Radiometry and Monte Carlo Integration

CS 184

Summer 2020

Your creative fellow classmates

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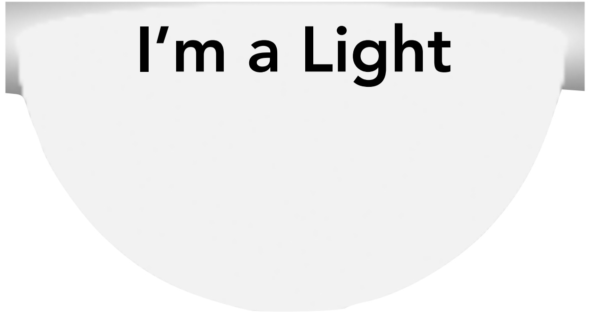
"My world's on fire, how 'bout yours?
That's the way I like it and I'll never get bored."
-- *All Star*, Smashmouth

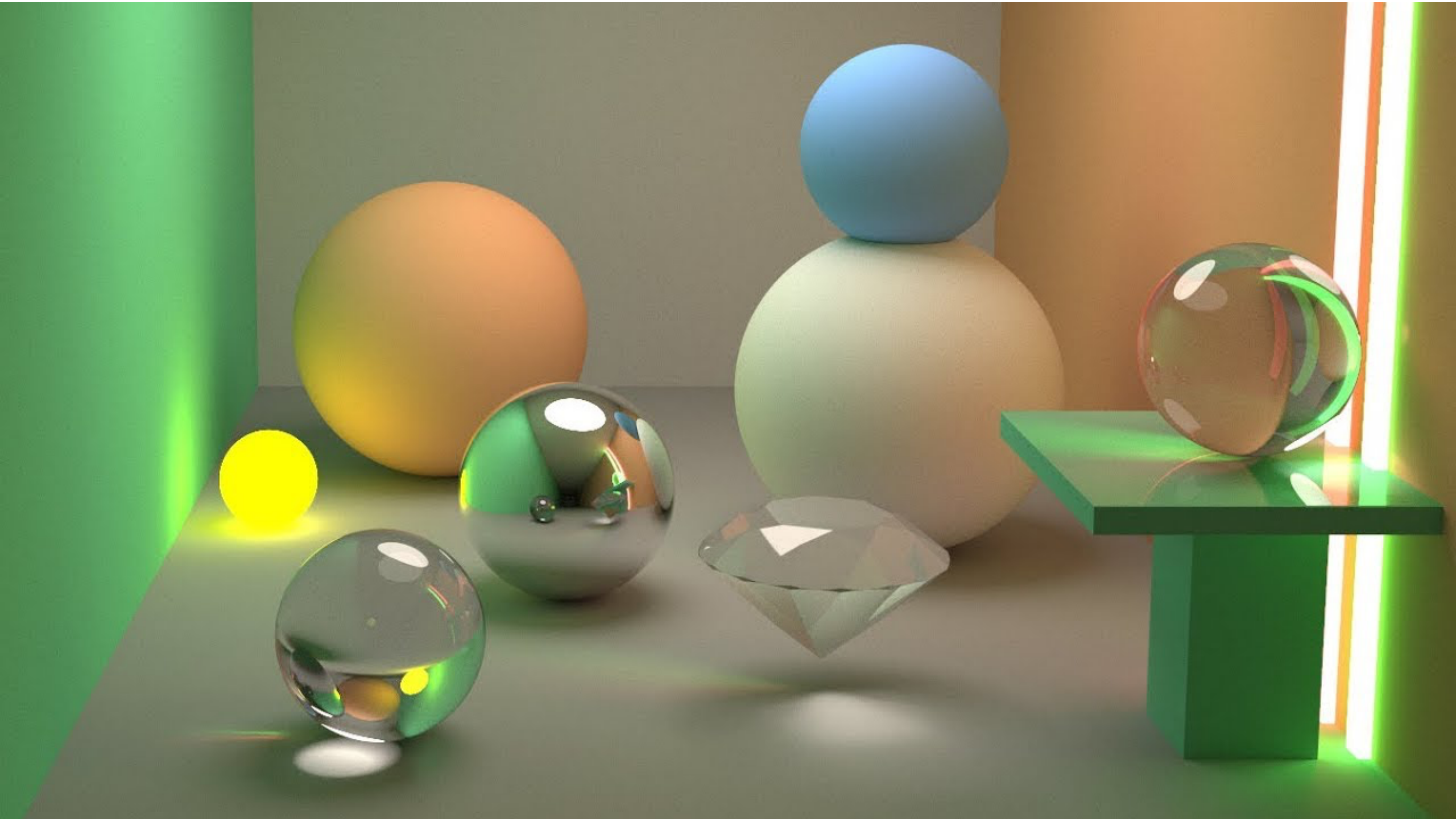
Eunice Chan

Agenda

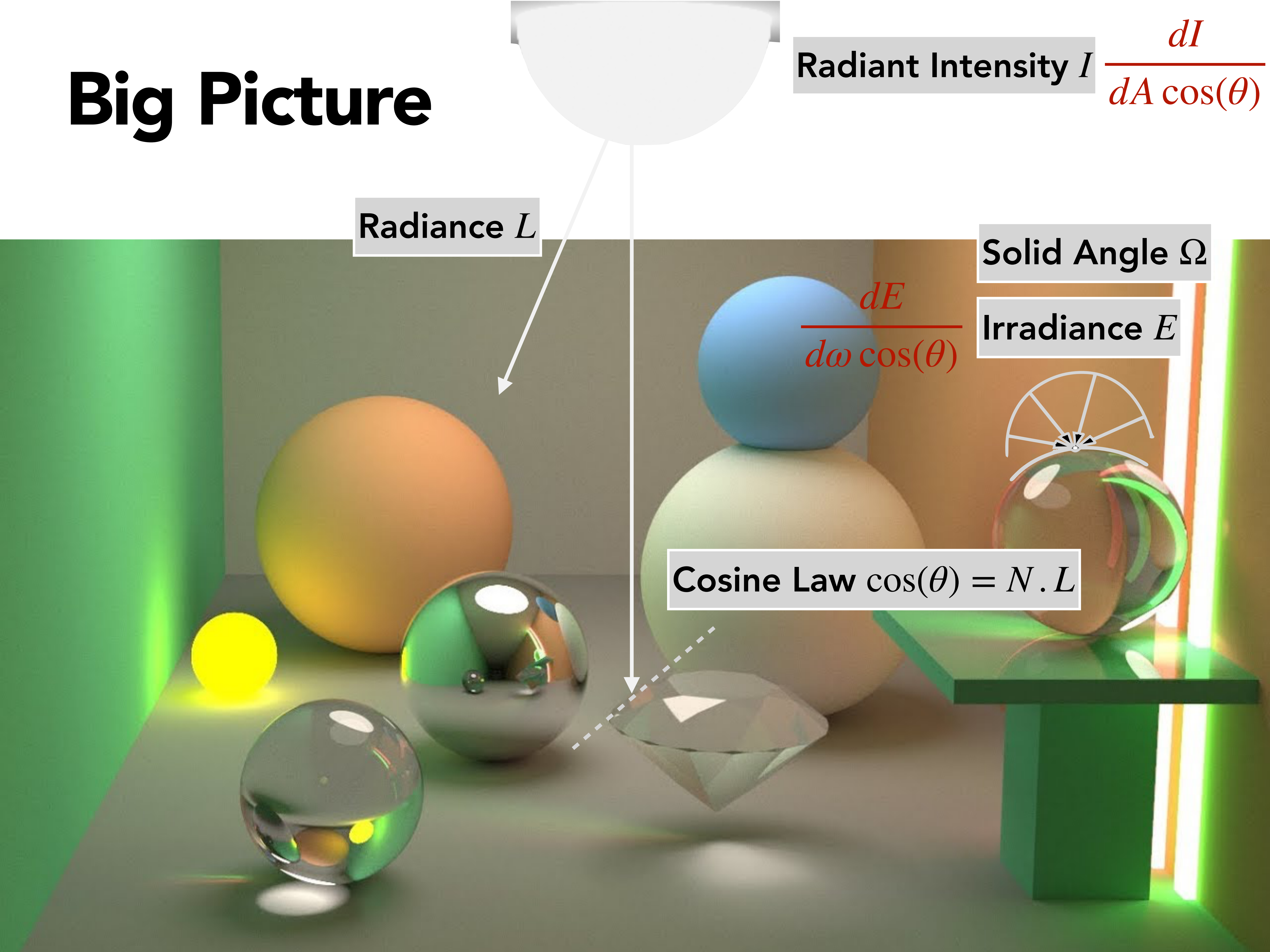
- Radiance and Irradiance
  - Solid angle (steradian), cosine law
- Monte Carlo Estimator
  - Biased, consistent
  - Reduce variance
  - Importance sampling
  - Stochastic, random sampling (inversion method)
- Demo

