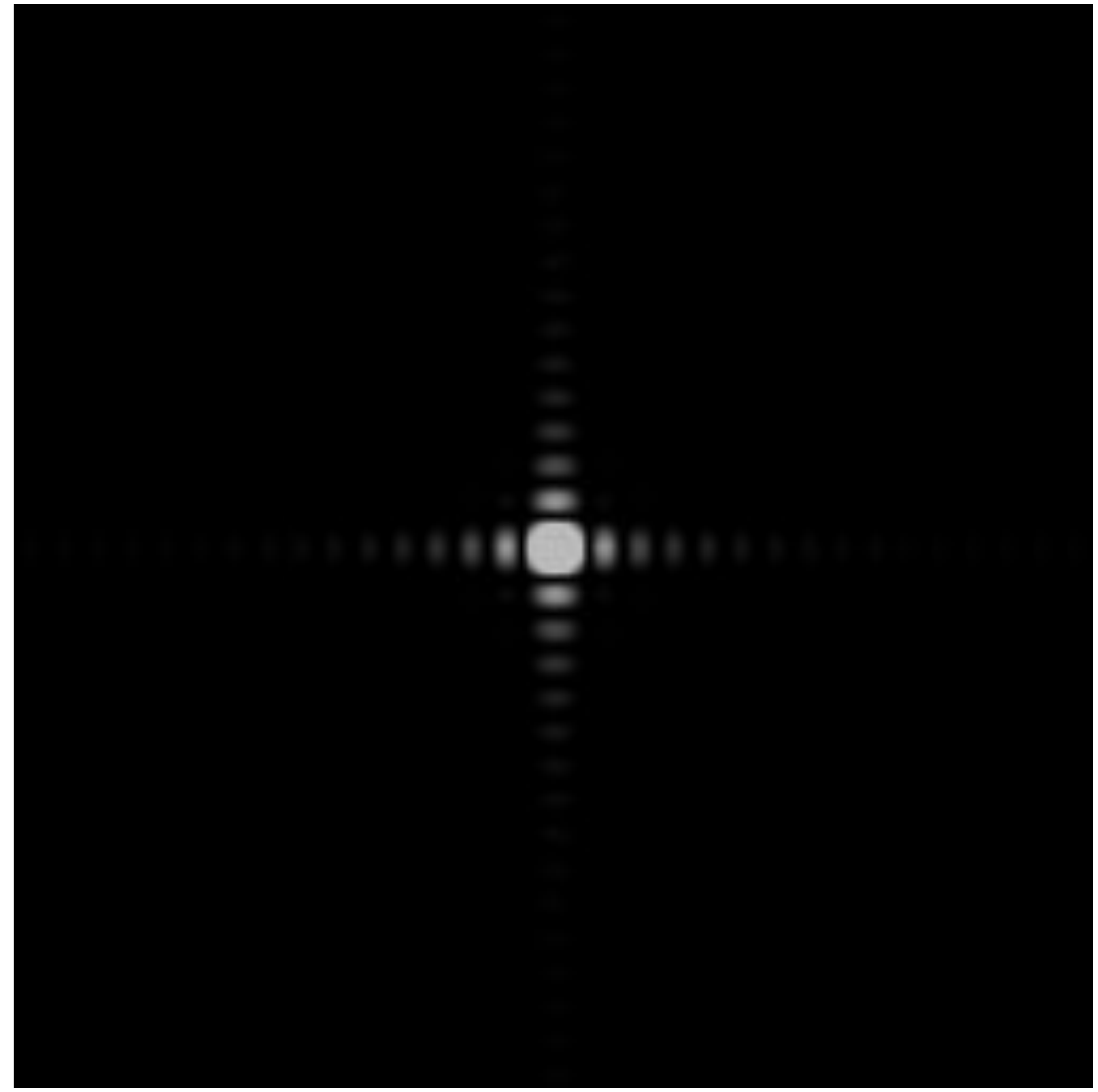
Frequency Domain

Ren Ng

Wider Filter Kernel = Lower Frequencies



Spatial Domain



Frequency Domain

Wider Filter Kernel = Lower Frequencies

As a filter is localized in the spatial domain,
it spreads out in frequency domain.

Conversely, as a filter is localized in frequency domain,
it spreads out in the spatial domain

Efficiency?

When is it faster to implement a filter by convolution in the spatial domain?

When is it faster to implement a filter by multiplication in the frequency domain?

Nyquist Frequency & Antialiasing

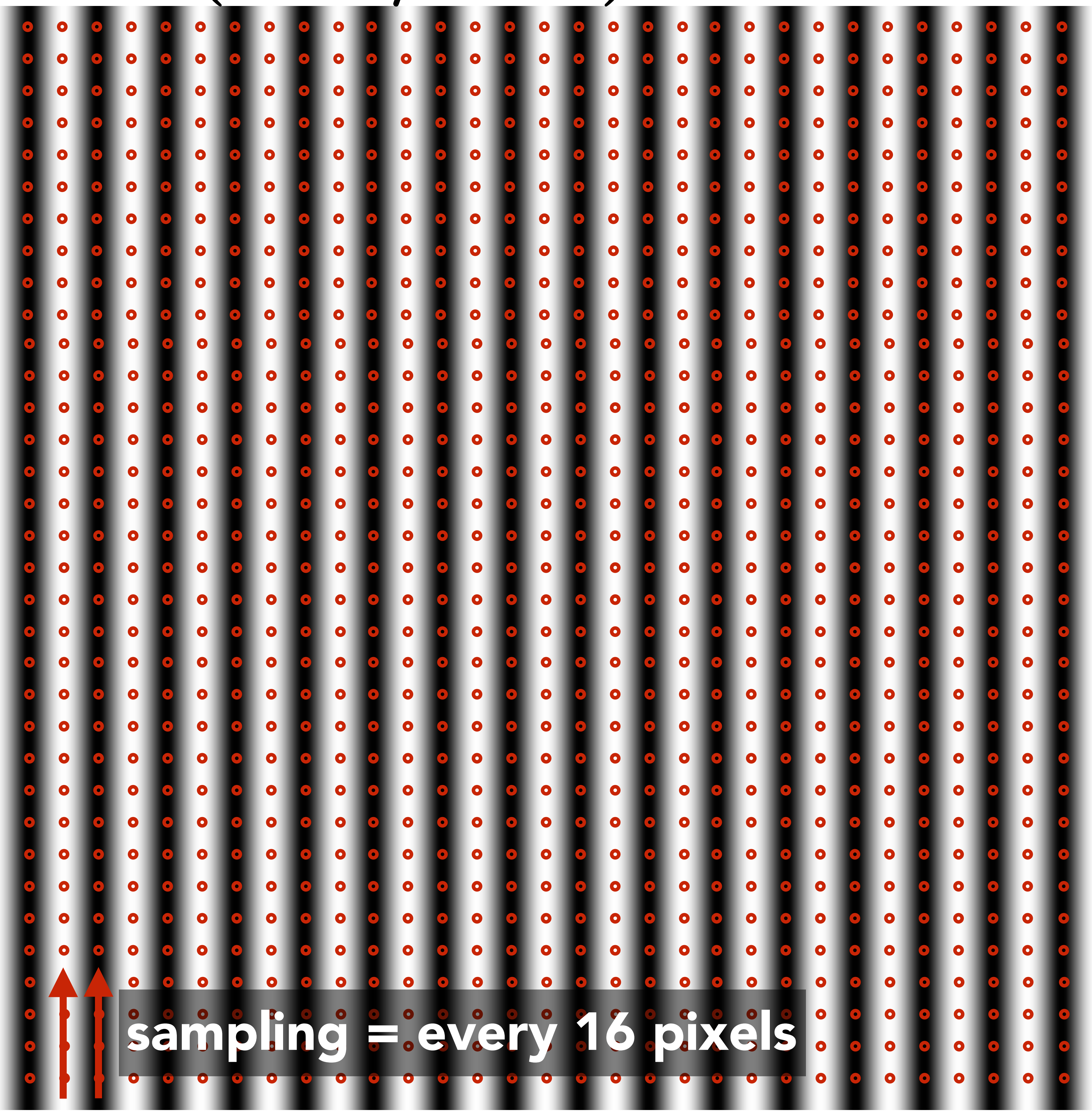
Nyquist Theorem

Theorem: We get no aliasing from frequencies in the signal that are less than the Nyquist frequency (which is defined as half the sampling frequency) *

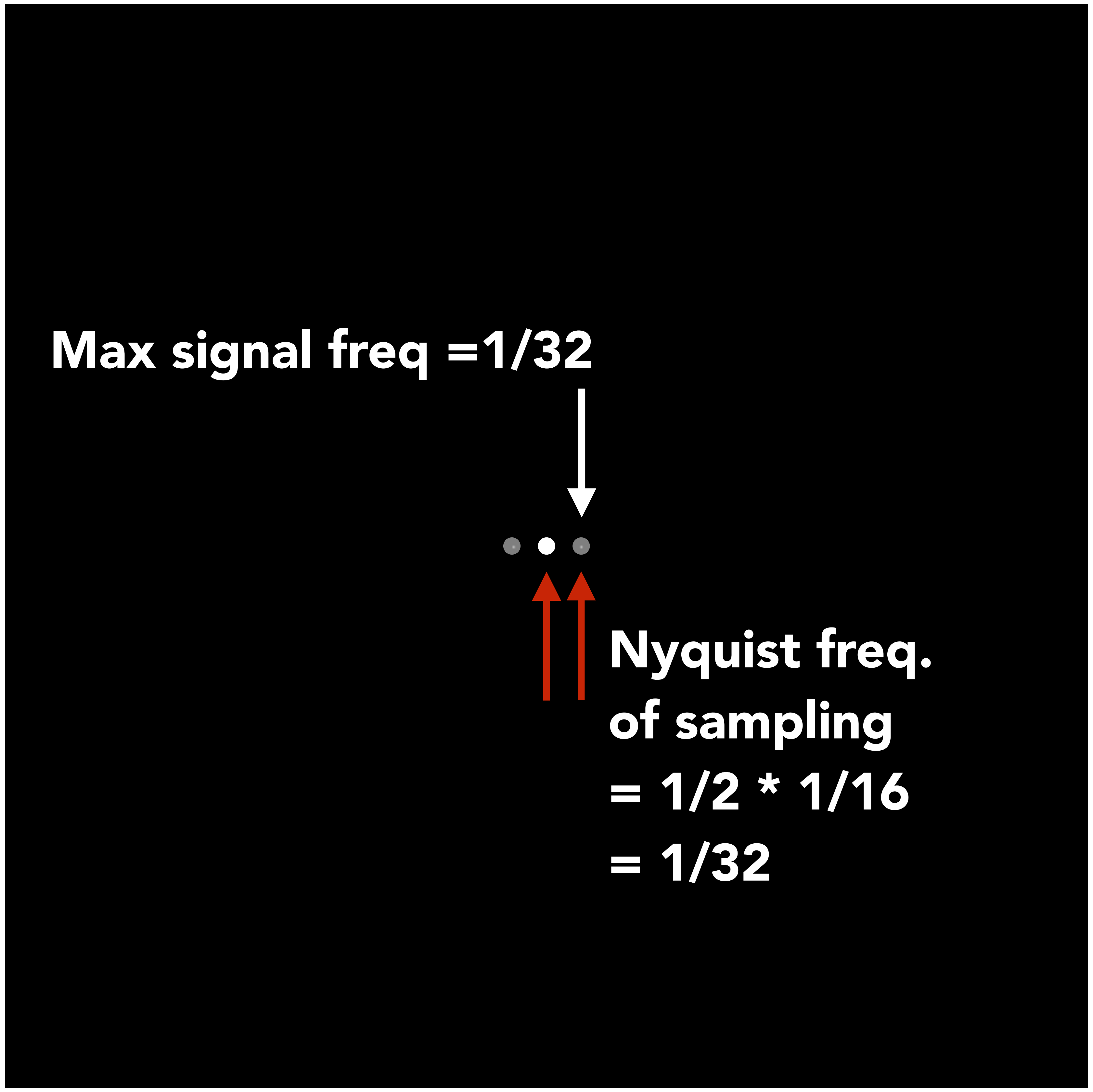
* Won't cover proof in course, see Shannon sampling theorem

Signal vs Nyquist Frequency: Example

$\sin(2\pi/32)x$ — **frequency 1/32; 32 pixels per cycle**



Spatial Domain

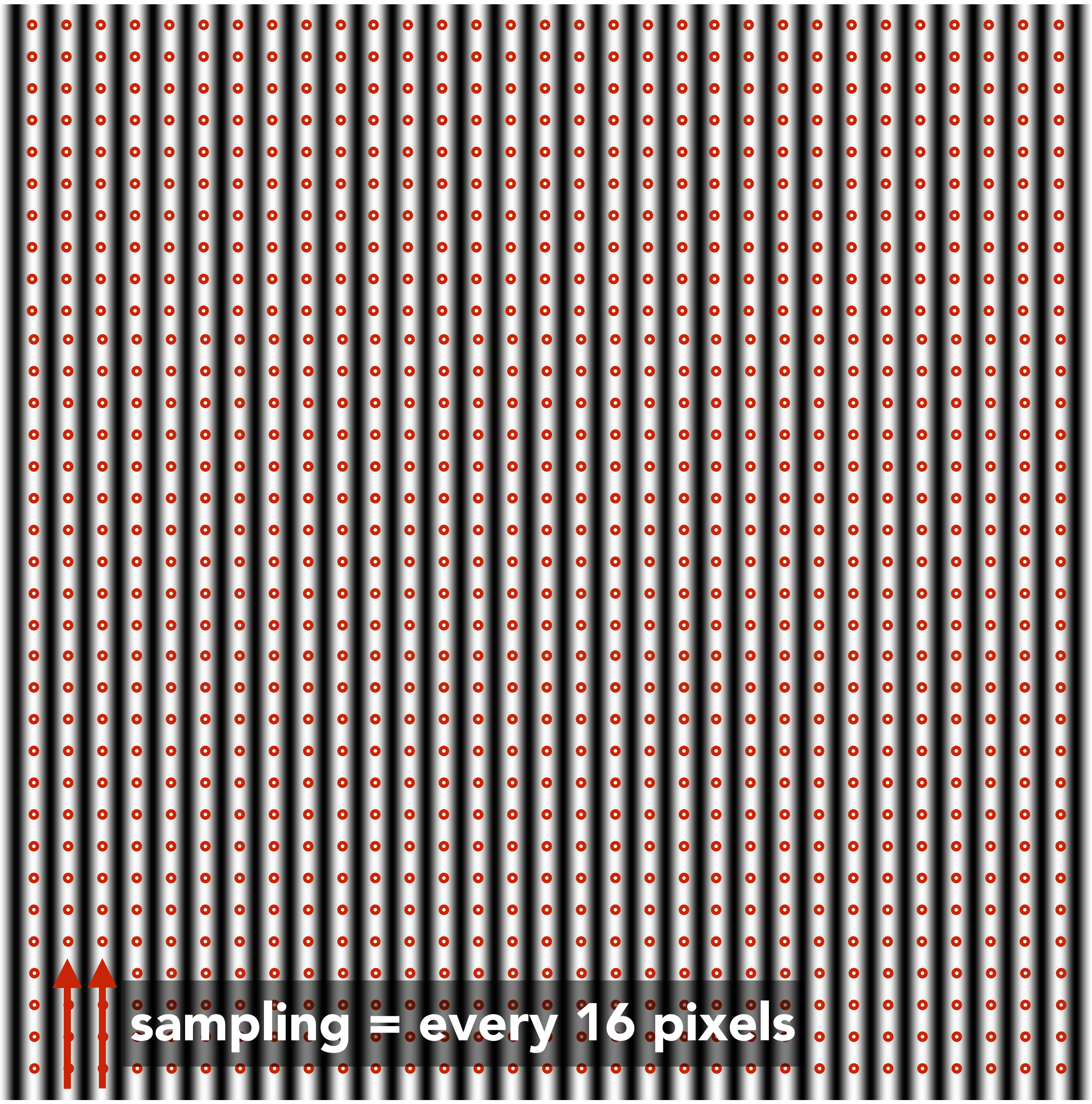


Frequency Domain

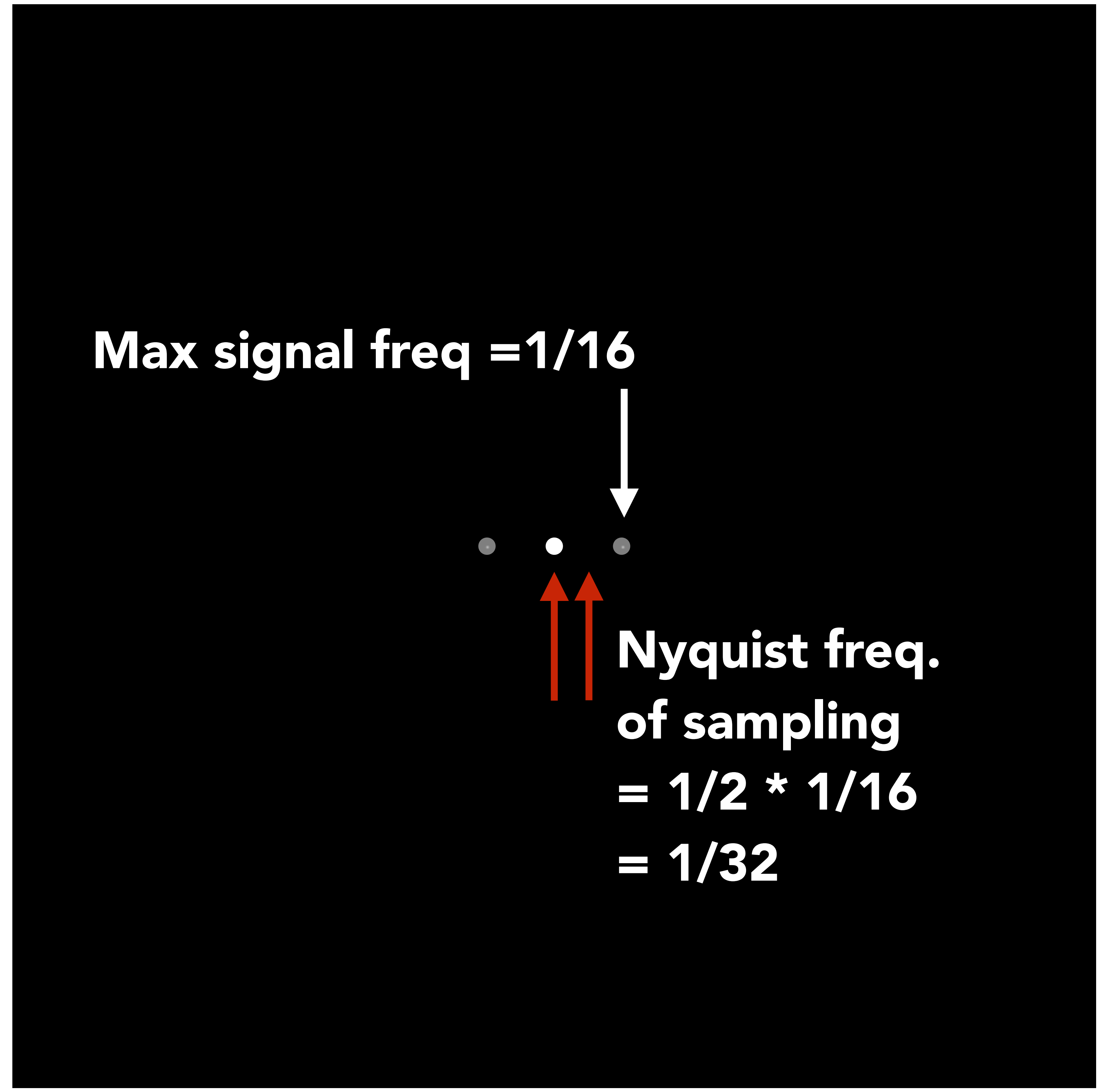
No Aliasing!

Signal vs Nyquist Frequency: Example

$\sin(2\pi/16)x$ — frequency 1/16; 16 pixels per cycle



Spatial Domain



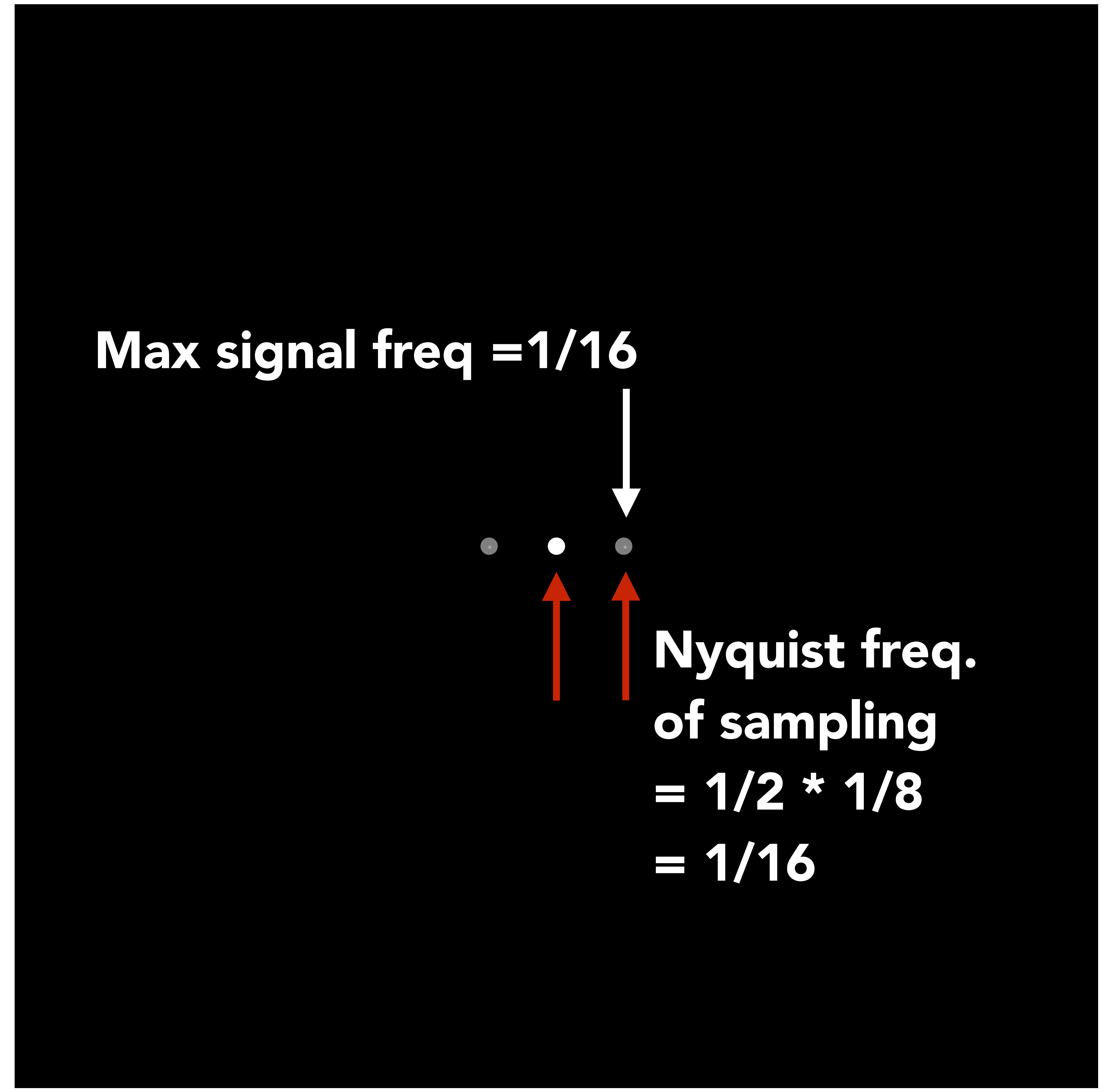
Frequency Domain

Aliasing!

Signal vs Nyquist Frequency: Example

$\sin(2\pi/16)x$ — **frequency 1/16; 16 pixels per cycle**

Spatial Domain



Frequency Domain

No Aliasing!