Big Picture



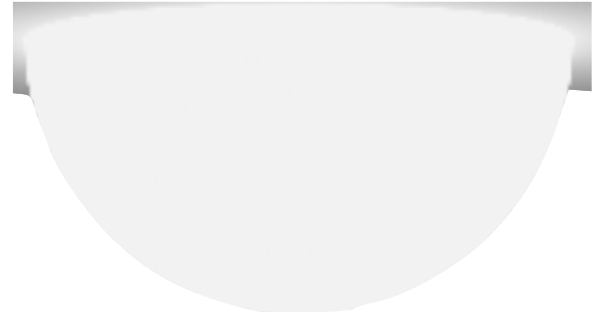


Big Picture



Radiant Intensity I $\frac{dI}{dA\cos(\theta)}$
Radiance L
Solid Angle Ω
$\frac{dE}{d\omega\cos(\theta)}$
Irradiance E
Cosine Law $\cos(\theta) = N \, . \, L$

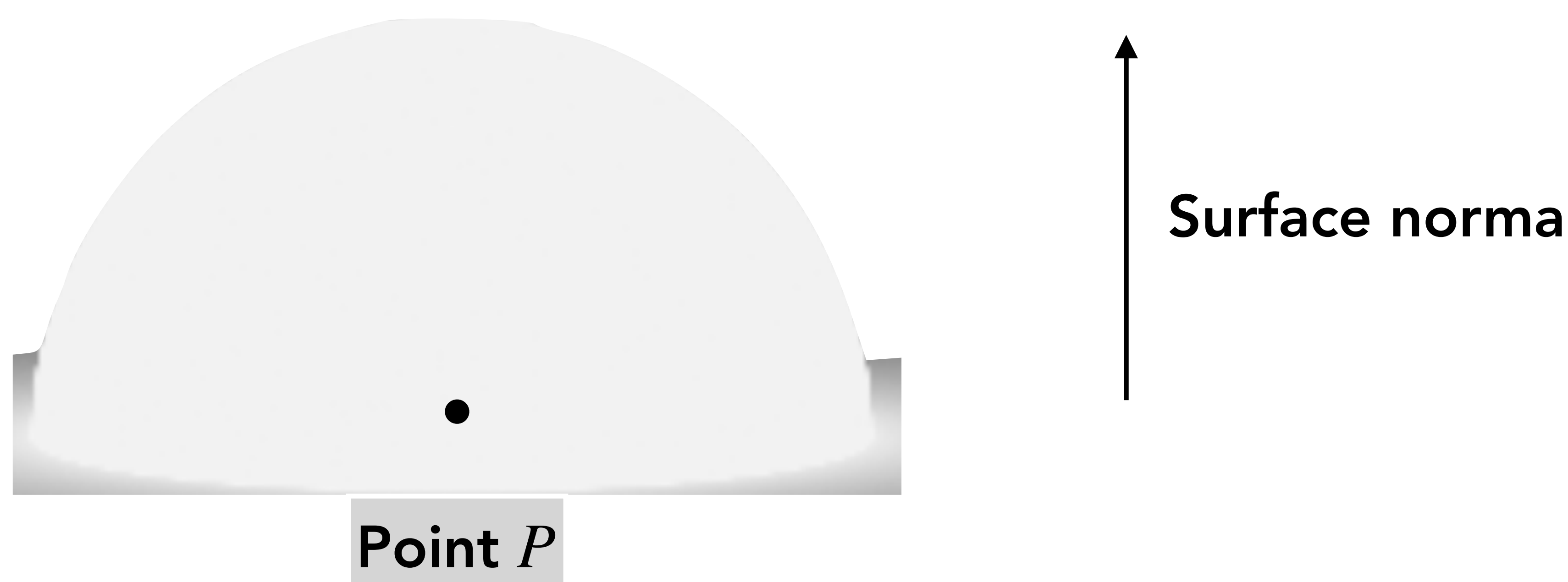
Big Picture



- Incident Radiance & Exiting Radiance Not the Same
  - Different materials respond to light in different ways
- Exiting Radiance Not Only for Light Sources
  - Objects can also been seen as light sources. Not because they emit light though but because they reflect light.
- Direct Illumination: only one bounce
  - VS. Indirect Illumination: multiple bounces — Global illumination (e.g. surface not exposed to light is not completely black)

Computing Radiance $\int_\Omega L$

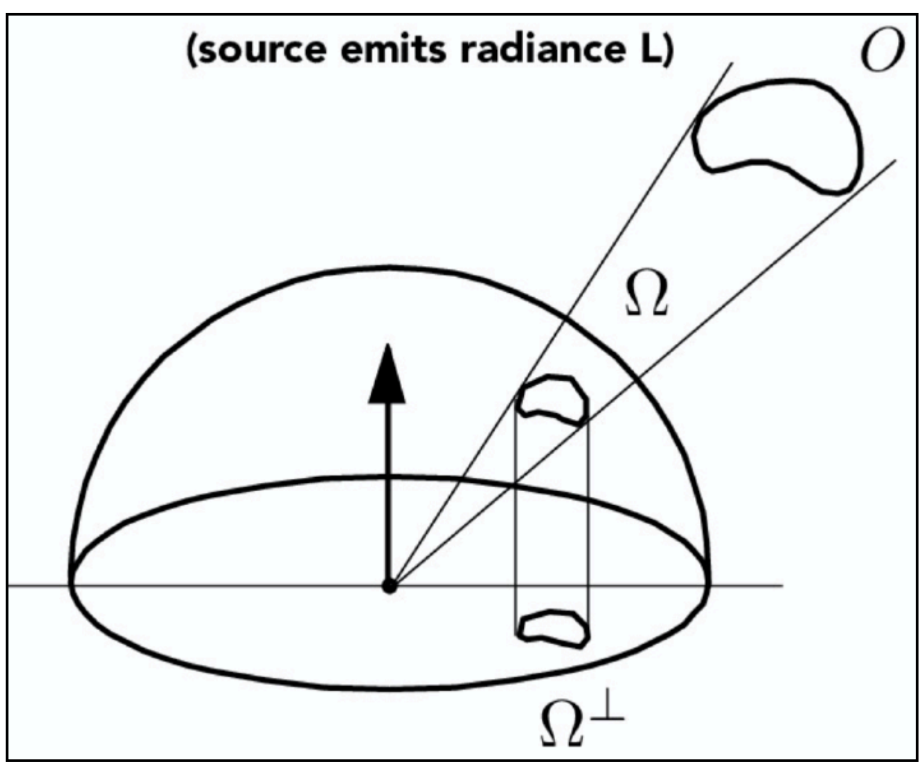
- What: Contribution to Irradiance at Point P
- How:
  - Analytical — Closed-Form Solution
  - Approximation — Monte Carlo Methods