Visual Example:
Image Frequencies &
Nyquist Frequency

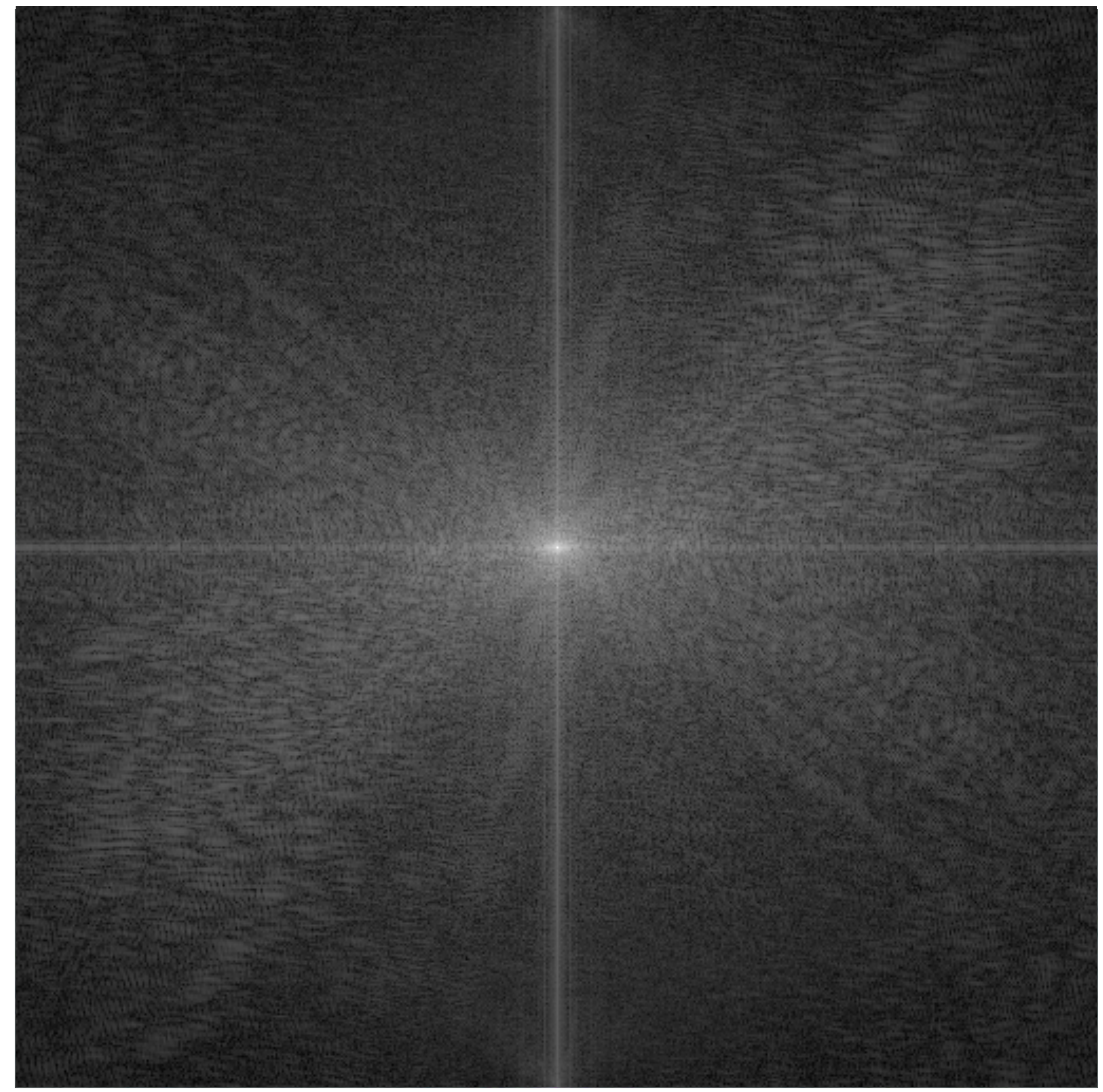
Image Frequency: Visual Example

In the following image sequence:

- Image is 512x512 pixels
- We will progressively blur the image, see how the frequency spectrum shrinks, and see what the maximum frequency is