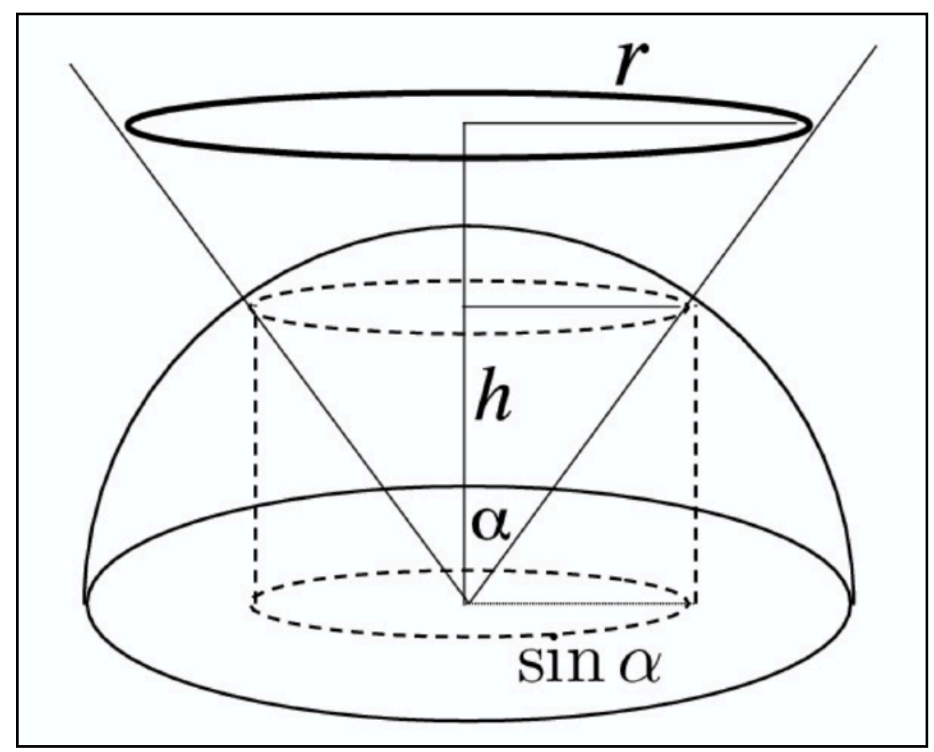
Computing Radiance

The analytical way

- Query position P, Solid Angle Ω subtend by the light source L

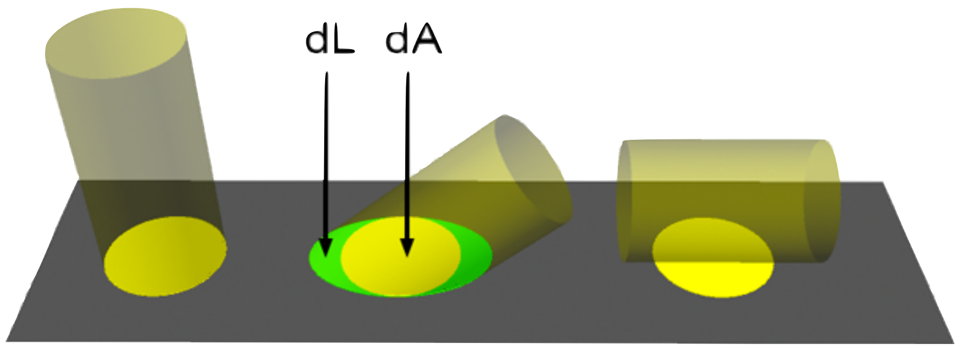


Example 1



Example 2

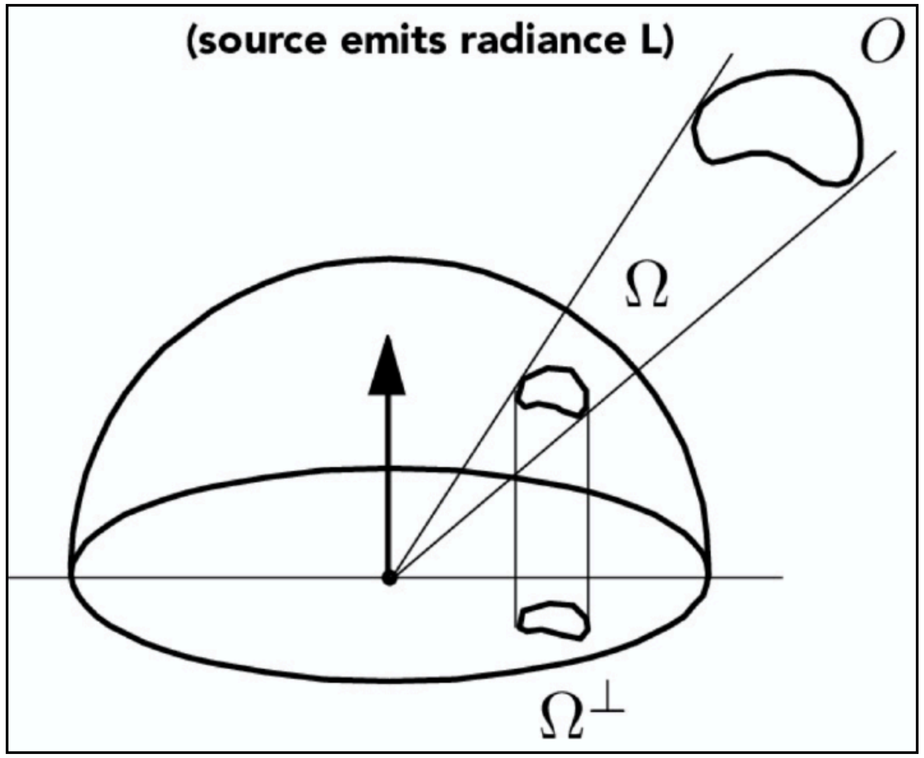