Image Frequency: Visual Example

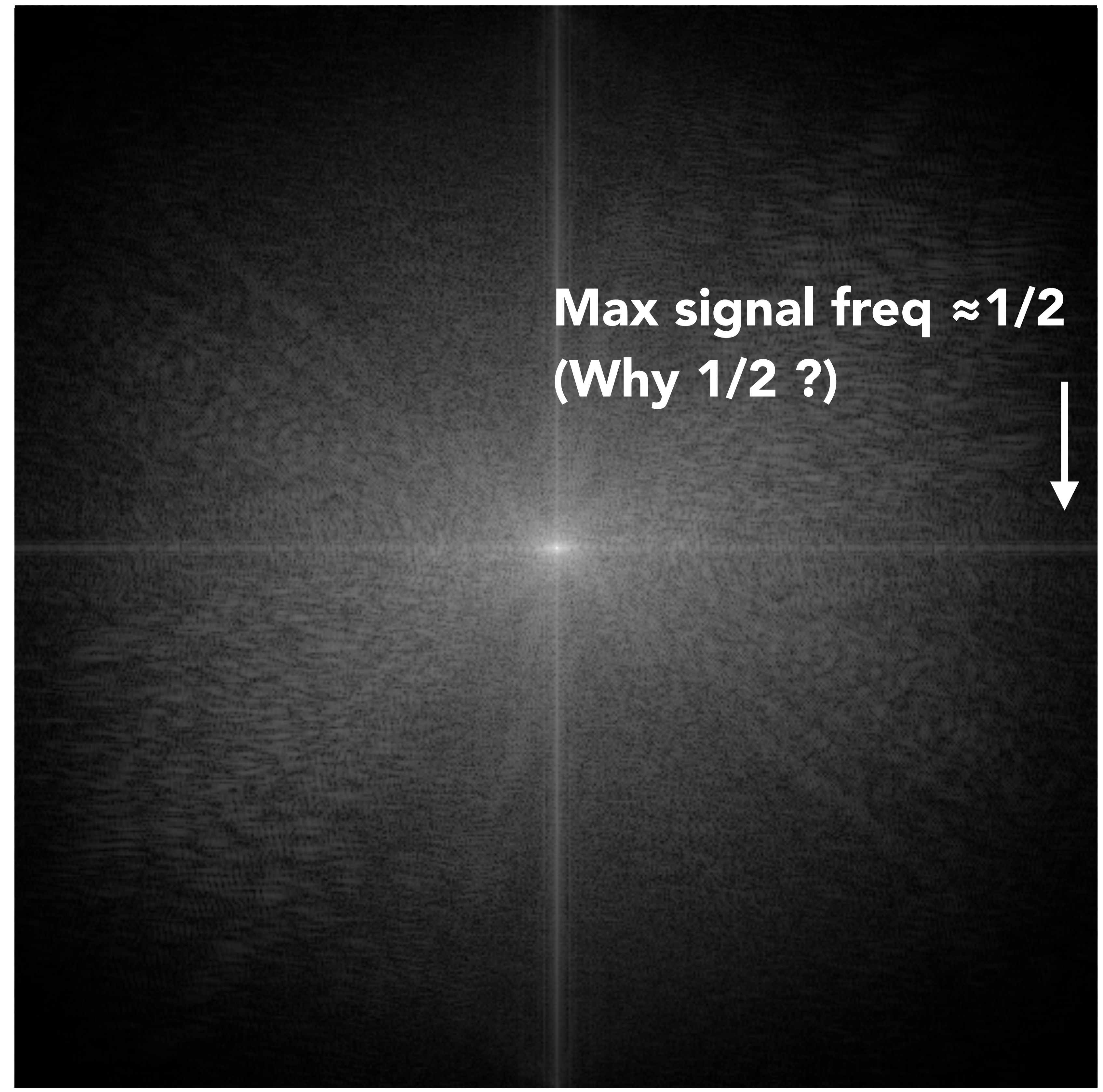
Spatial Domain



Frequency Domain

Image Frequency: Visual Example

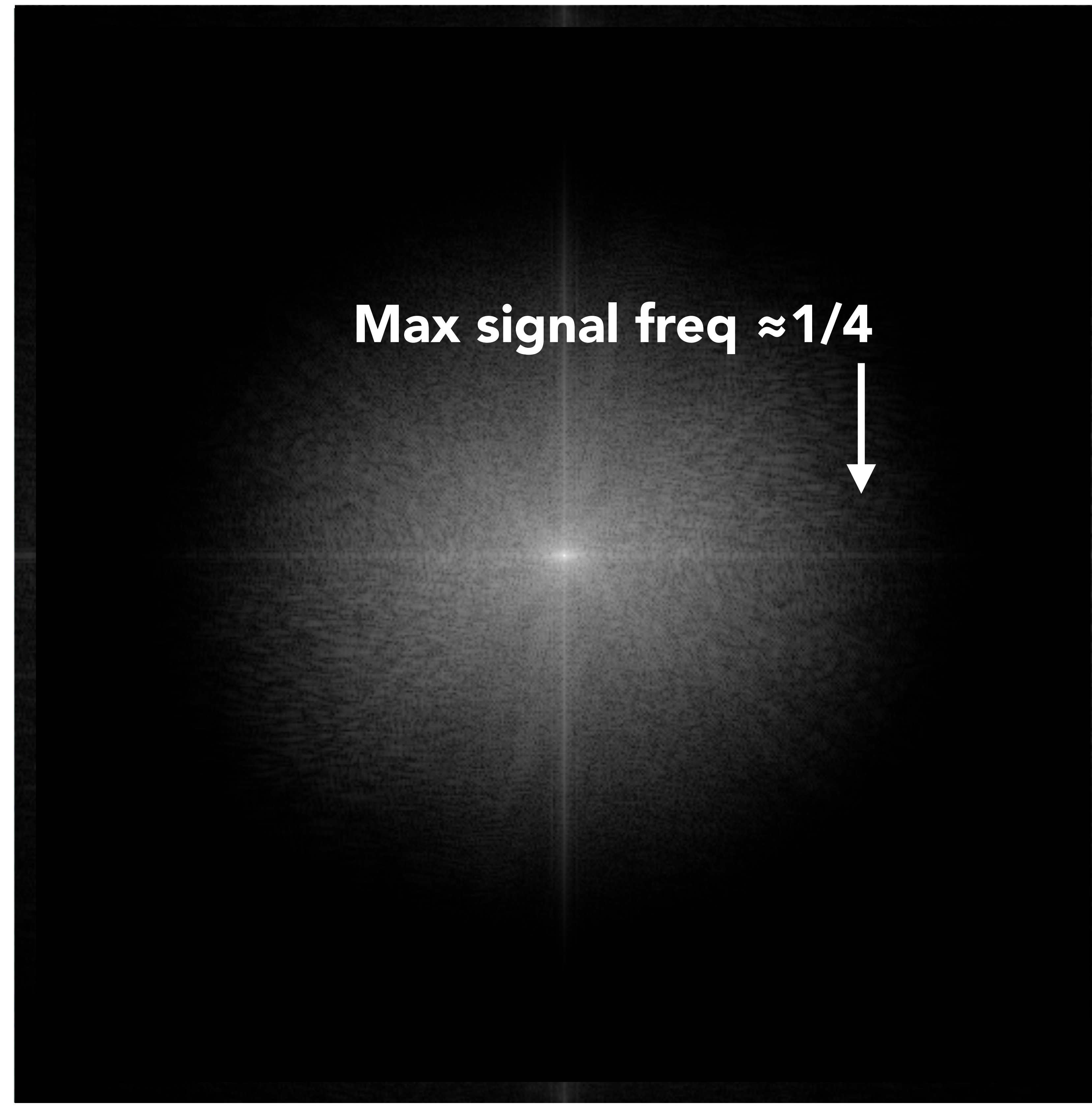
Spatial Domain



Frequency Domain

Image Frequency: Visual Example

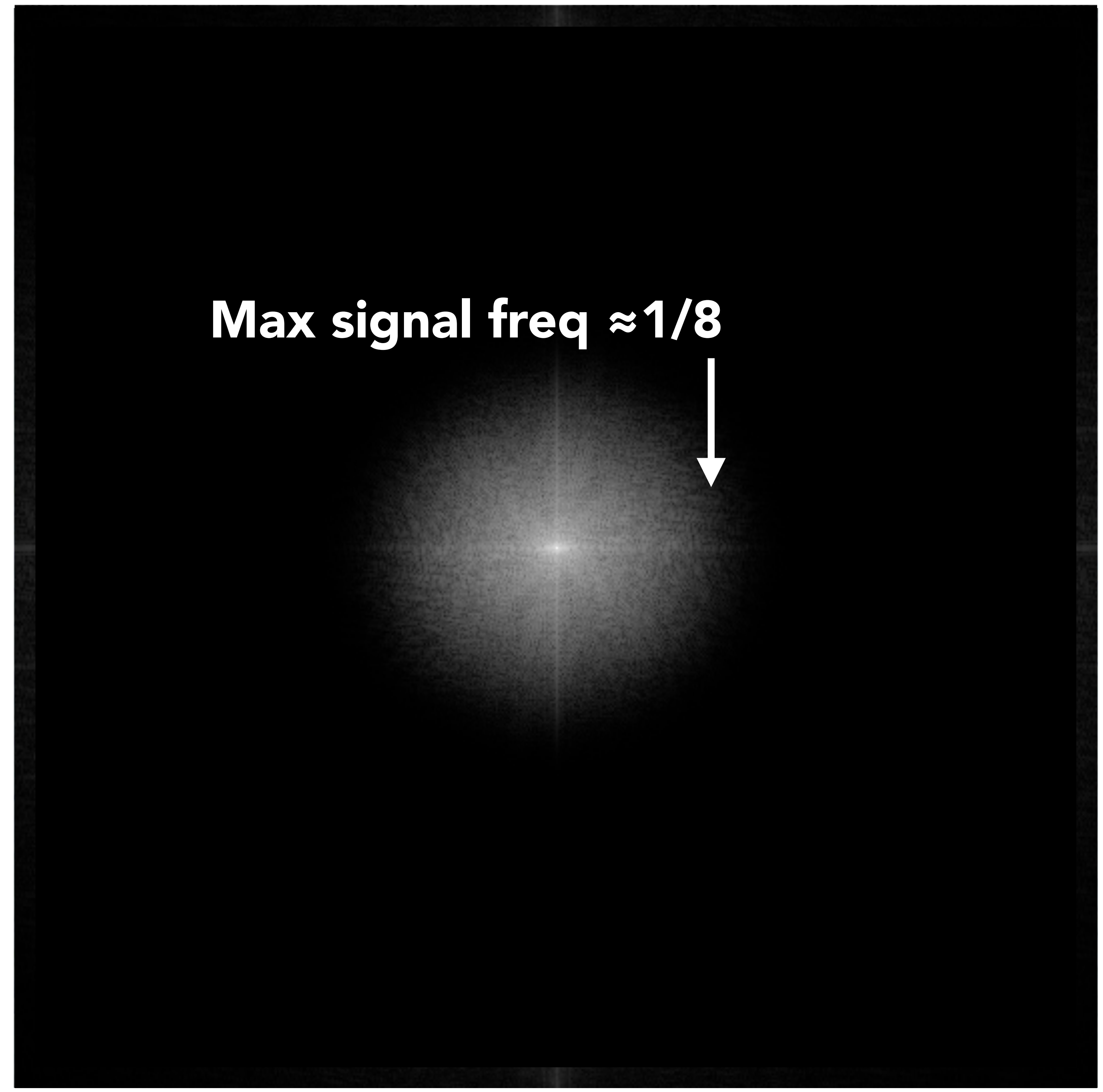
Spatial Domain



Frequency Domain

Image Frequency: Visual Example

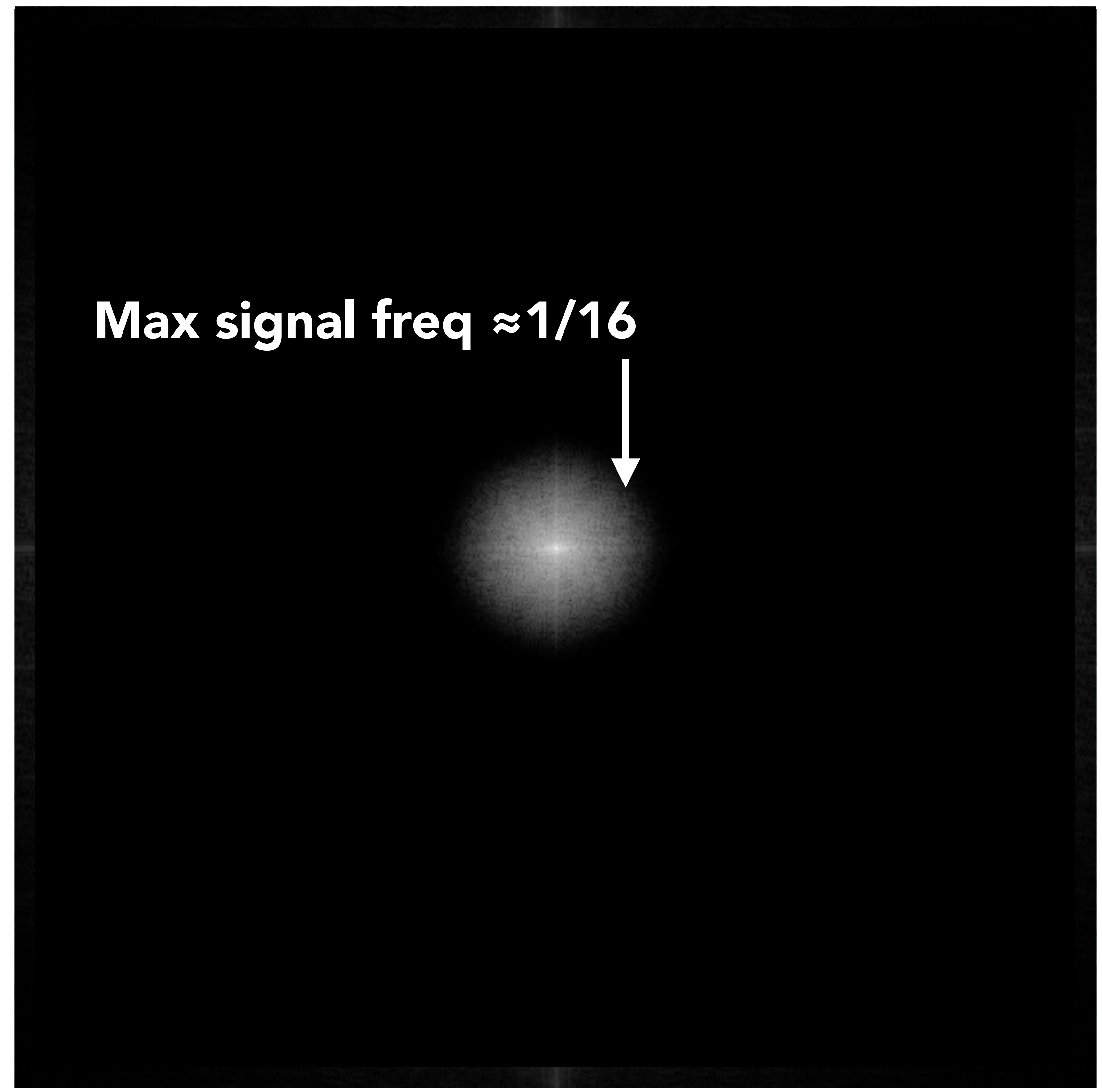
Spatial Domain



Frequency Domain

Image Frequency: Visual Example

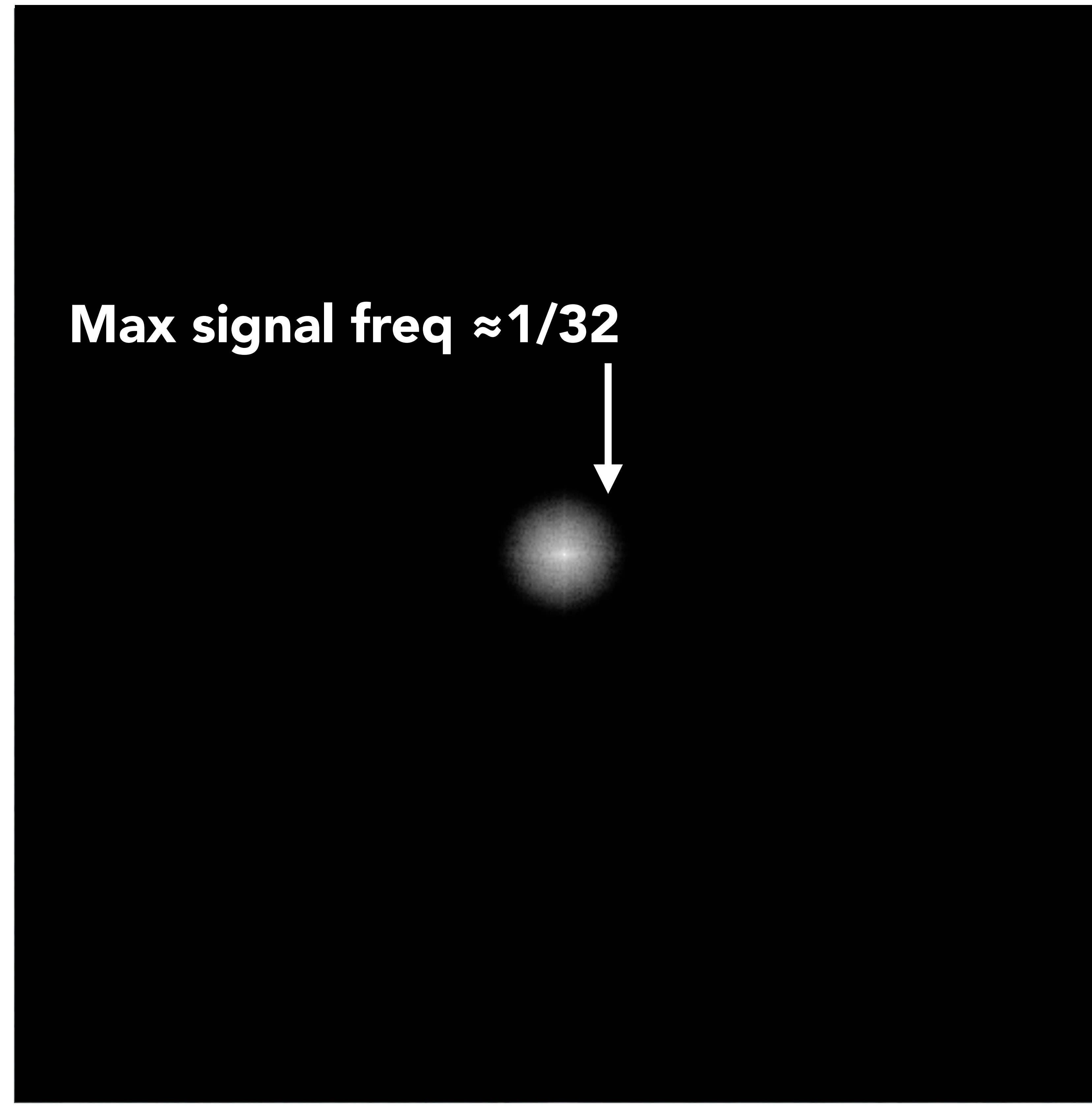
Spatial Domain



Frequency Domain

Image Frequency: Visual Example

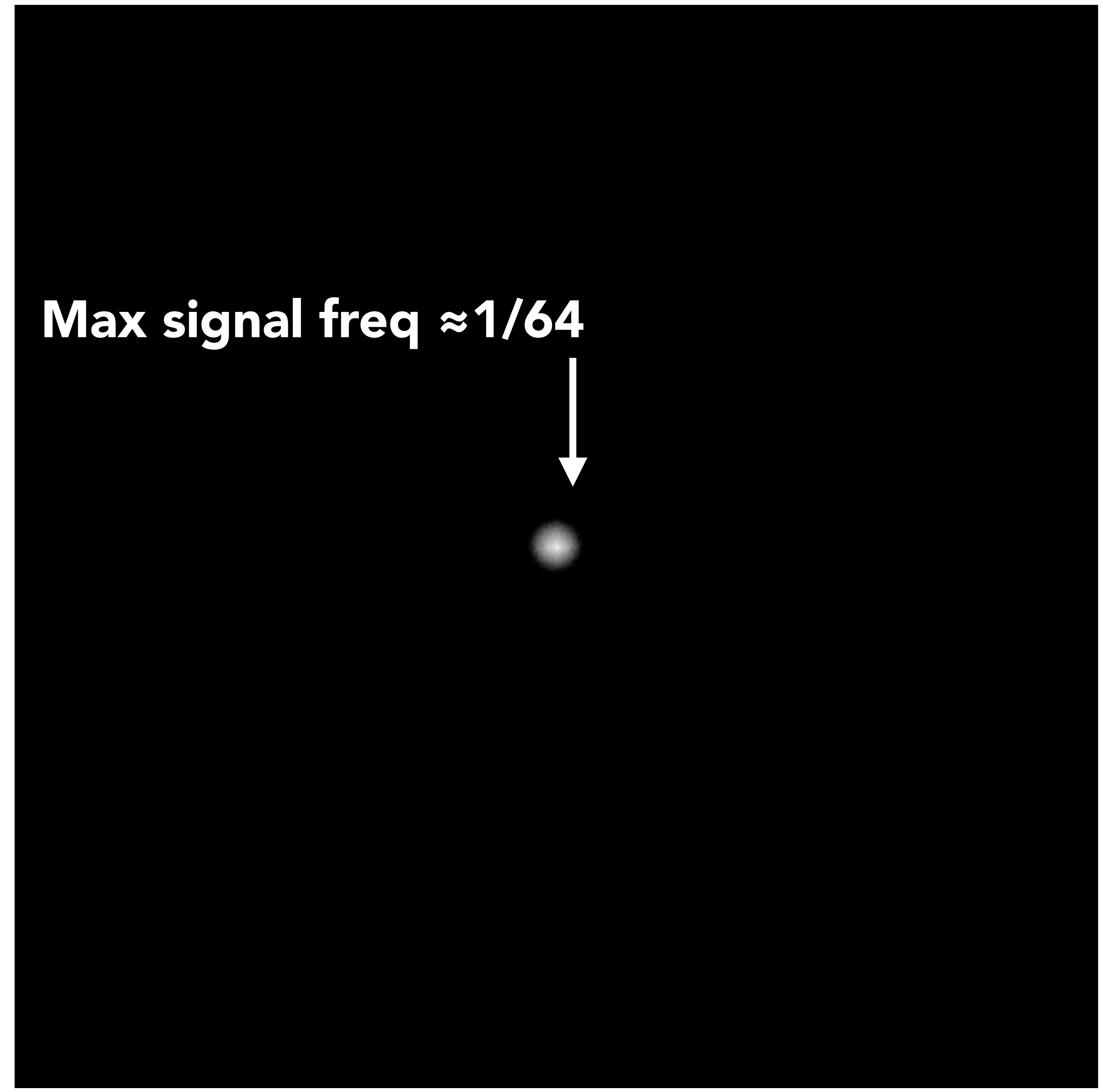
Spatial Domain



Frequency Domain

Image Frequency: Visual Example

Spatial Domain



Frequency Domain

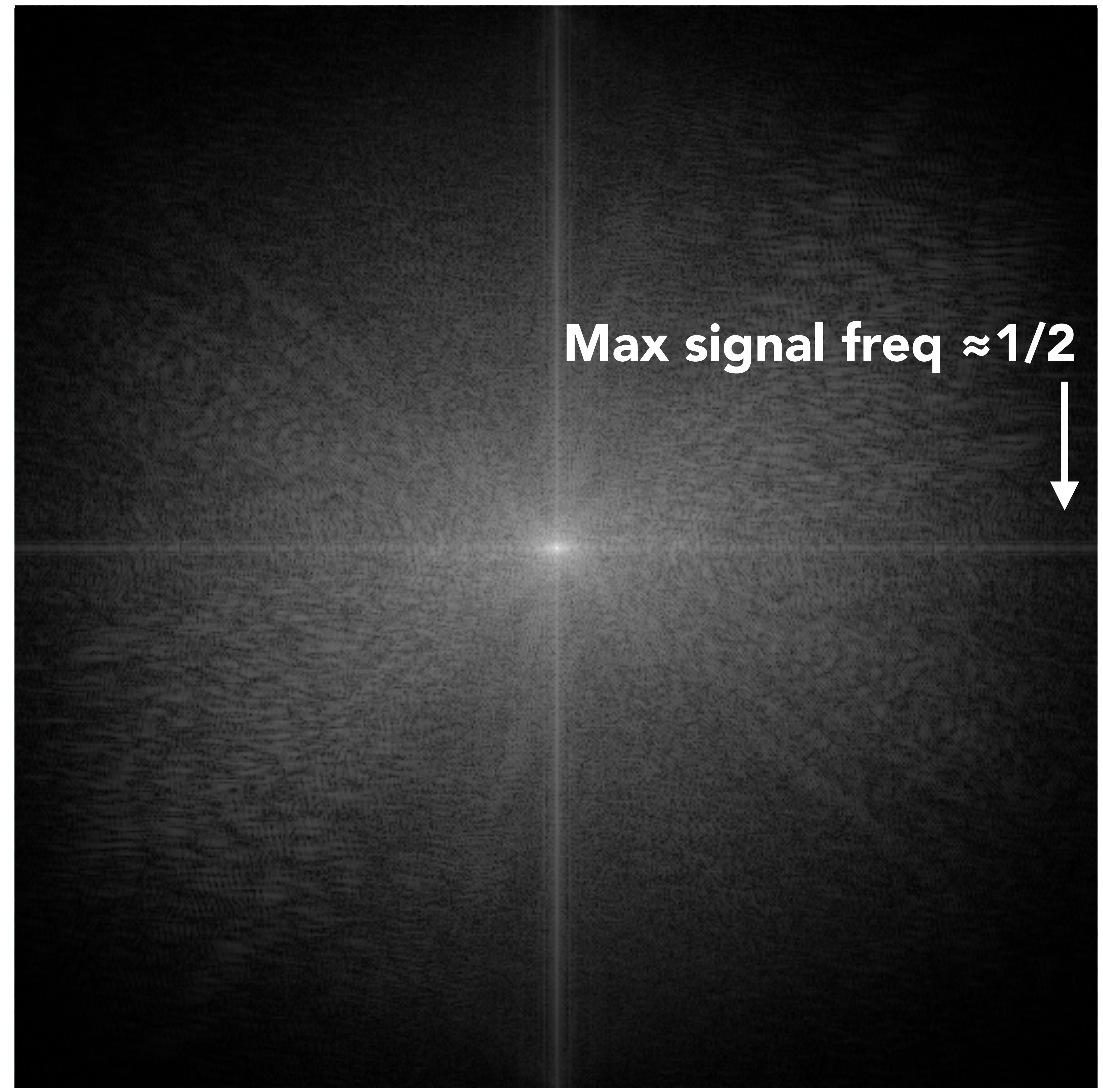
Nyquist Frequency: Visual Example

In next sequence:

- Visualize sampling an image every 16 pixels
- Visualize when image is blurred enough that image frequencies match Nyquist frequency (no aliasing)

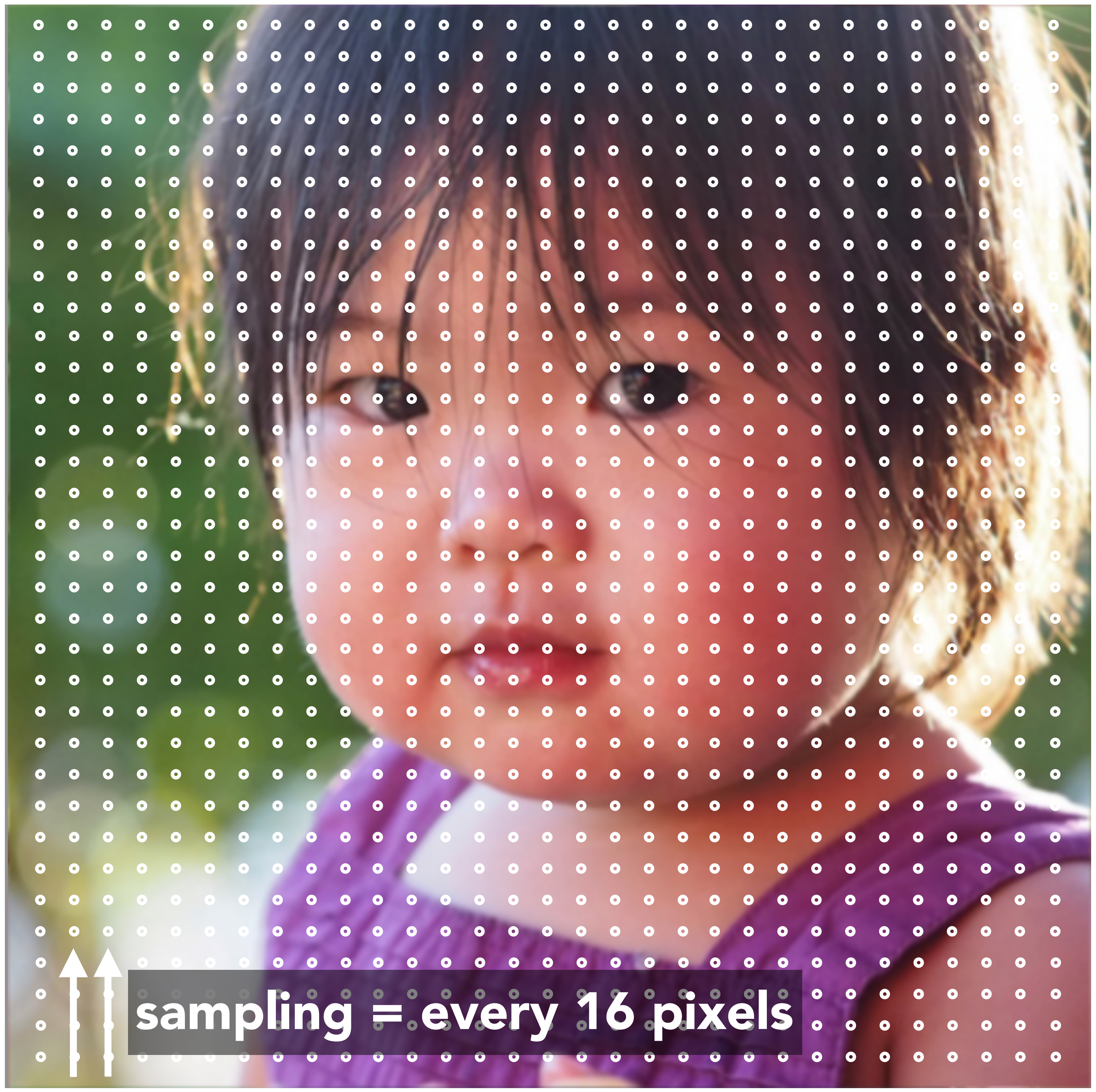
Nyquist Frequency: Visual Example

Spatial Domain

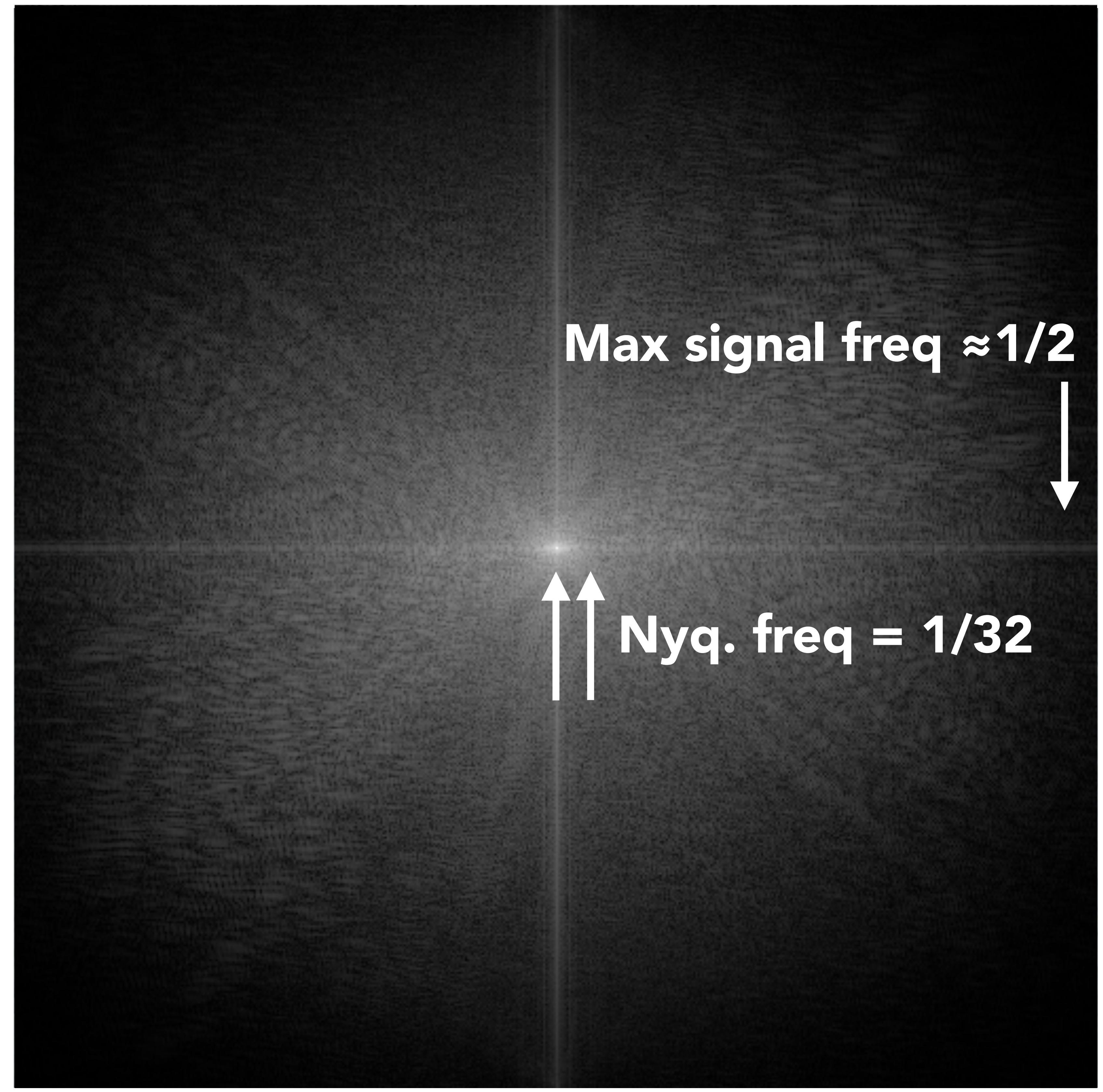


Frequency Domain

Nyquist Frequency: Visual Example

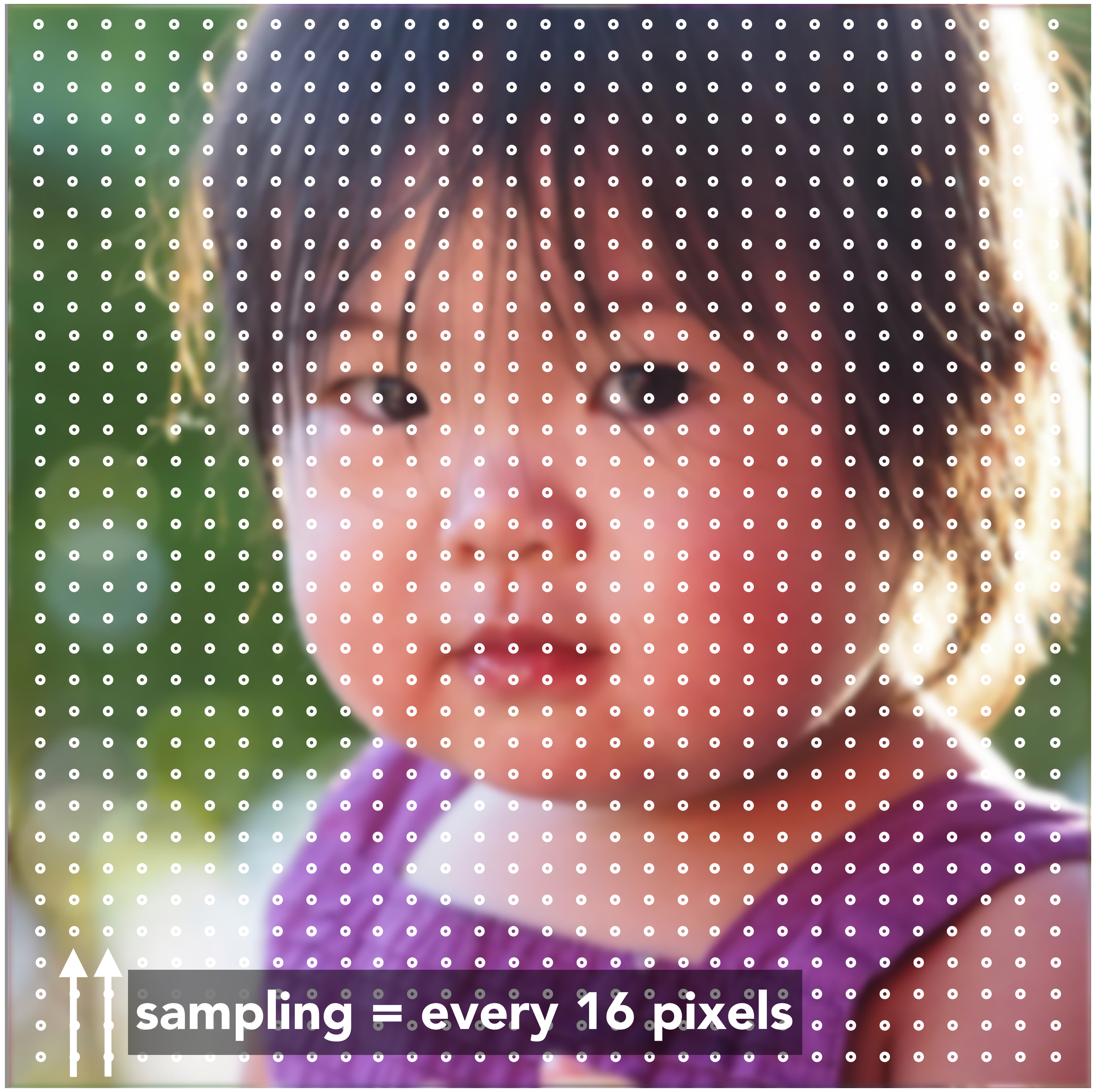


Spatial Domain

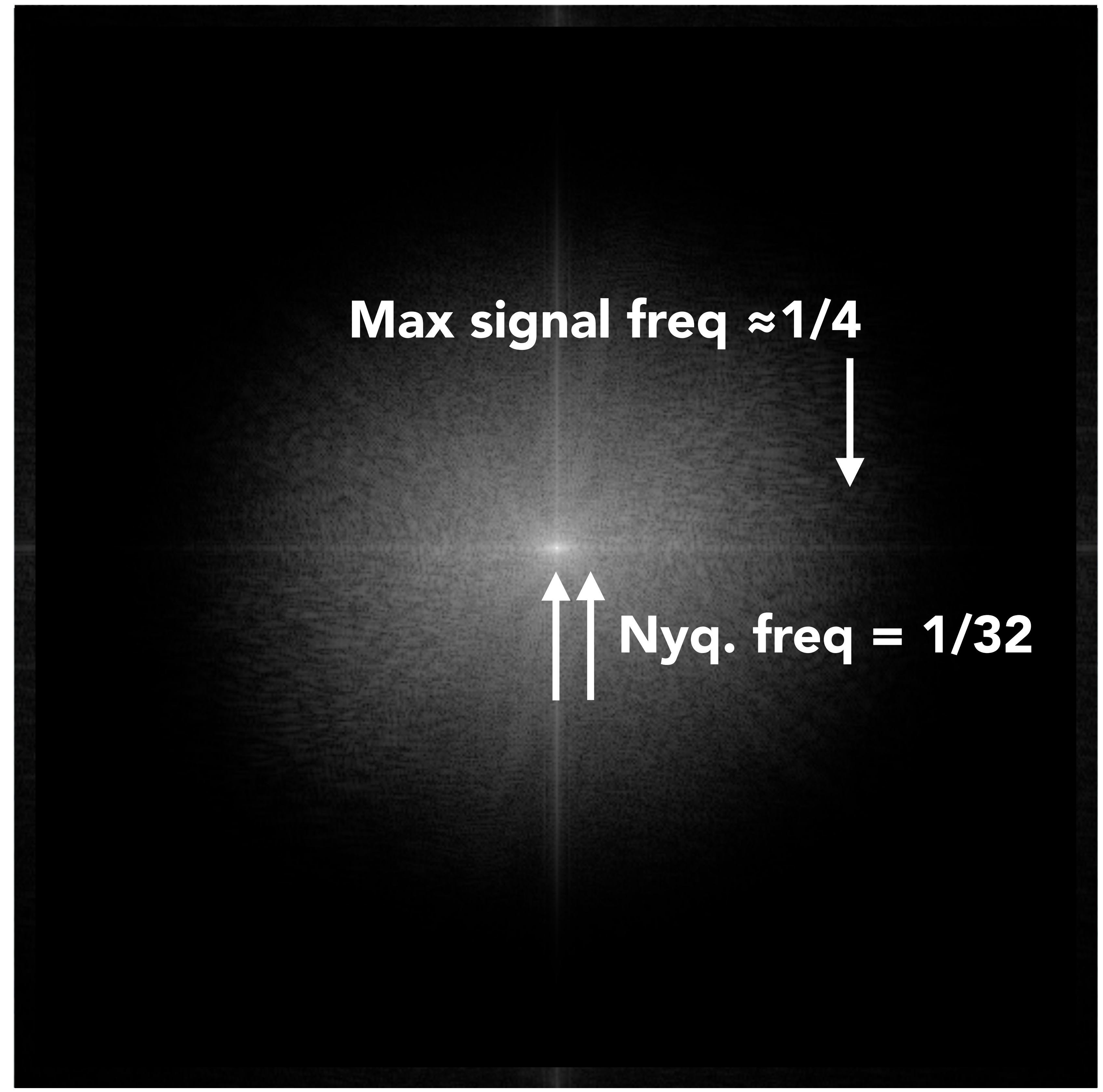


Frequency Domain

Nyquist Frequency: Visual Example

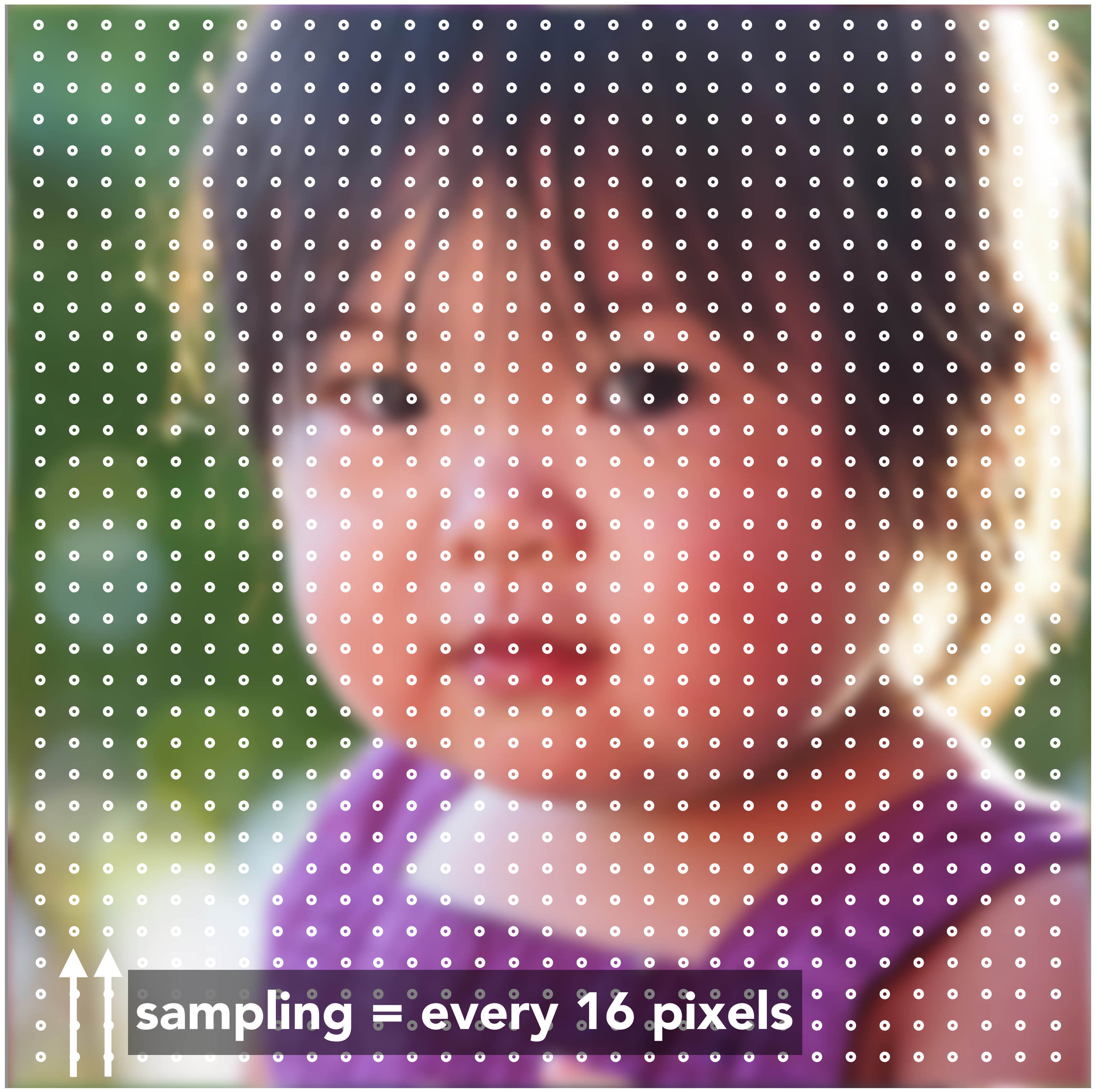


Spatial Domain

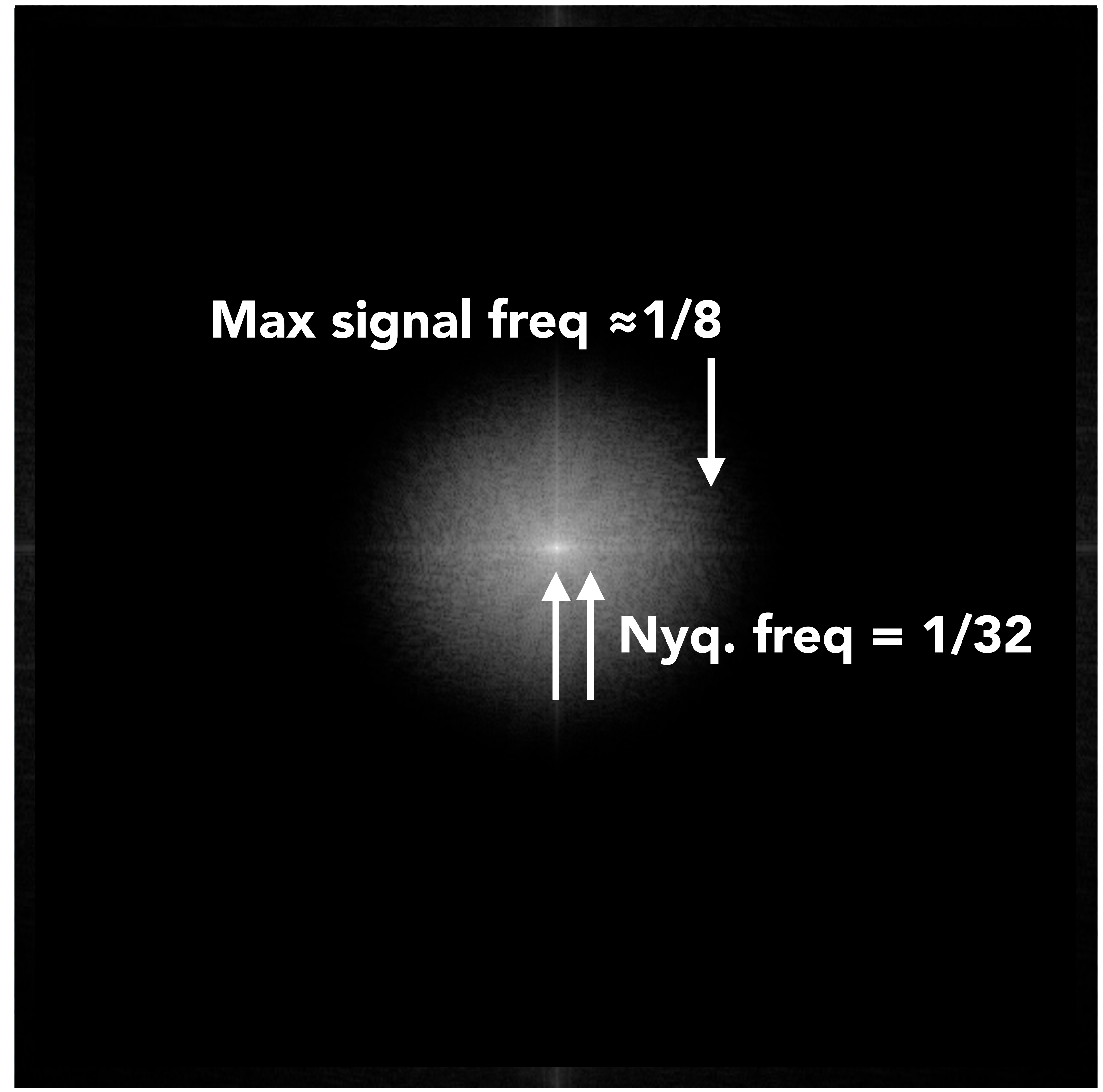


Frequency Domain

Nyquist Frequency: Visual Example

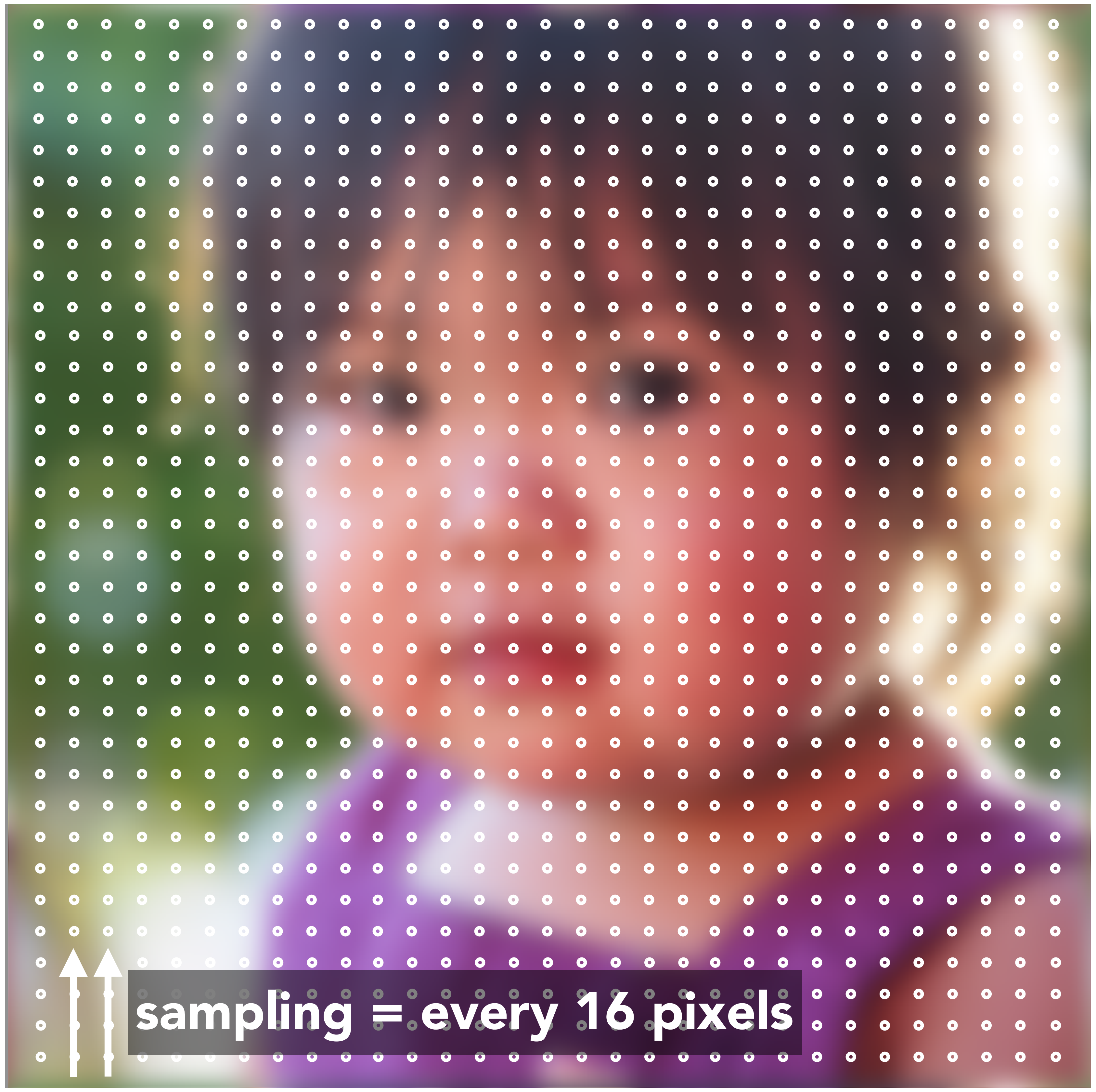


Spatial Domain

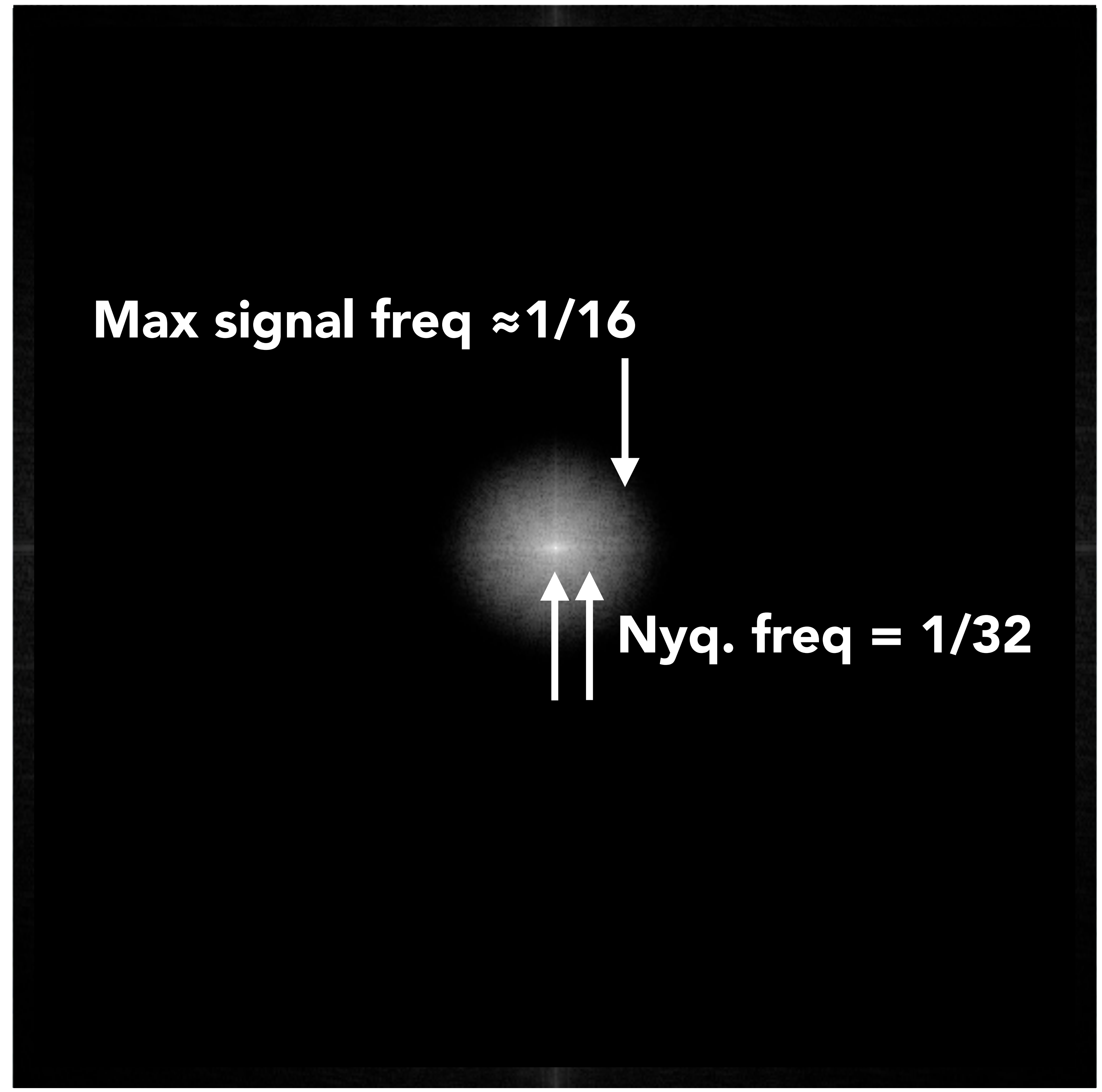


Frequency Domain

Nyquist Frequency: Visual Example

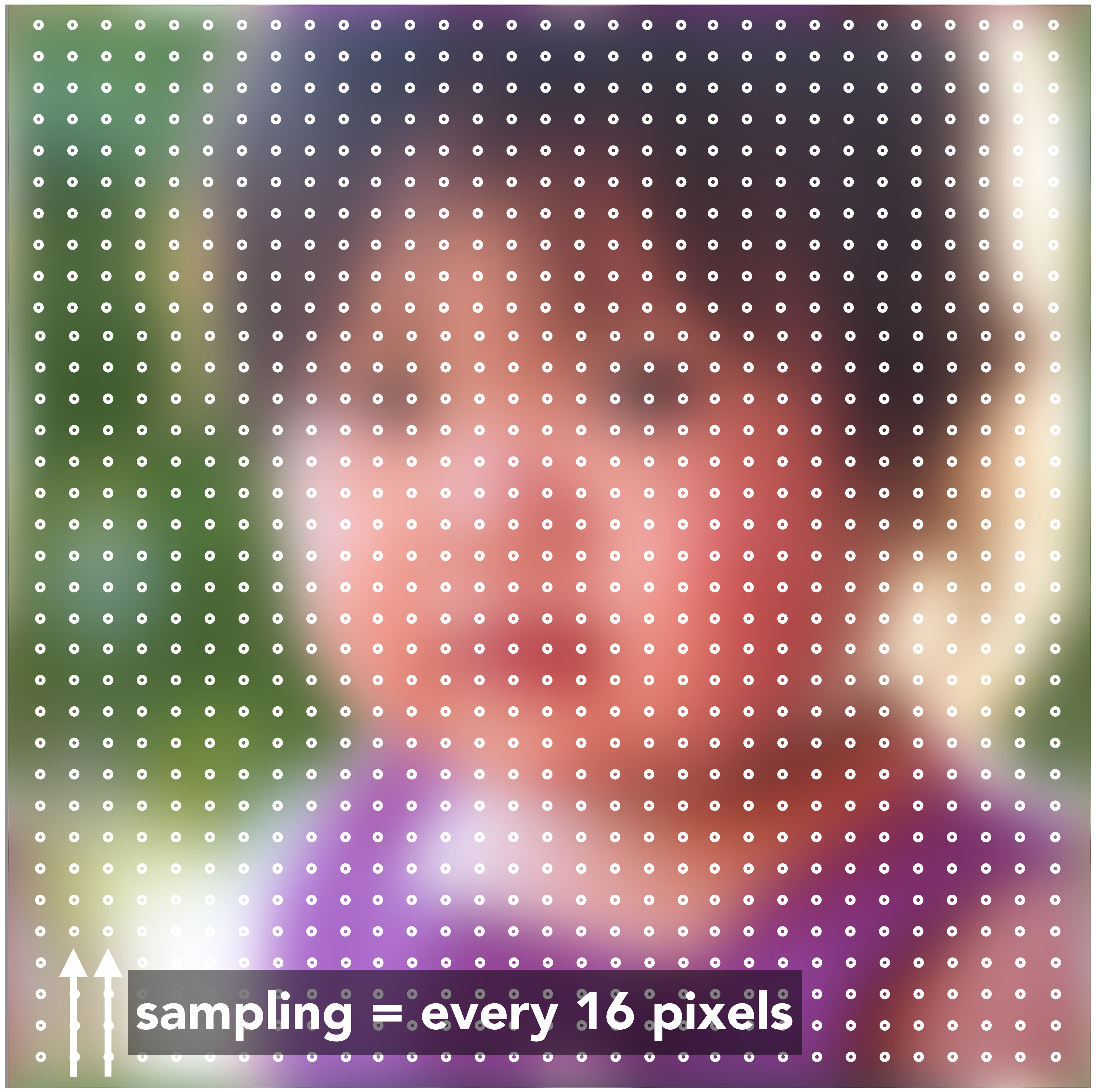


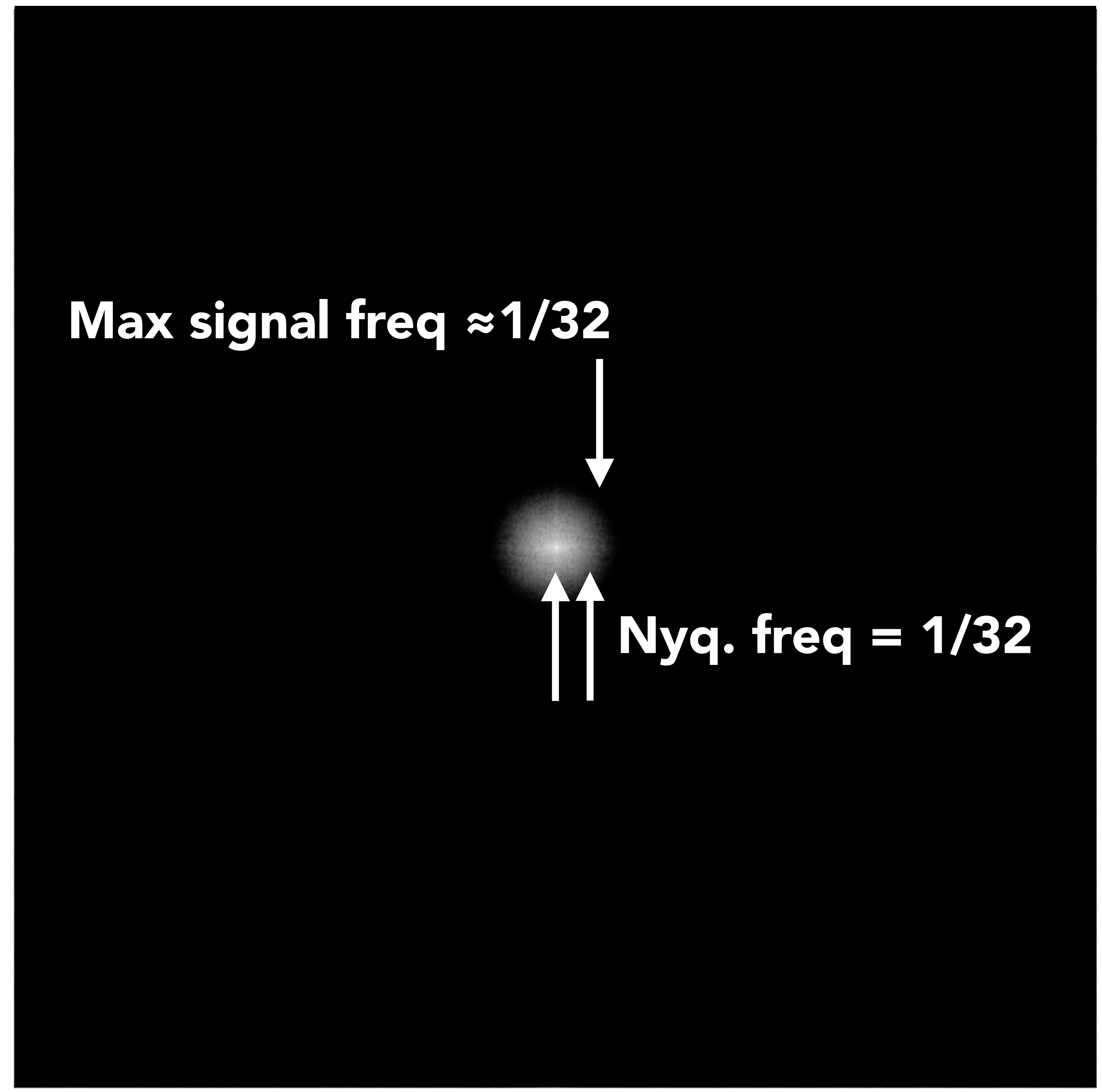
Spatial Domain



Frequency Domain

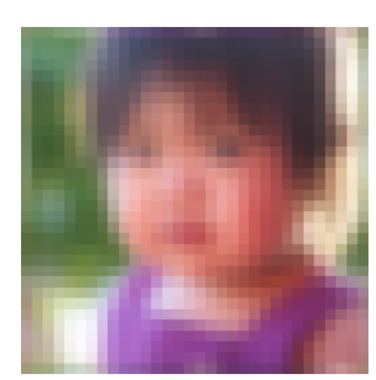
Nyquist Frequency: Visual Example



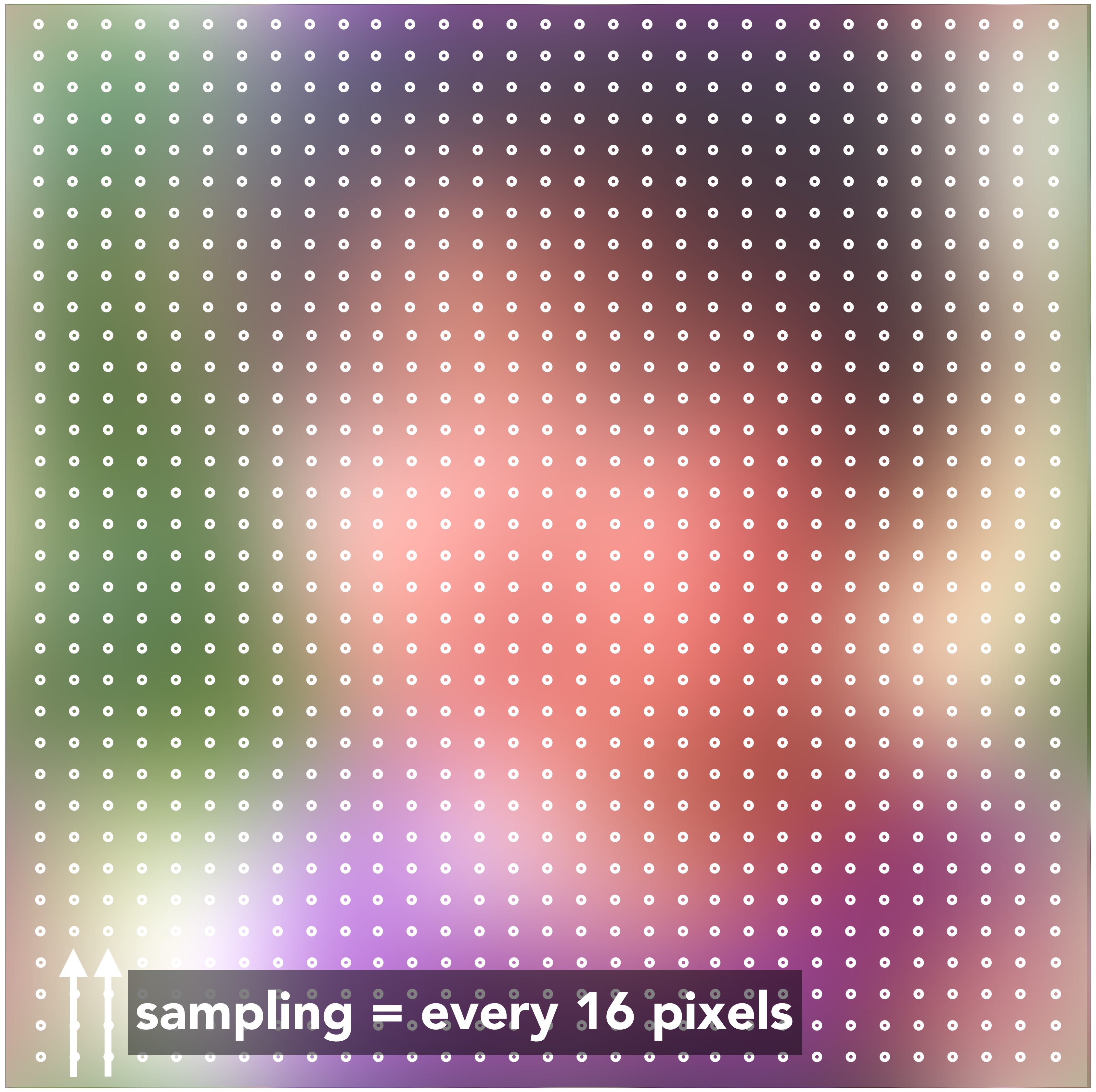


Spatial Domain

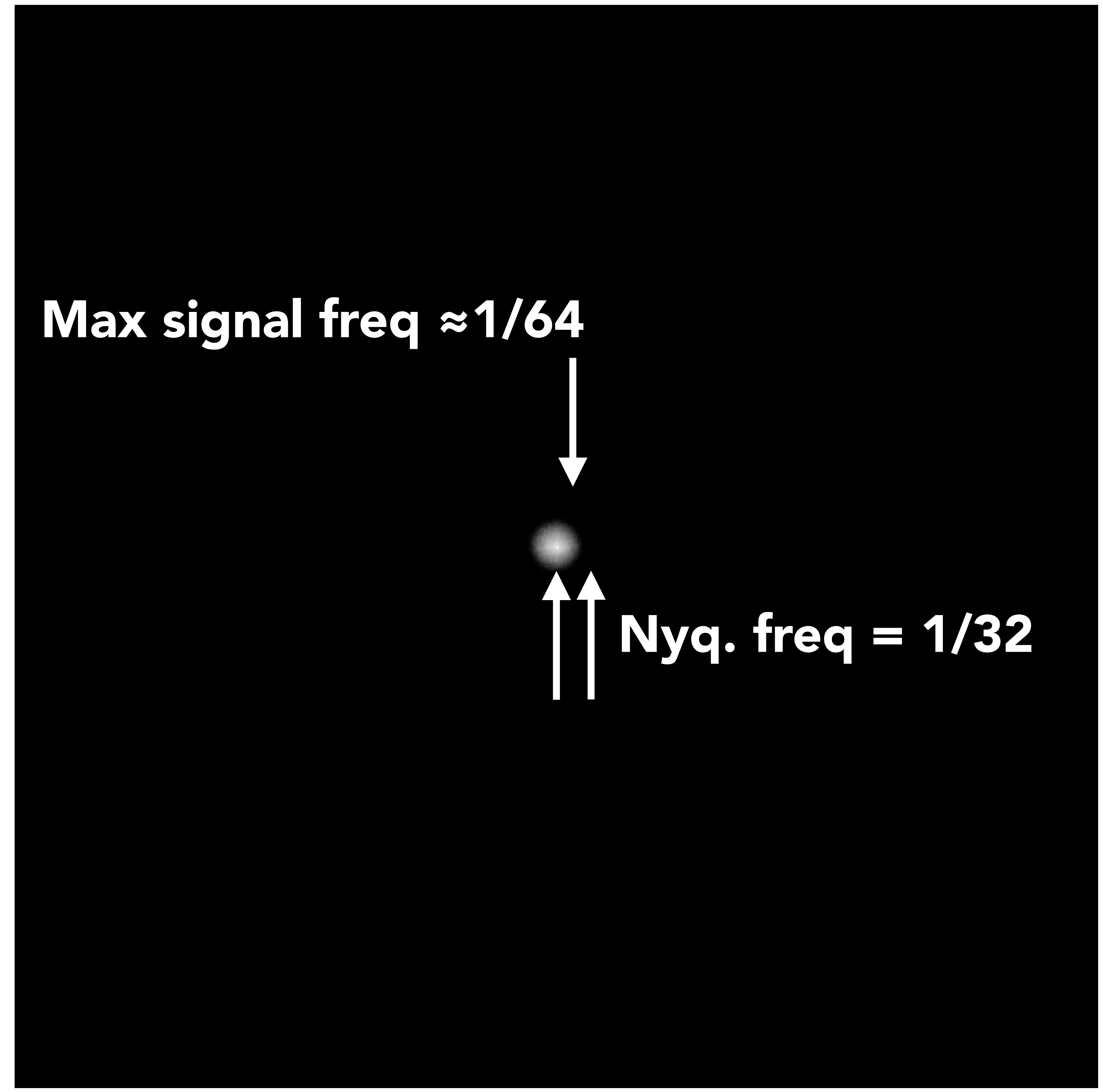
Frequency Domain



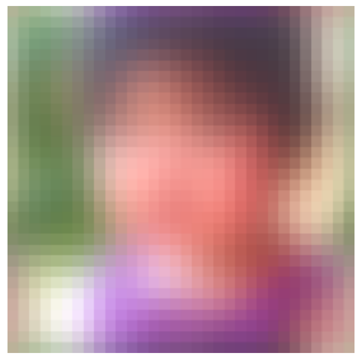
Nyquist Frequency: Visual Example



Spatial Domain



Frequency Domain



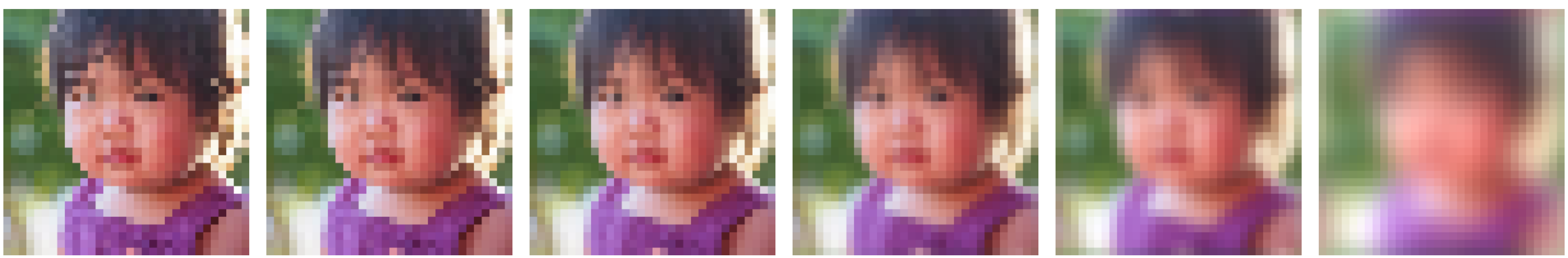
Nyquist Frequency: Visual Example

Recap:

- Filter (blur) original image to reduce maximum signal frequency
- Create low-resolution image by sampling only every 16 pixels
  - (Sampling frequency is 1/16, and Nyquist frequency is 1/32)

Less blur ⟵⟶ **More blur**

1 **2** **3** **4** **5** **6**



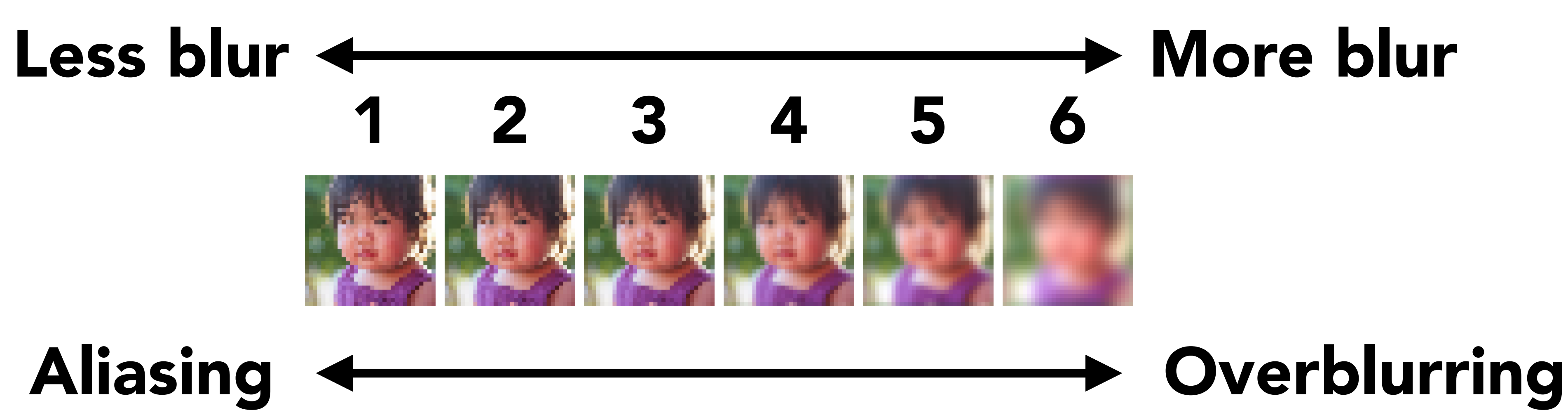
Nyquist

Aliasing ⟵ **Which do you prefer?** ⟶ **Overblurring**

Nyquist Frequency: Visual Example

Recap:

- Filter (blur) original image to reduce maximum signal frequency
- Create low-resolution image by sampling only every 16 pixels
  - (Sampling frequency is 1/16, and Nyquist frequency is 1/32)



Which do you prefer?

Aliasing and over blurring can be objectionable even at small image sizes

Antialiasing

Reminder: Nyquist Theorem

Theorem: We get no aliasing from frequencies in the signal that are less than the Nyquist frequency (which is defined as half the sampling frequency)

Consequence: sampling at twice the highest frequency in the signal will eliminate aliasing

How Can We Reduce Aliasing Error?

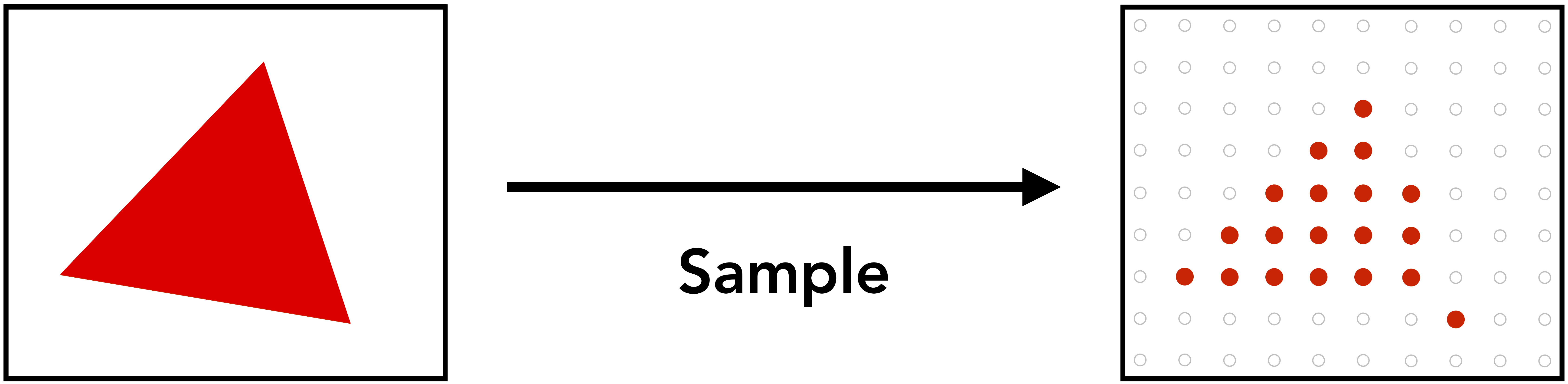
Increase sampling rate (increase Nyquist frequency)

- Higher resolution displays, sensors, framebuffers…
- But: costly & may need very high resolution

Antialiasing

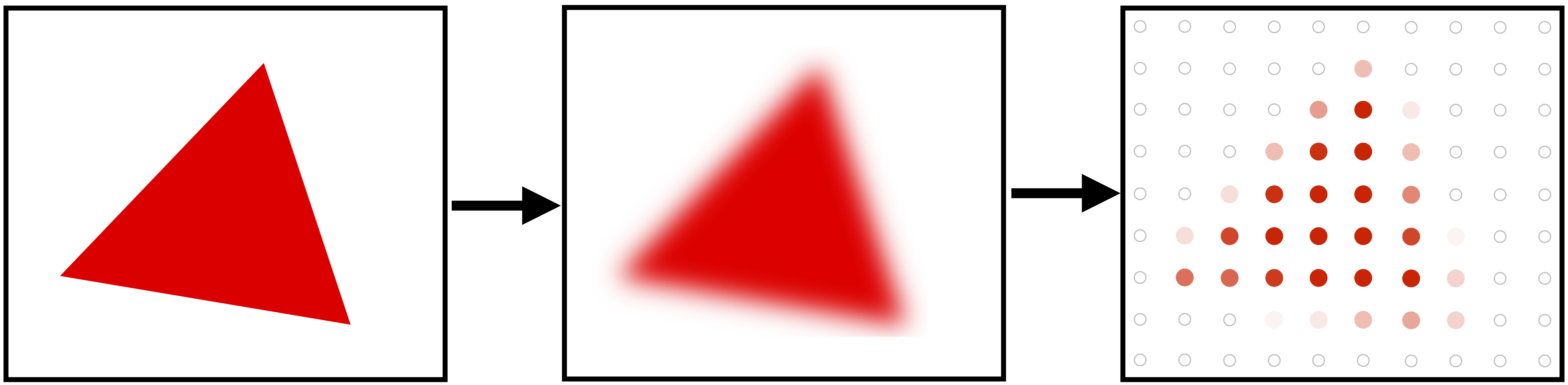
- Simple idea: remove (or reduce) signal frequencies above the Nyquist frequency before sampling
- How? Filter out high frequencies before sampling.

Regular Sampling



Note jaggies in rasterized triangle
where pixel values are pure red or white

Antialiased Sampling



Pre-Filter

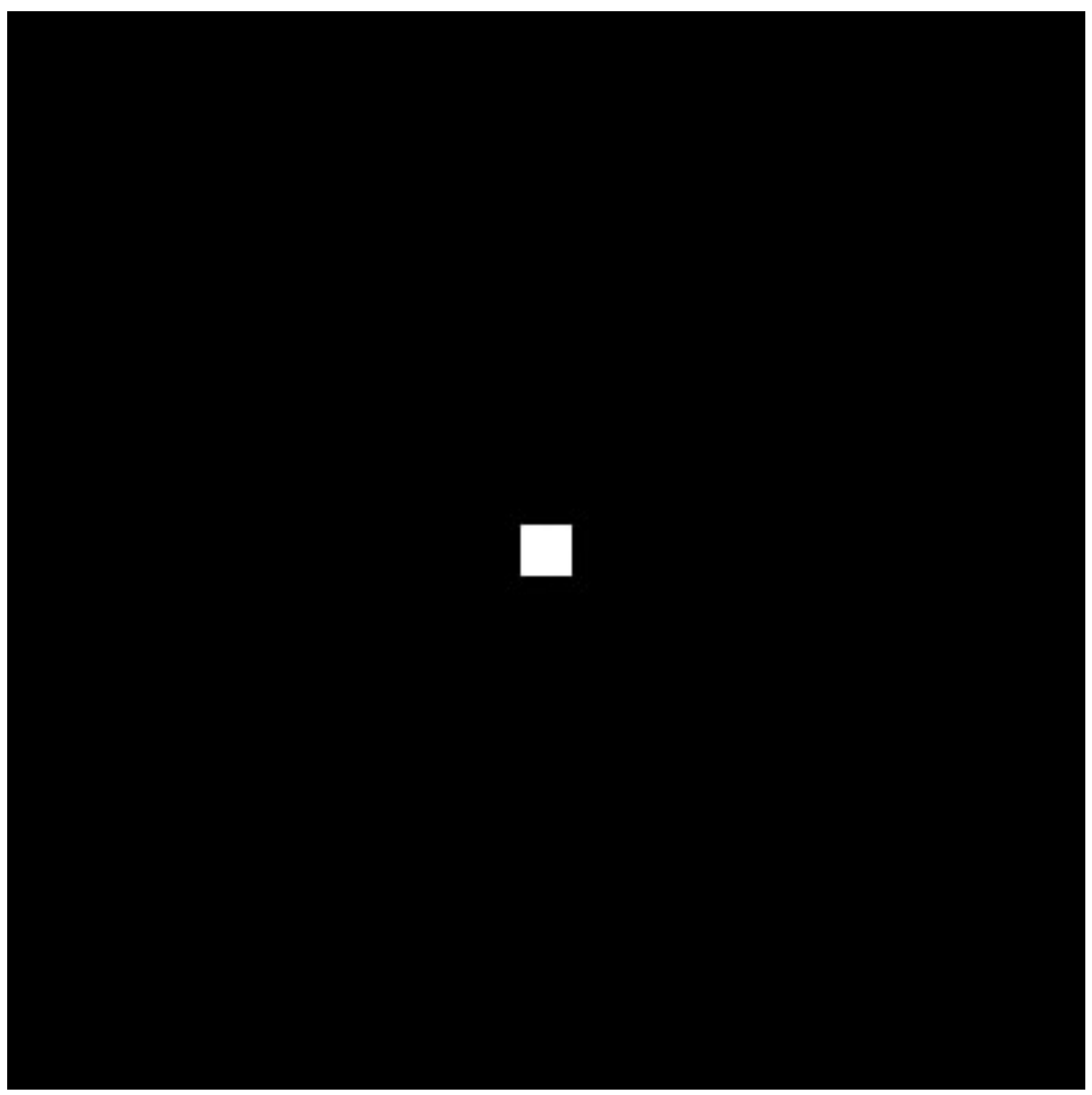
(remove frequencies above Nyquist)

Sample

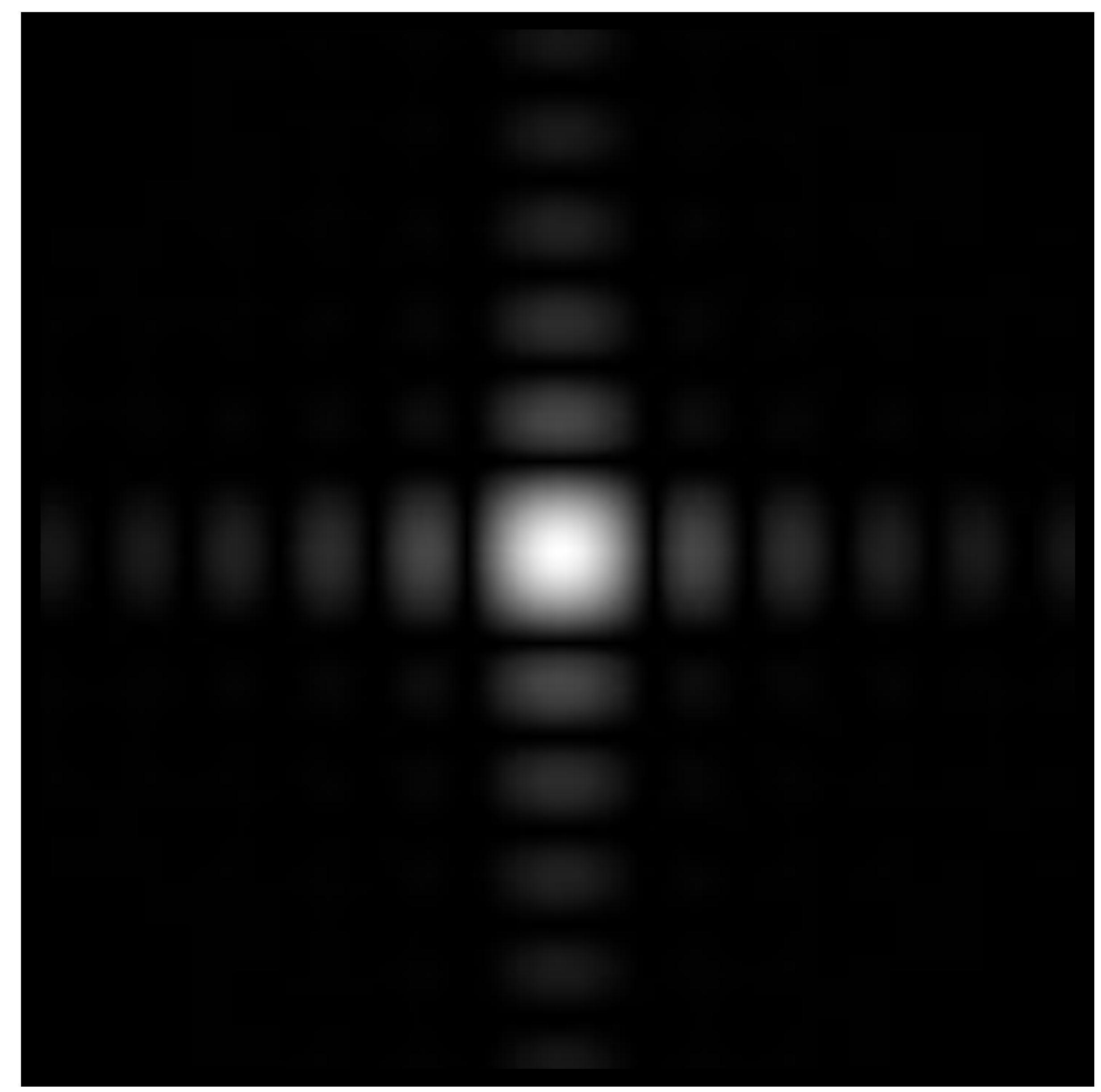
Note antialiased edges in rasterized triangle where pixel values take intermediate values

A Practical Pre-Filter

A 1 pixel-width box filter will attenuate frequencies whose period is less than or equal to 1 pixel-width



Spatial Domain



Frequency Domain

This is practical to implement — why?

Antialiasing By Averaging Values in Pixel Area

Convince yourself the following are the same:

Option 1:

- Convolve f(x,y) by a 1-pixel box-blur
- Then sample at every pixel

Option 2:

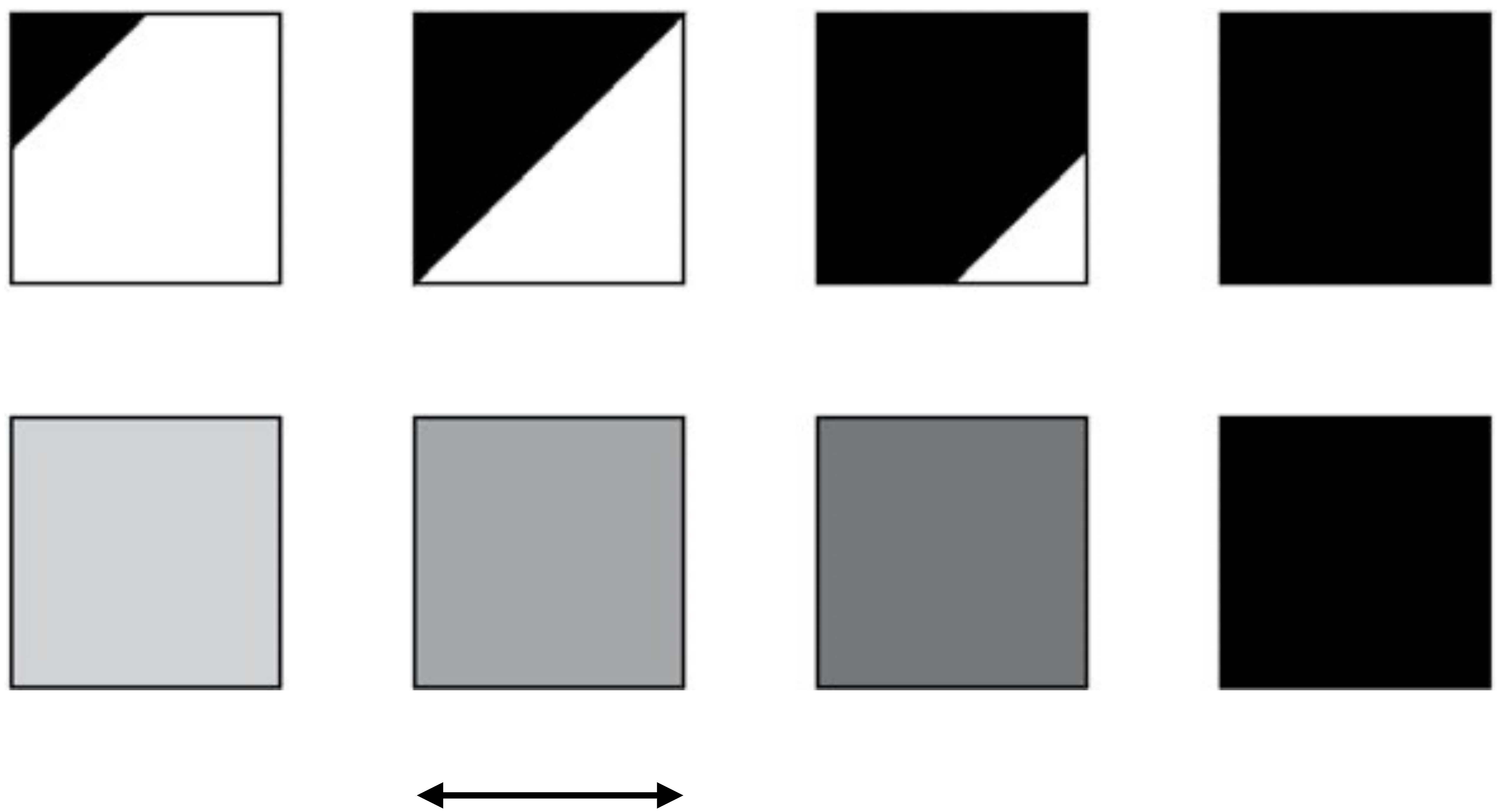
- Compute the average value of f(x,y) in the pixel

Antialiasing by Computing Average Pixel Value

In rasterizing one triangle, the average value inside a pixel area of f(x,y) = inside(triangle,x,y) is equal to the area of the pixel covered by the triangle.

Original

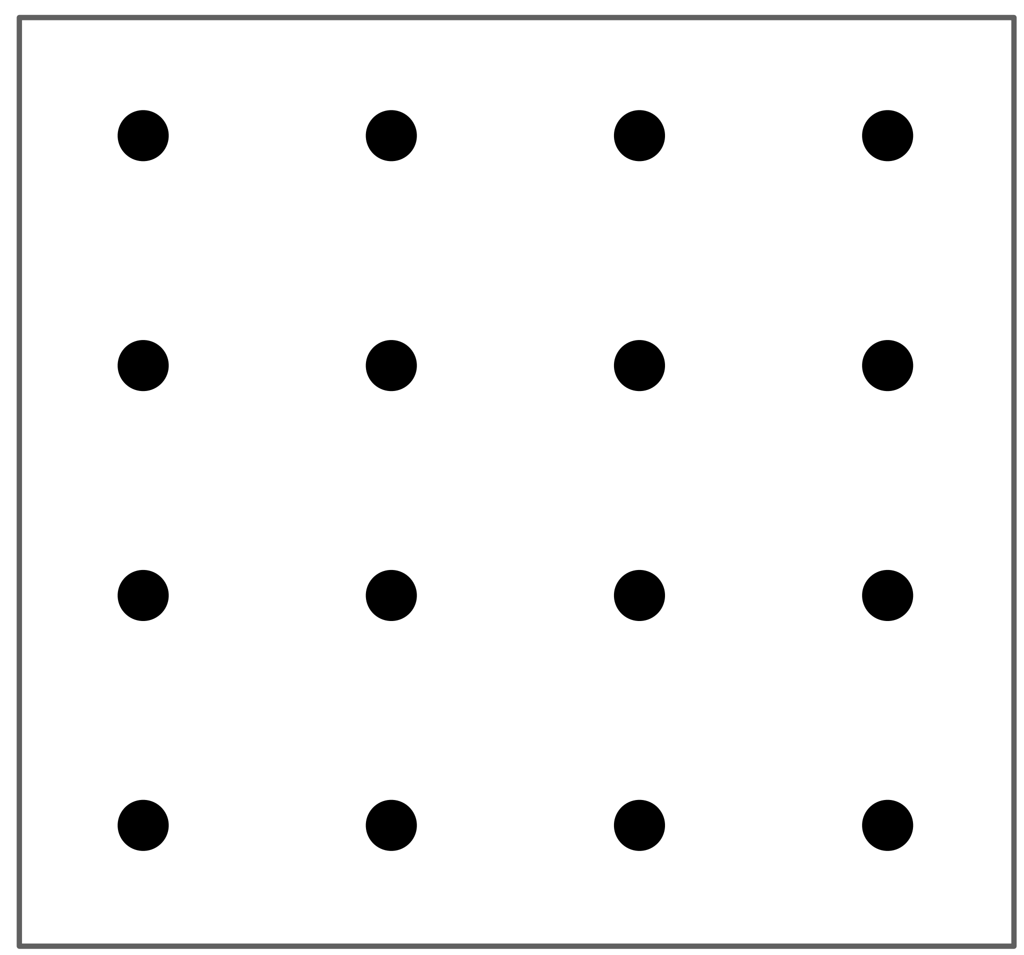
Filtered



Antialiasing By Supersampling

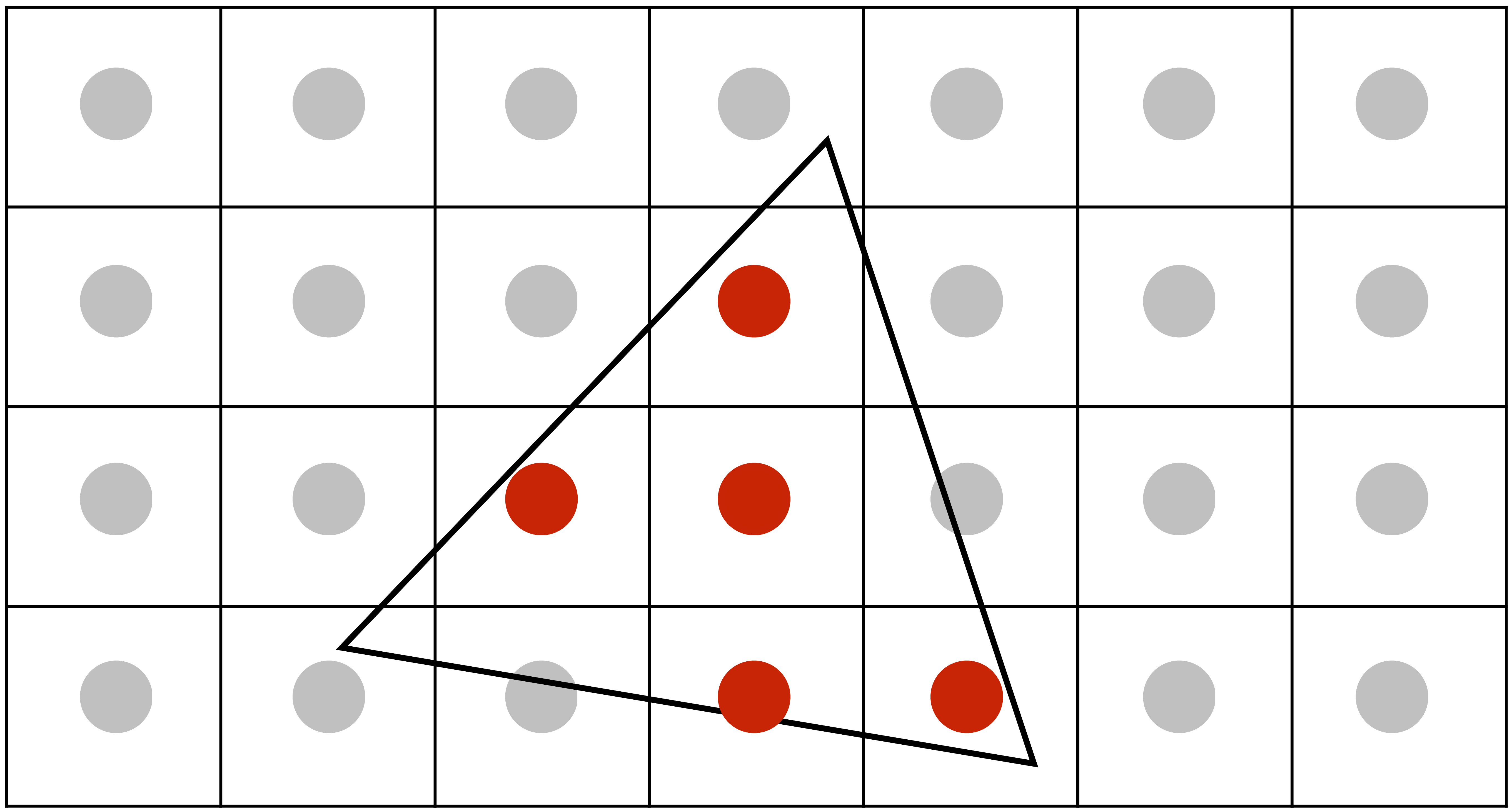
Supersampling

We can approximate the effect of the 1-pixel box filter by sampling multiple locations within a pixel and averaging their values:



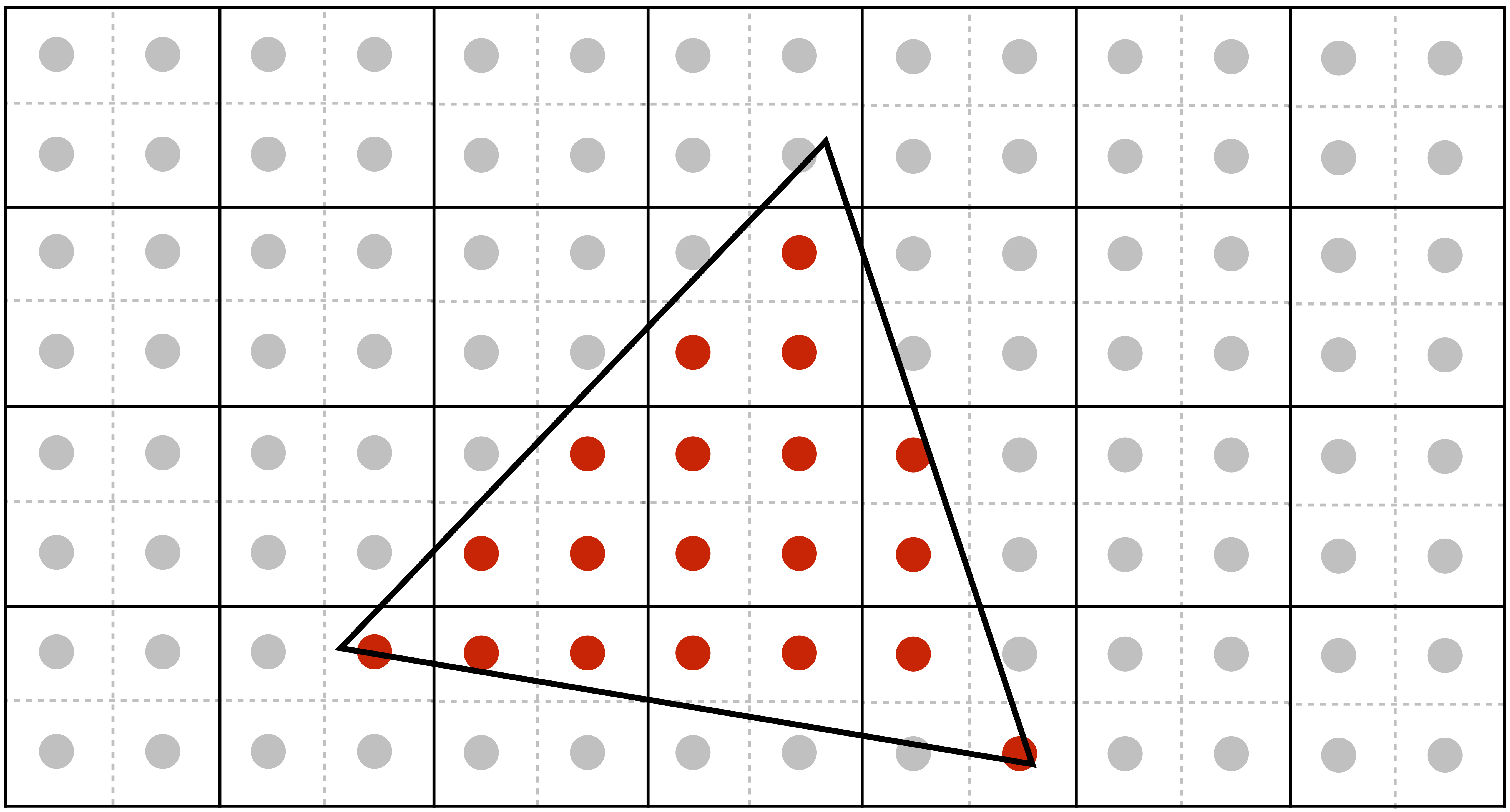
4x4 supersampling

Point Sampling: One Sample Per Pixel



Supersampling: Step 1

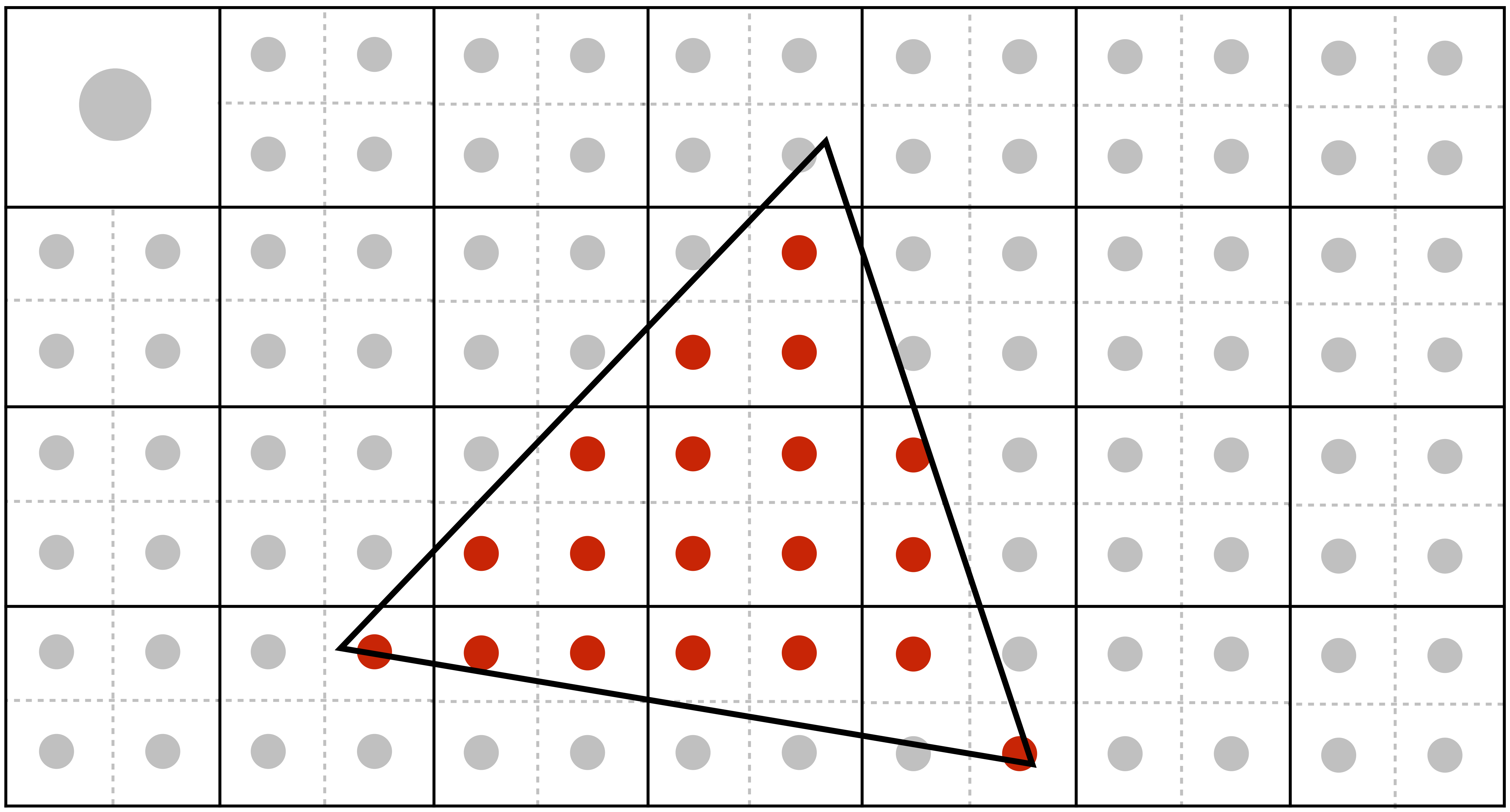
Take NxN samples in each pixel.



2x2 supersampling

Supersampling: Step 2

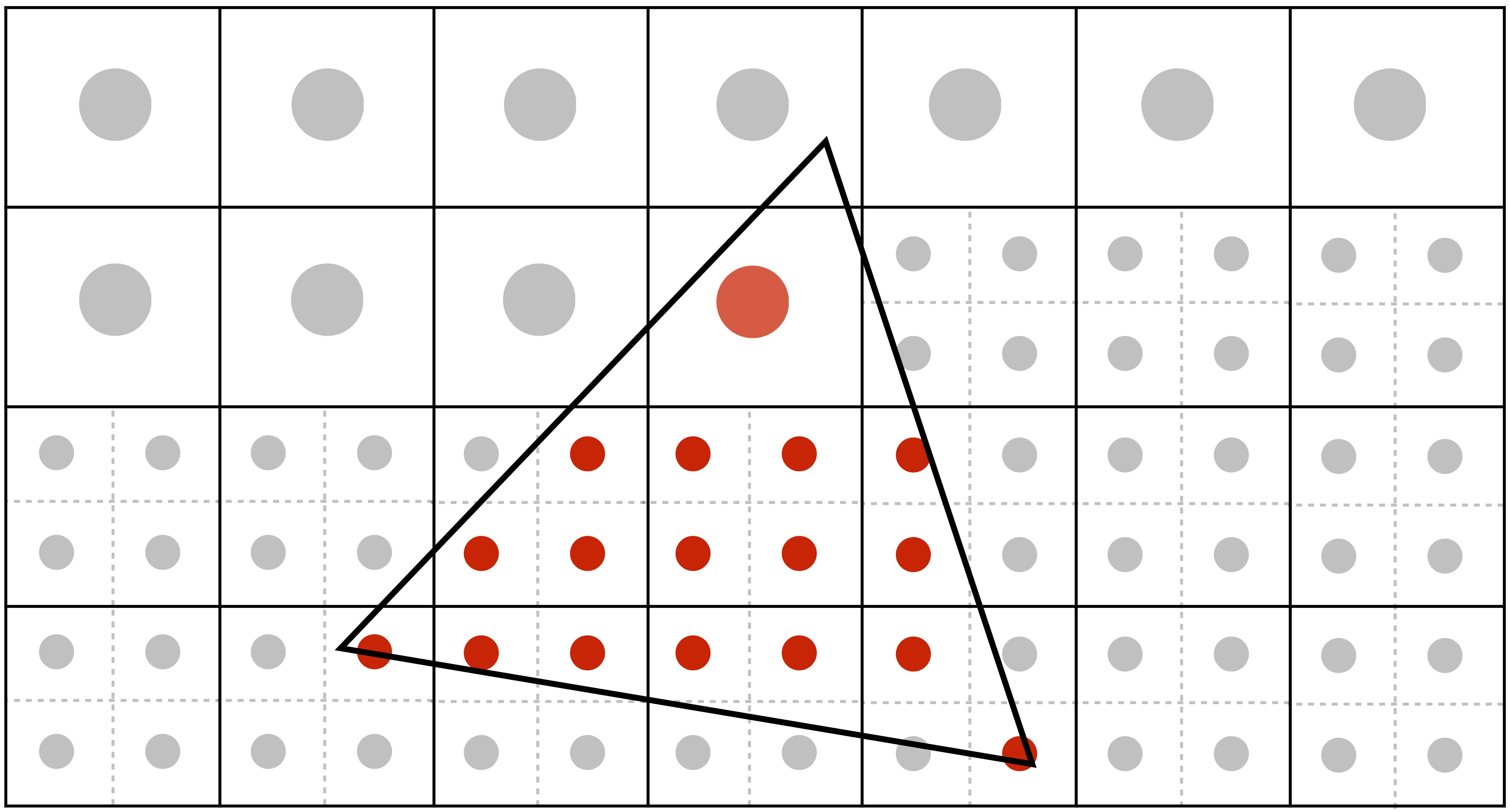
Average the NxN samples “inside” each pixel.



Averaging down

Supersampling: Step 2

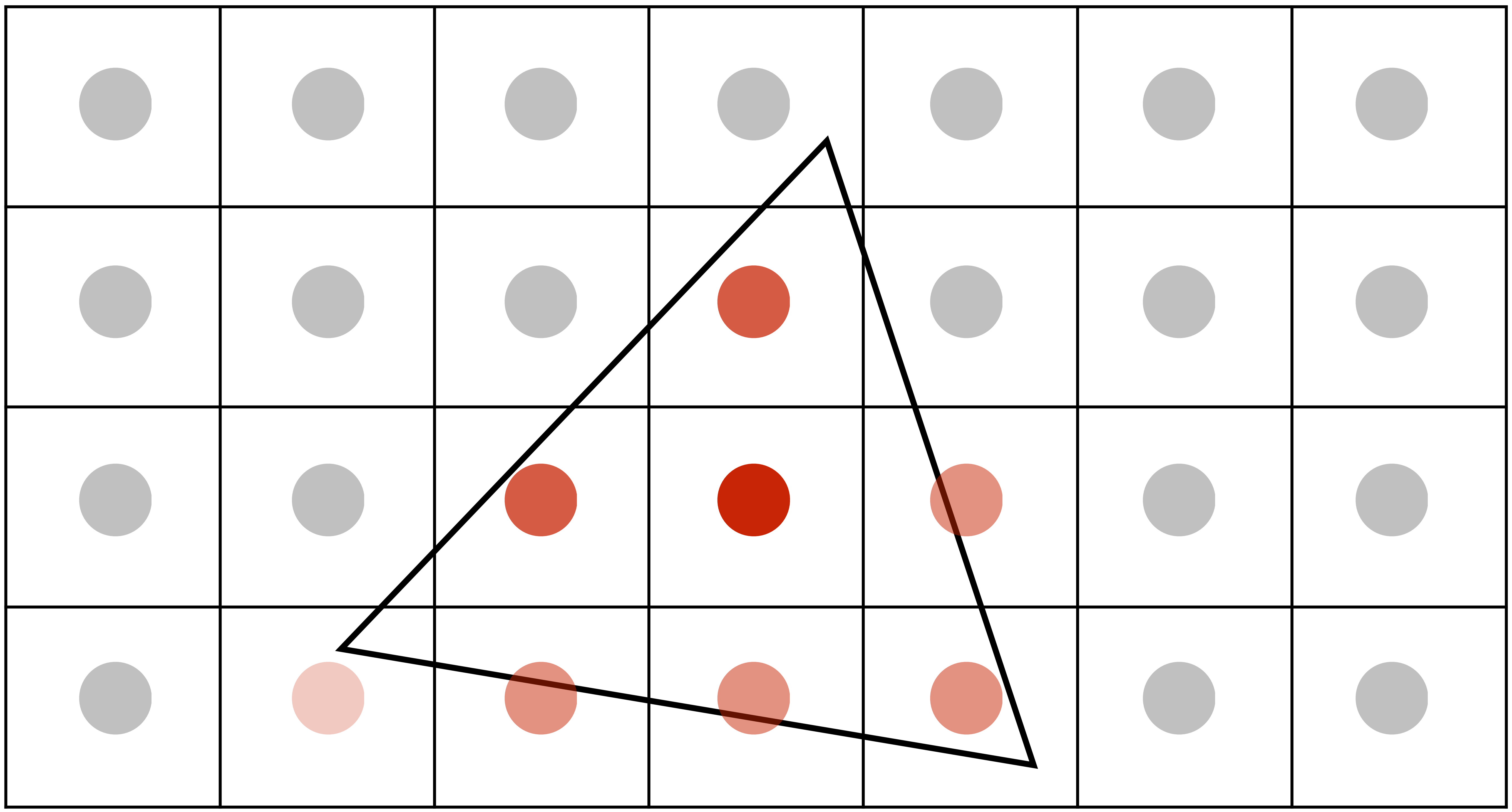
Average the NxN samples “inside” each pixel.



Averaging down

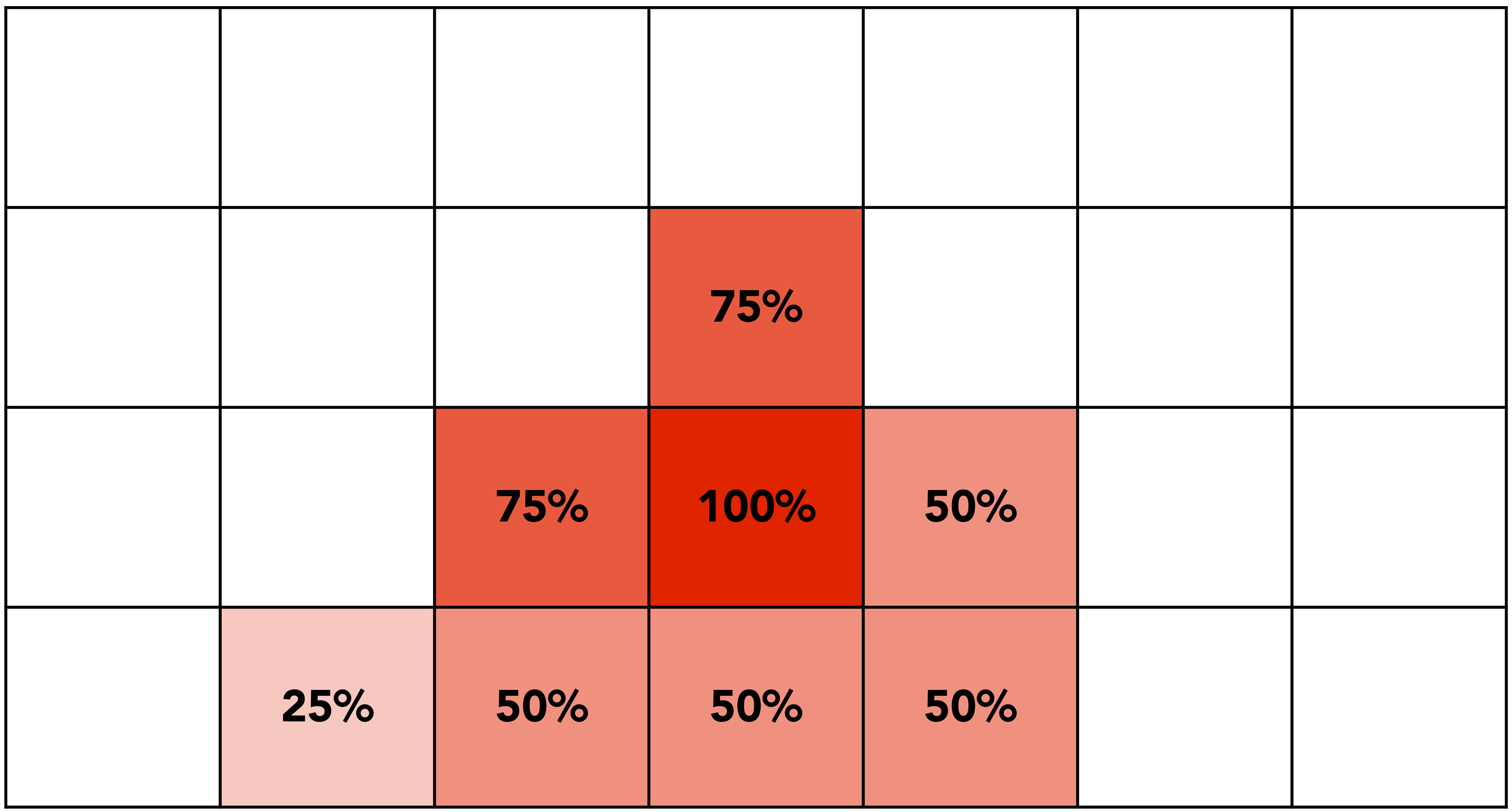
Supersampling: Step 2

Average the NxN samples “inside” each pixel.

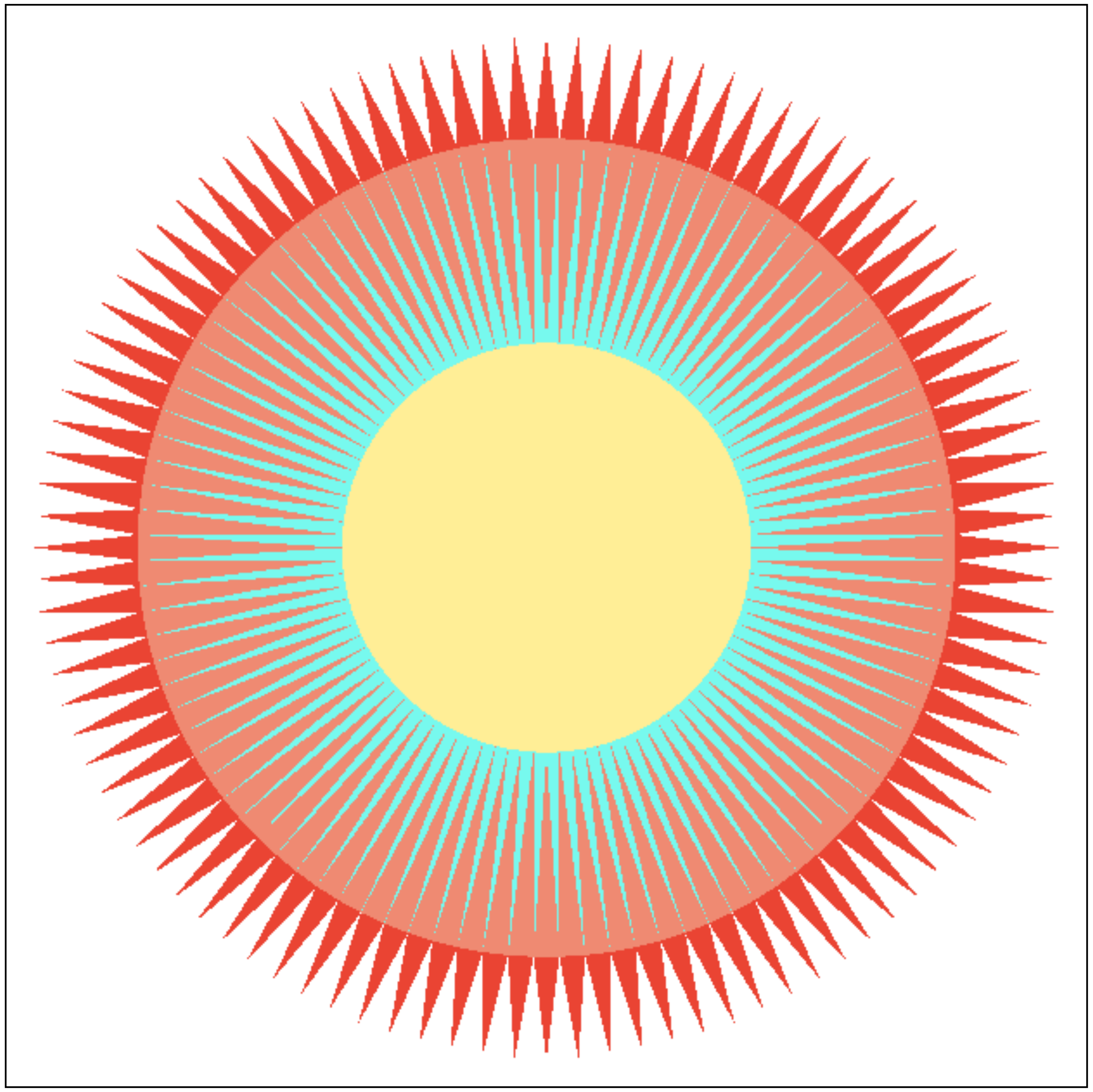


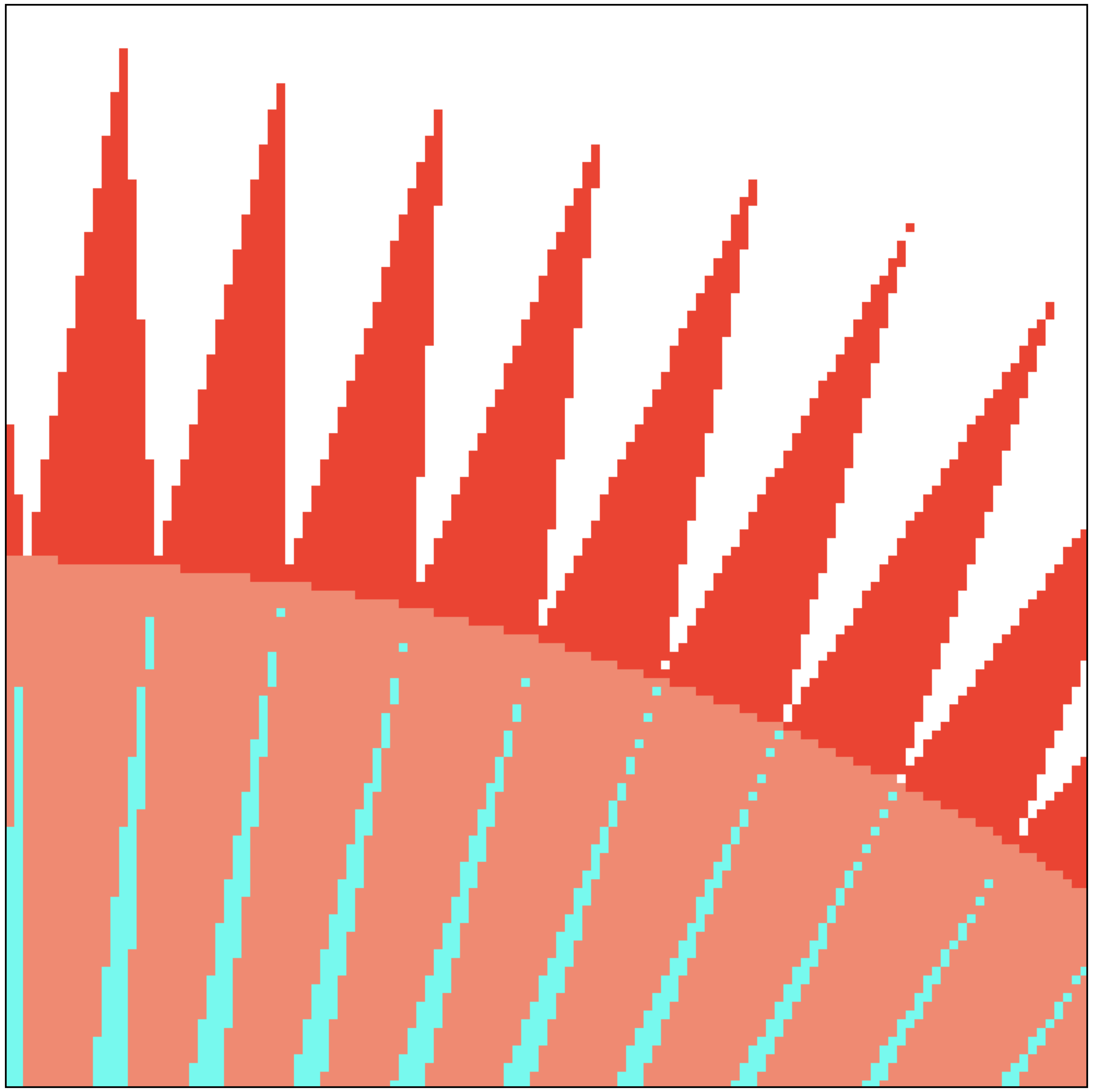
Supersampling: Result

This is the corresponding signal emitted by the display



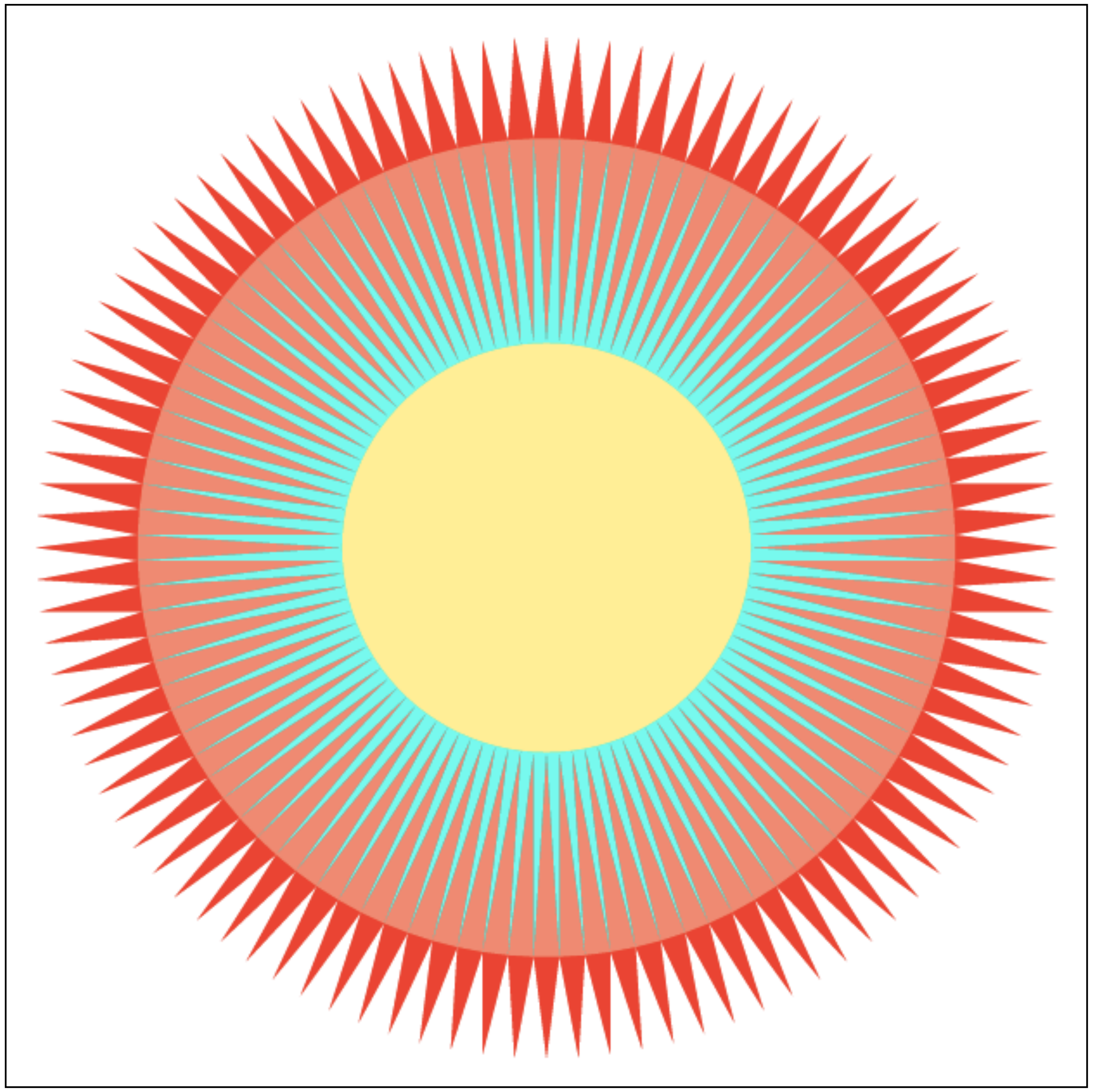
Point Sampling

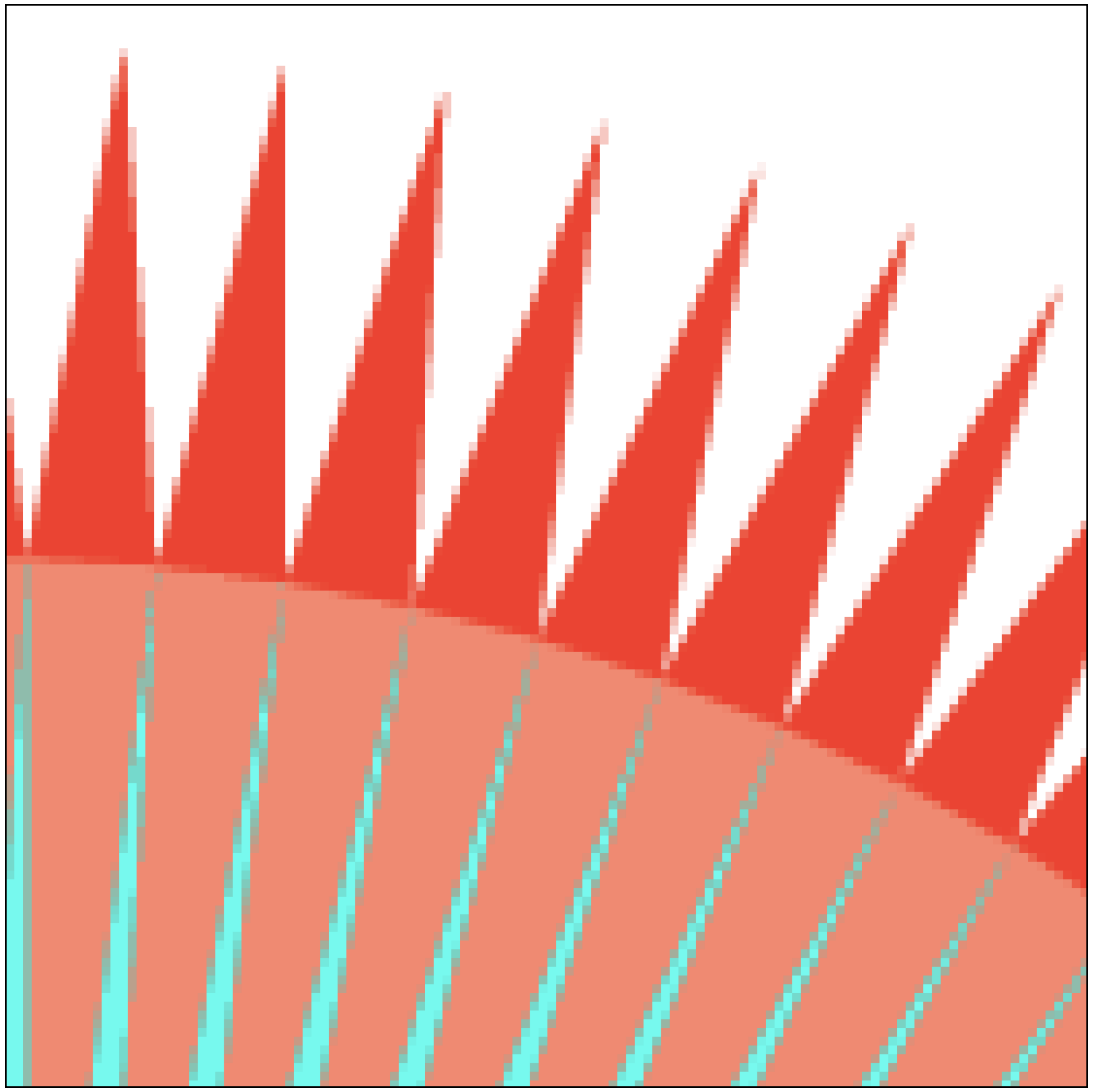




One sample per pixel

4x4 Supersampling + Downsampling





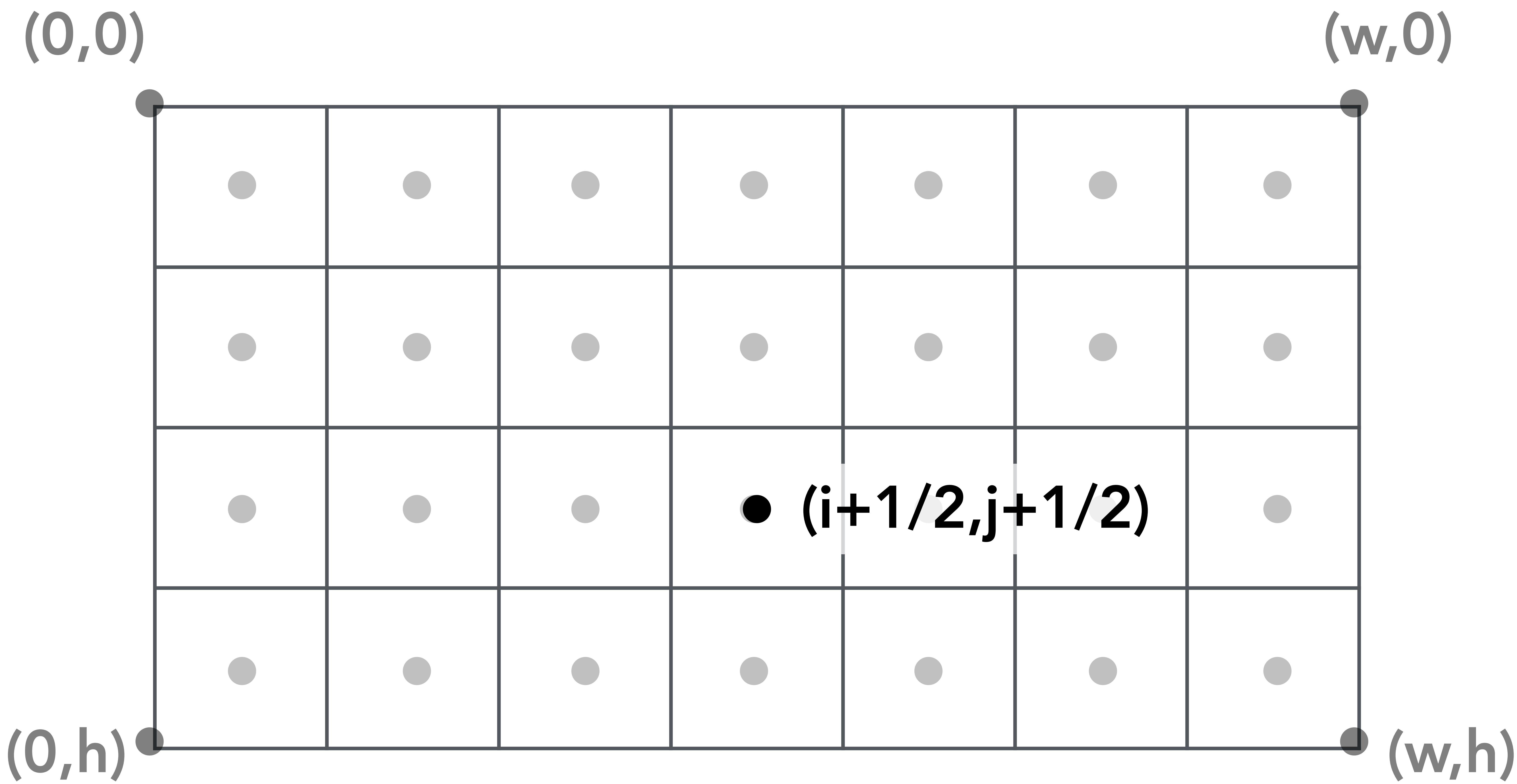
Pixel value is average of 4x4 samples per pixel

Antialiasing By Supersampling - Summary

- Antialiasing = remove frequencies above Nyquist before sampling
- We can attenuate these frequencies quite well with a 1-pixel box filter (convolution)
- We approximated the 1-pixel box sampling by supersampling and averaging
- Simple, good idea - high image quality, but costly
- May feel “right”, but can get even higher quality!

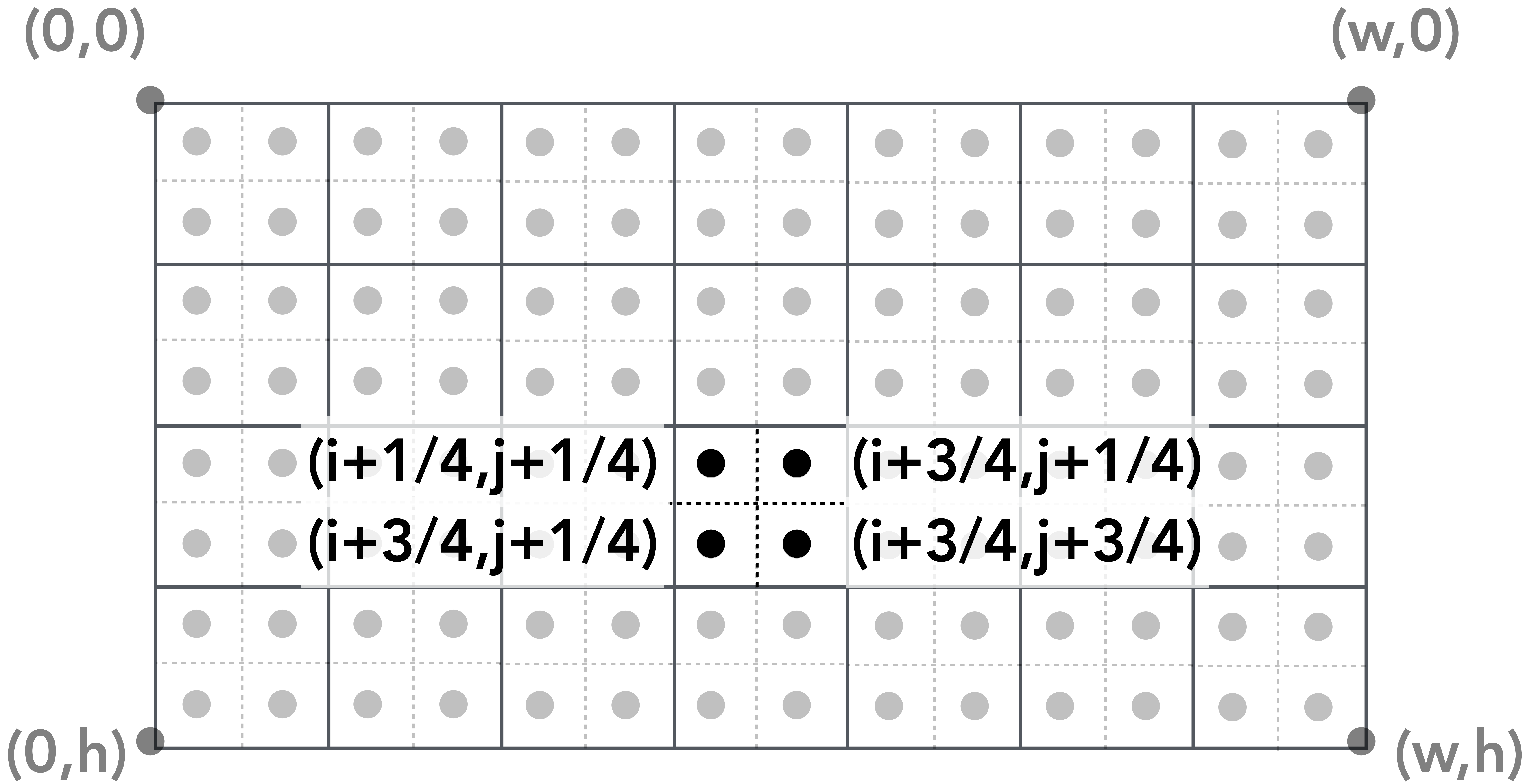
Supersampling Implementation Tips

Tip 1: Sample Locations



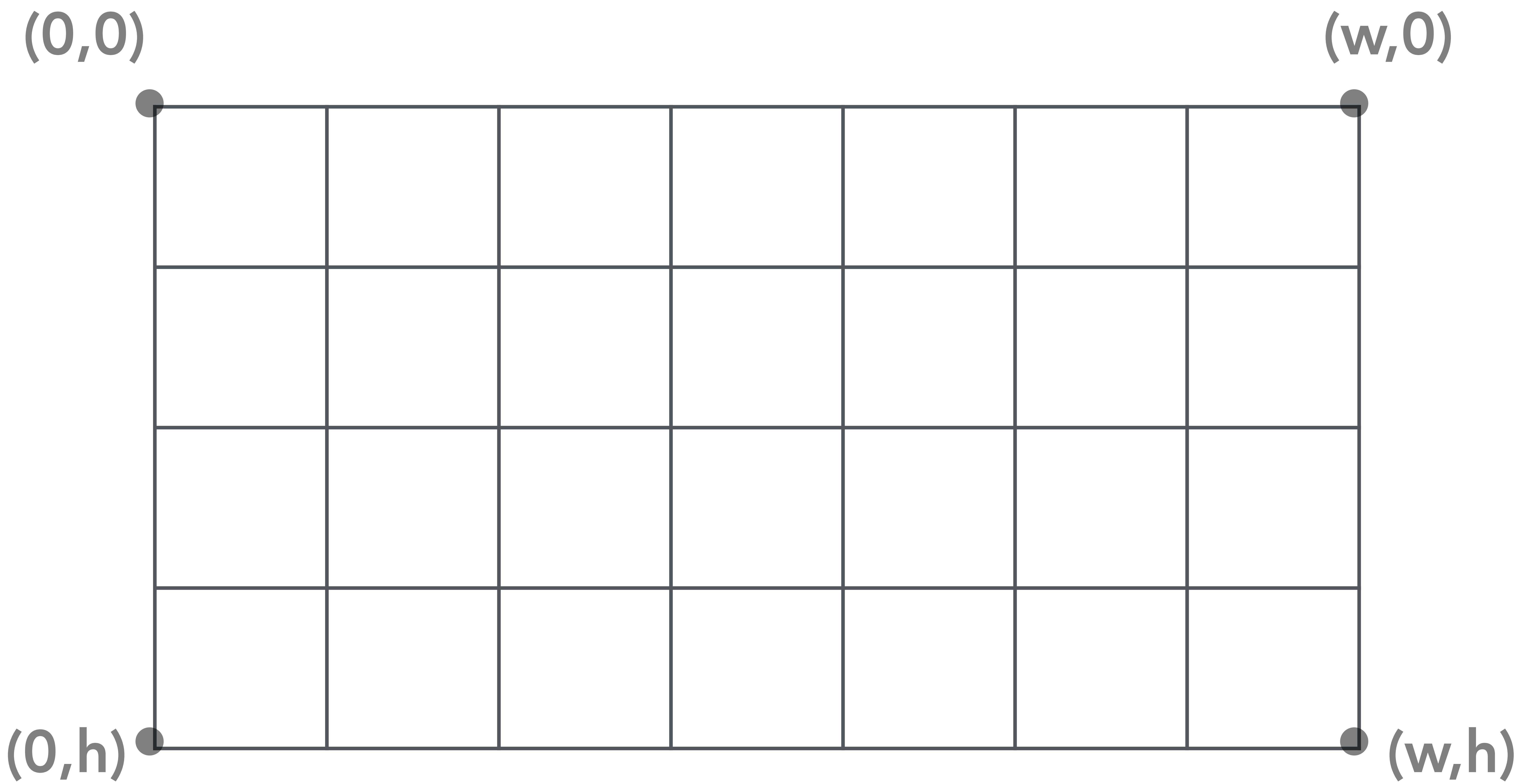
Regular sampling: sample location for pixel (i,j)

Tip 1: Sample Locations



2x2 supersampling: locations for pixel (i,j)

Tip 1: Sample Locations



Sample locations for NxN supersampling?

Tip 2: Supersampling Multiple Triangles

So far, we rasterized only a single triangle:

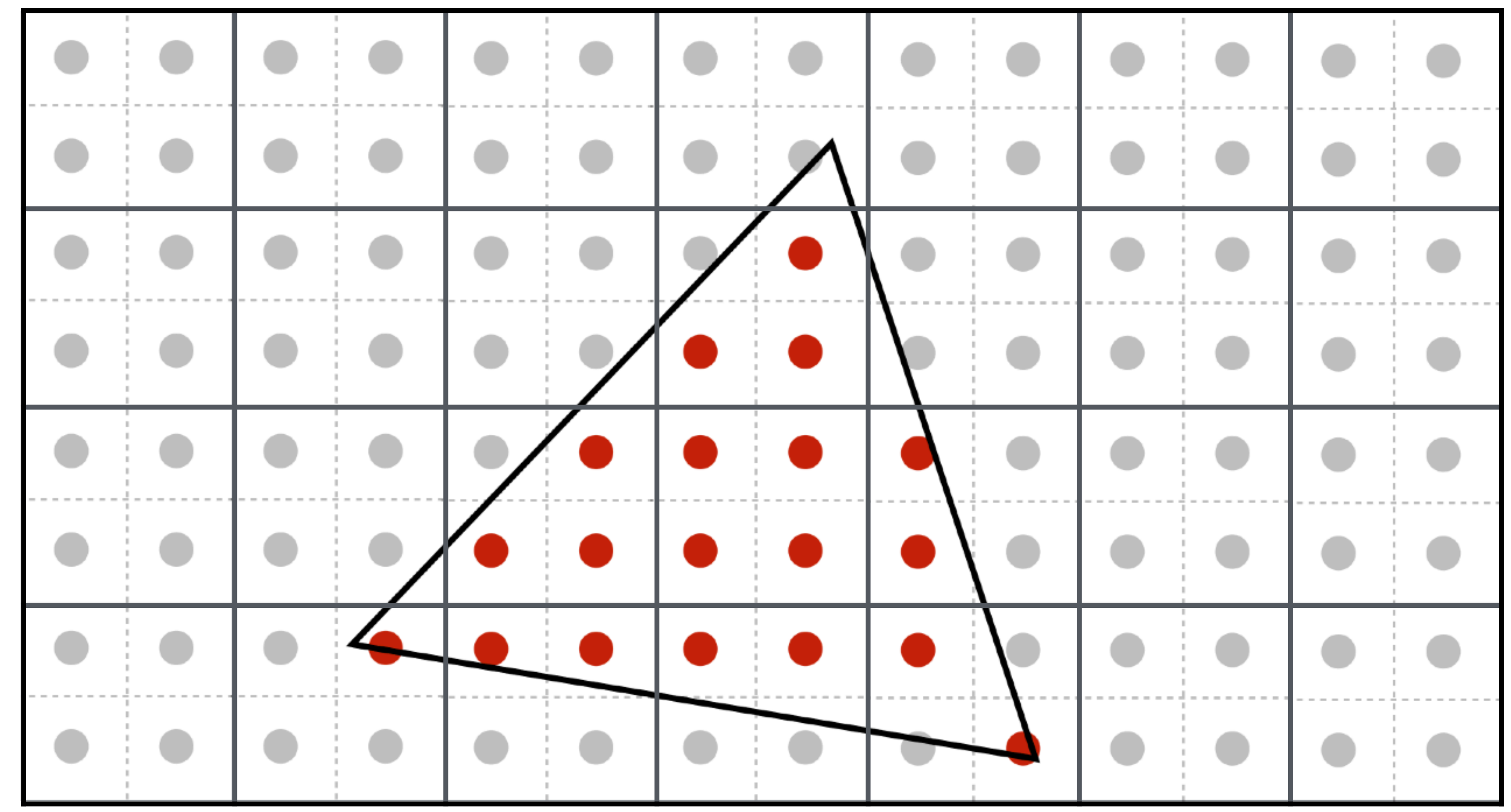
- Supersample
- Then average down

How should this change when we rasterize N triangles in the same image?

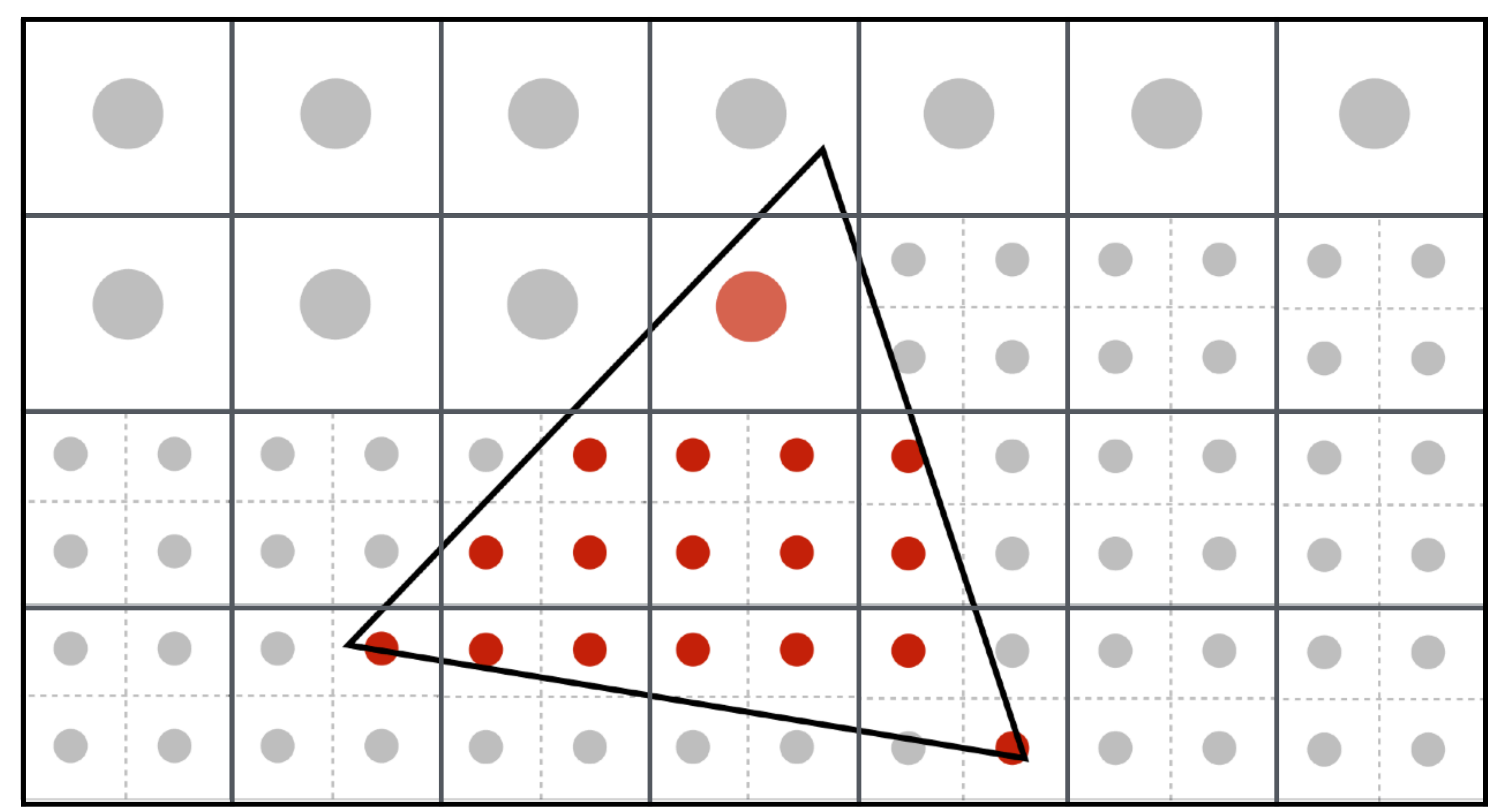
- Supersample and average down each triangle, one by one?
- Or supersample all N triangles onto a high-res grid, then average down?

What are the algorithmic implications?

- E.g. what is the minimum memory needed?



Supersample



Average Down

Note: There is Much, Much More To Sampling Theory & Practice!

Things to Remember

Signal processing key concepts:

- Frequency domain vs spatial domain
- Filters in the frequency domain scale frequencies
- Filters in the sampling domain = convolution

Sampling and aliasing

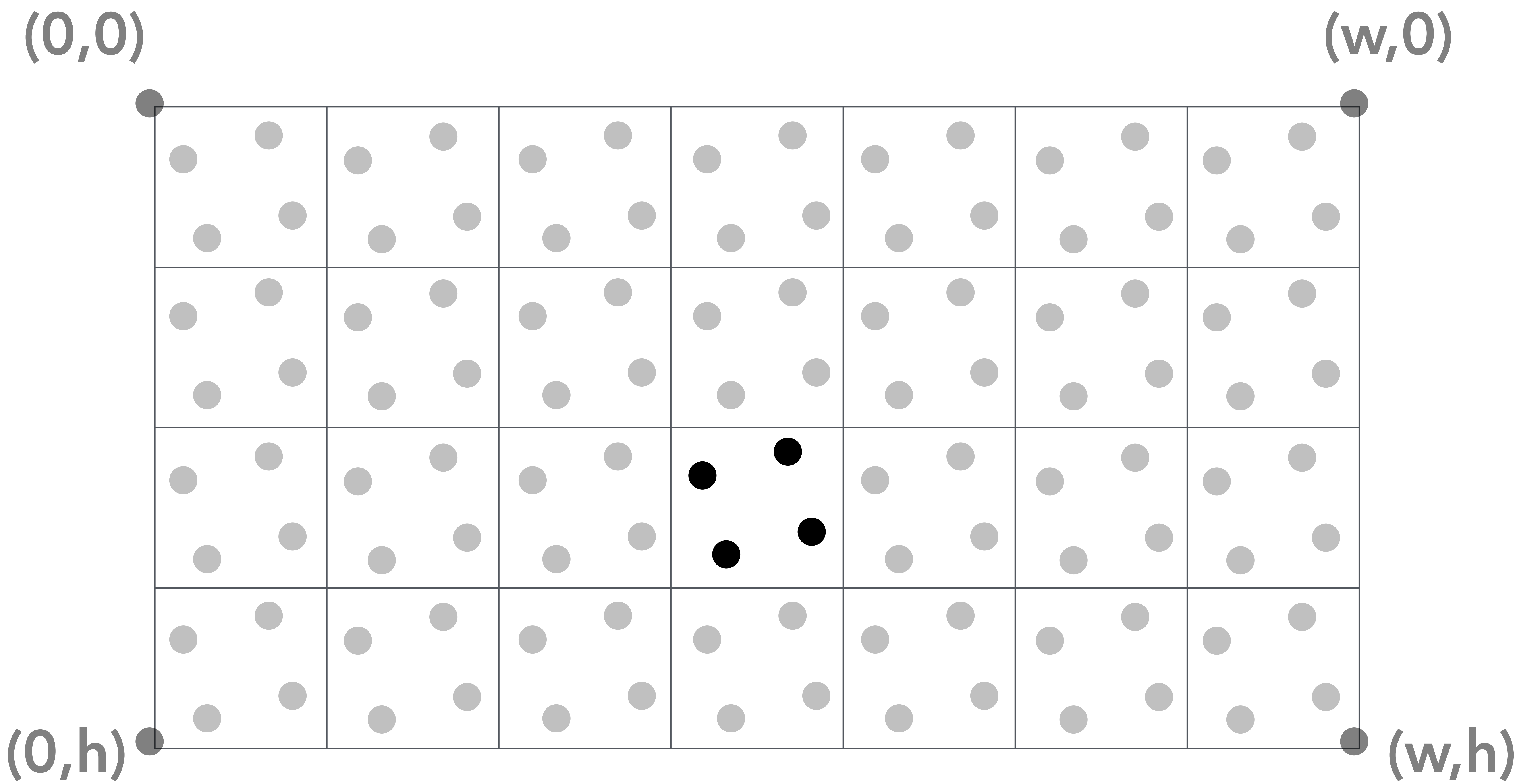
- Image generation involves sampling
- Nyquist frequency is half the sampling rate
- Frequencies above Nyquist appear as aliasing artifacts
- Antialiasing = filter out high frequencies before sampling
- Interpret supersampling as (approx) box pre-filter antialiasing

Acknowledgments

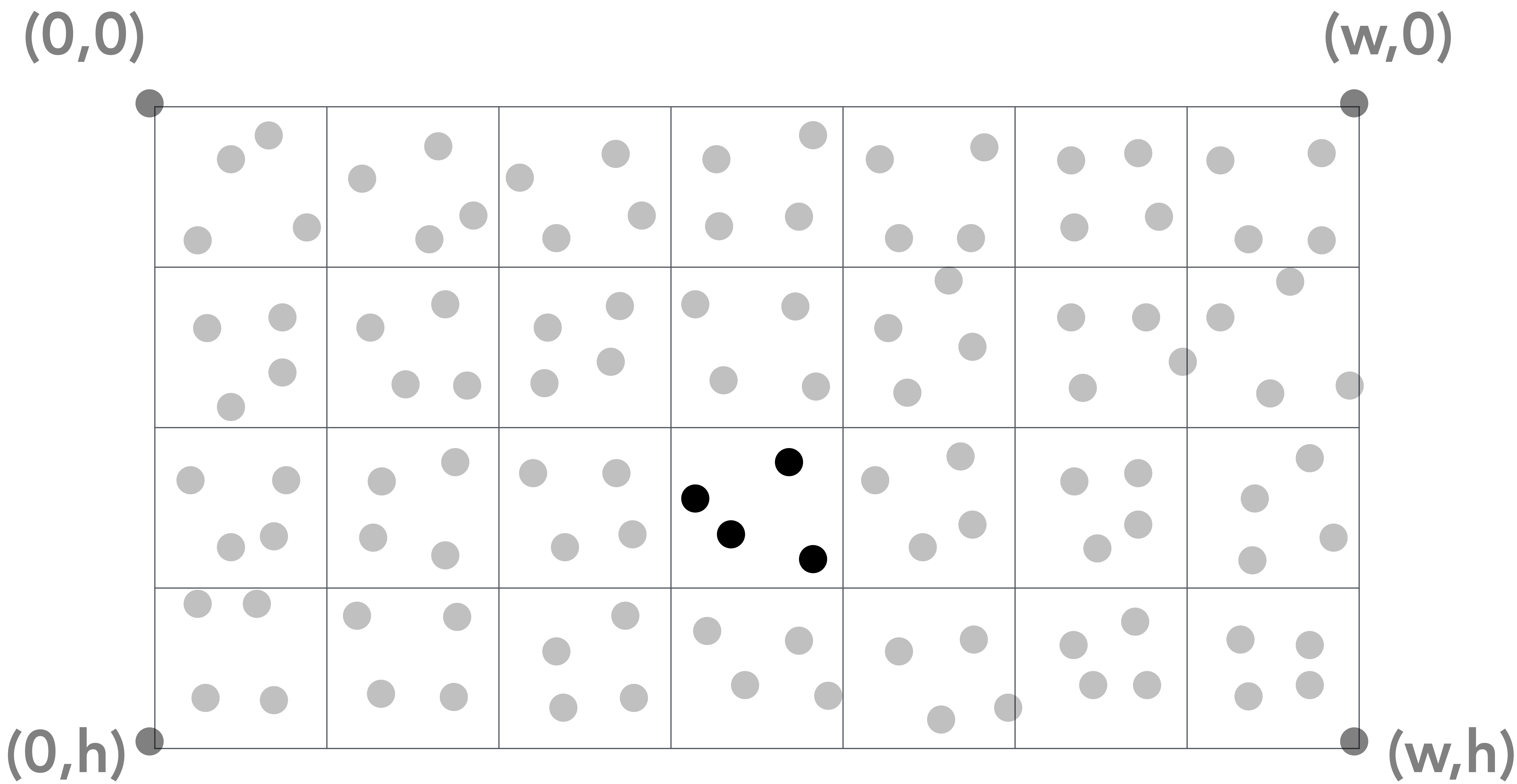
Thanks to Kayvon Fatahalian, Pat Hanrahan, Mark Pauly and Steve Marschner for slide resources.

Sampling Food for Thought

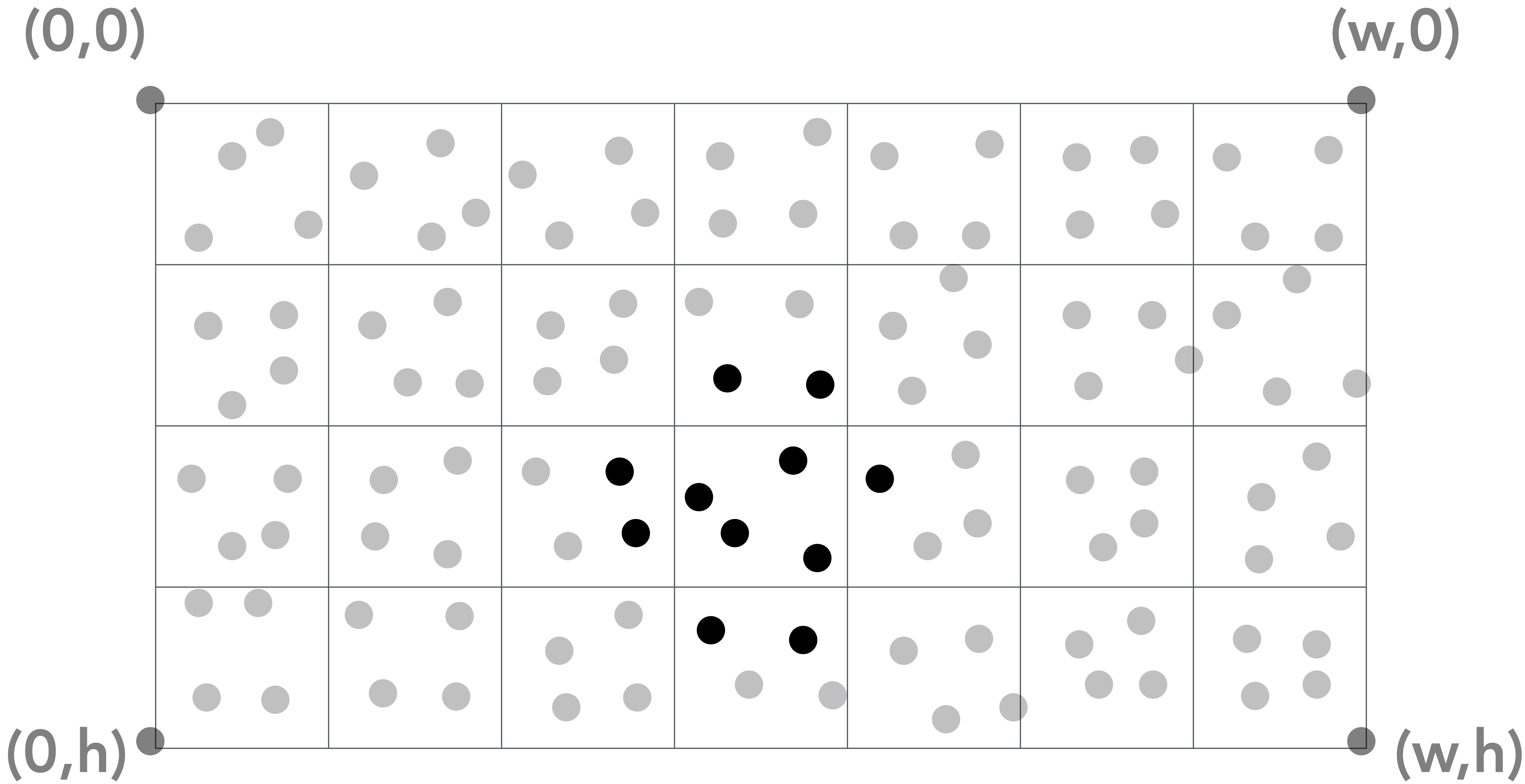
Off-Grid Sampling?



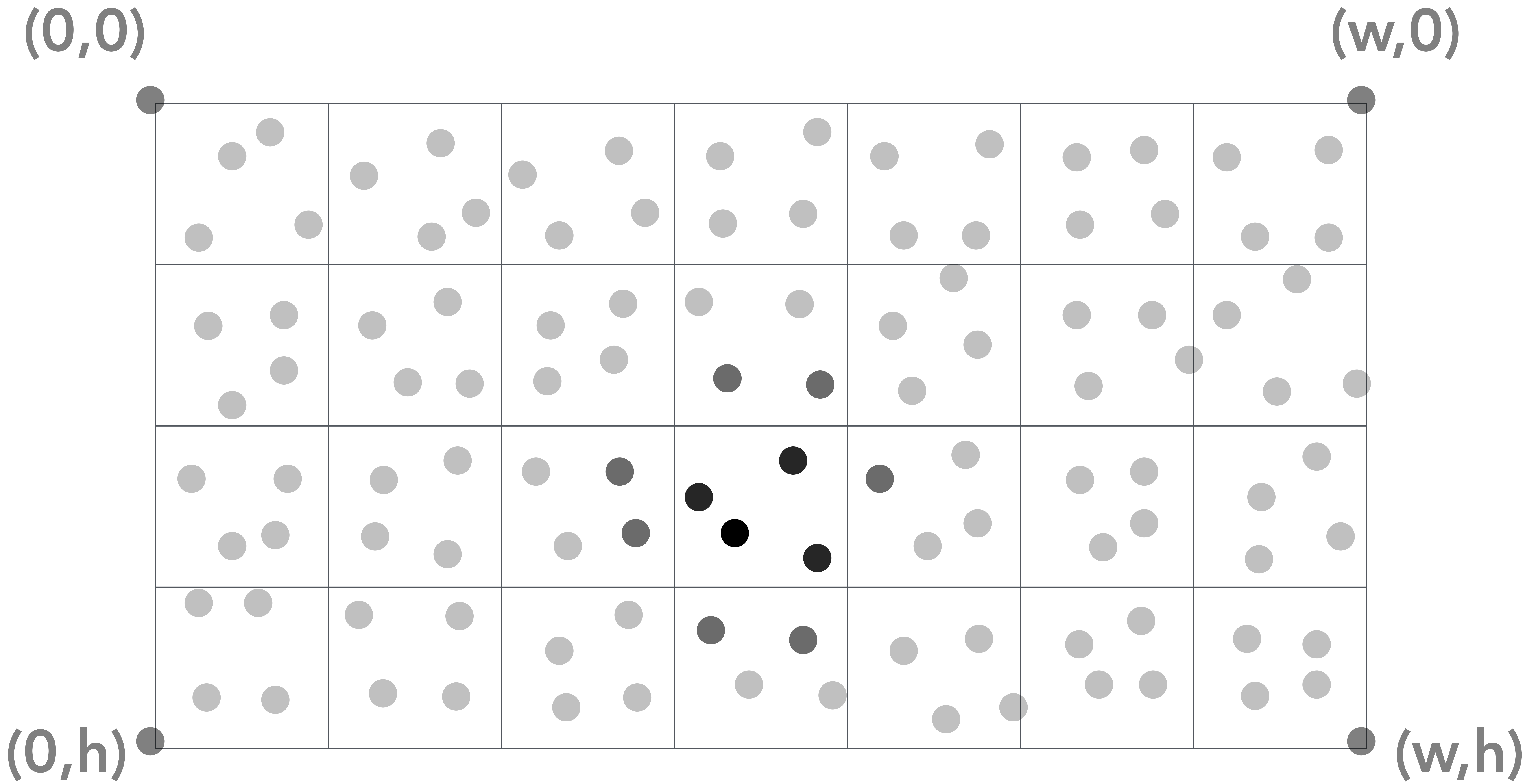
Random Sampling?



Use Samples "Outside" Pixel?



Non-Uniform Sample Weighting?



What weighting function k(x,y)?

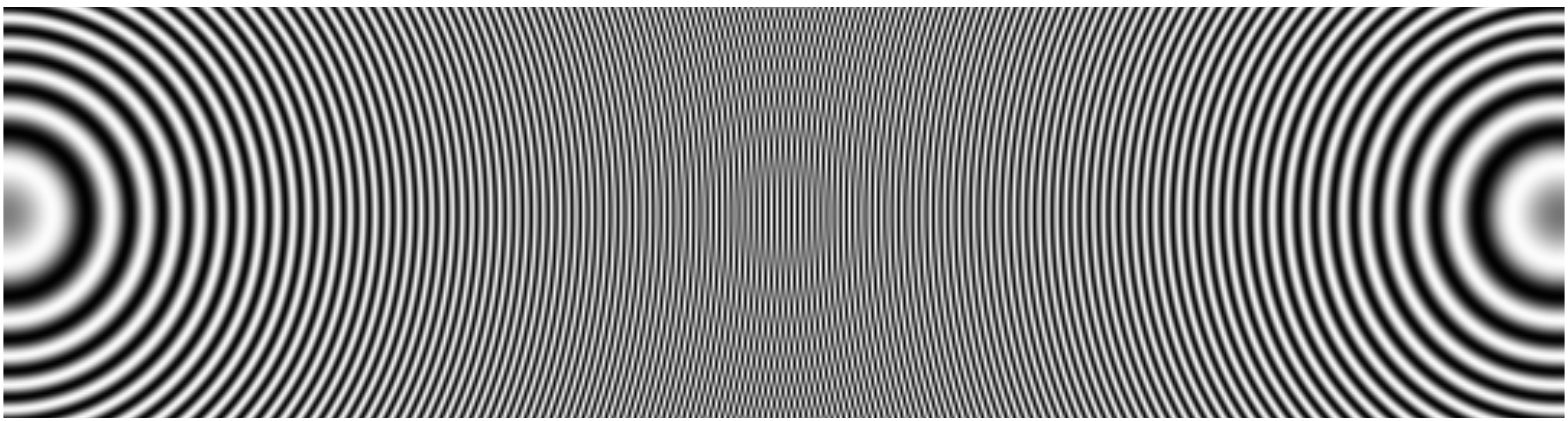
What frequency spectrum would a desirable k(x,y) have?

Sampling Stress Test: Zone Plate

$$f(x,y) = \sin(x^2+y^2)$$

What should this look like?

(0,0)



Real signal
(low frequency oscillation)

Aliasing from undersampling
increasingly high frequencies

Figure credit: Pat Hanrahan and Bryce Summers