Computing Radiance



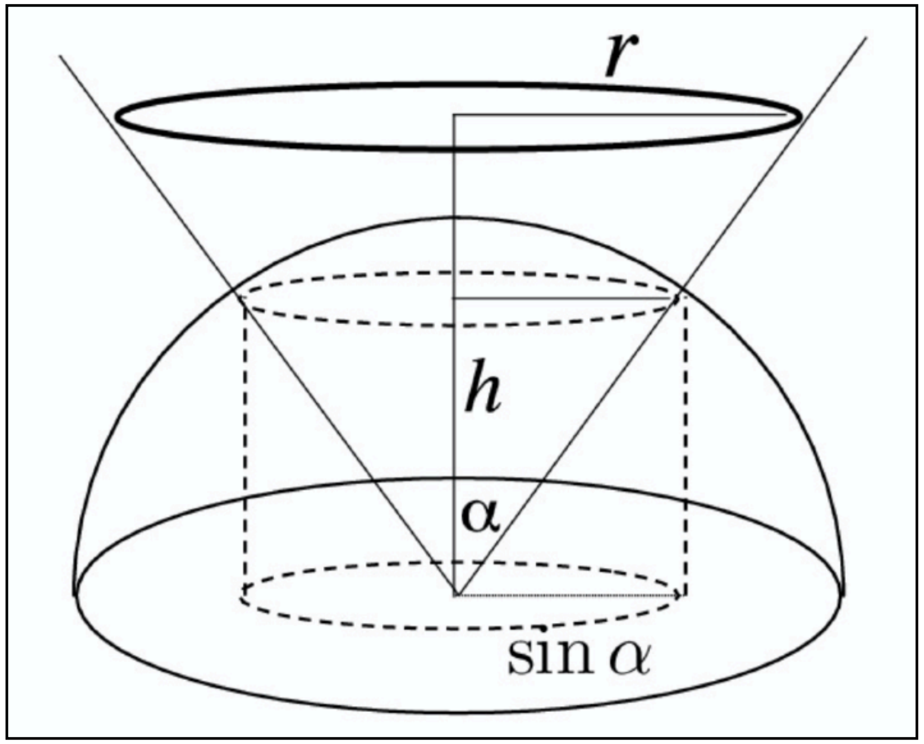
The analytical way

- $\int_{H^2} L(p,\omega)\cos\theta d\omega$ — Projected area to the unit sphere and then to the ground

$$\int_0^{2\pi}\int_0^{\alpha}\cos\theta\sin\theta d\theta d\phi$$



Example 1

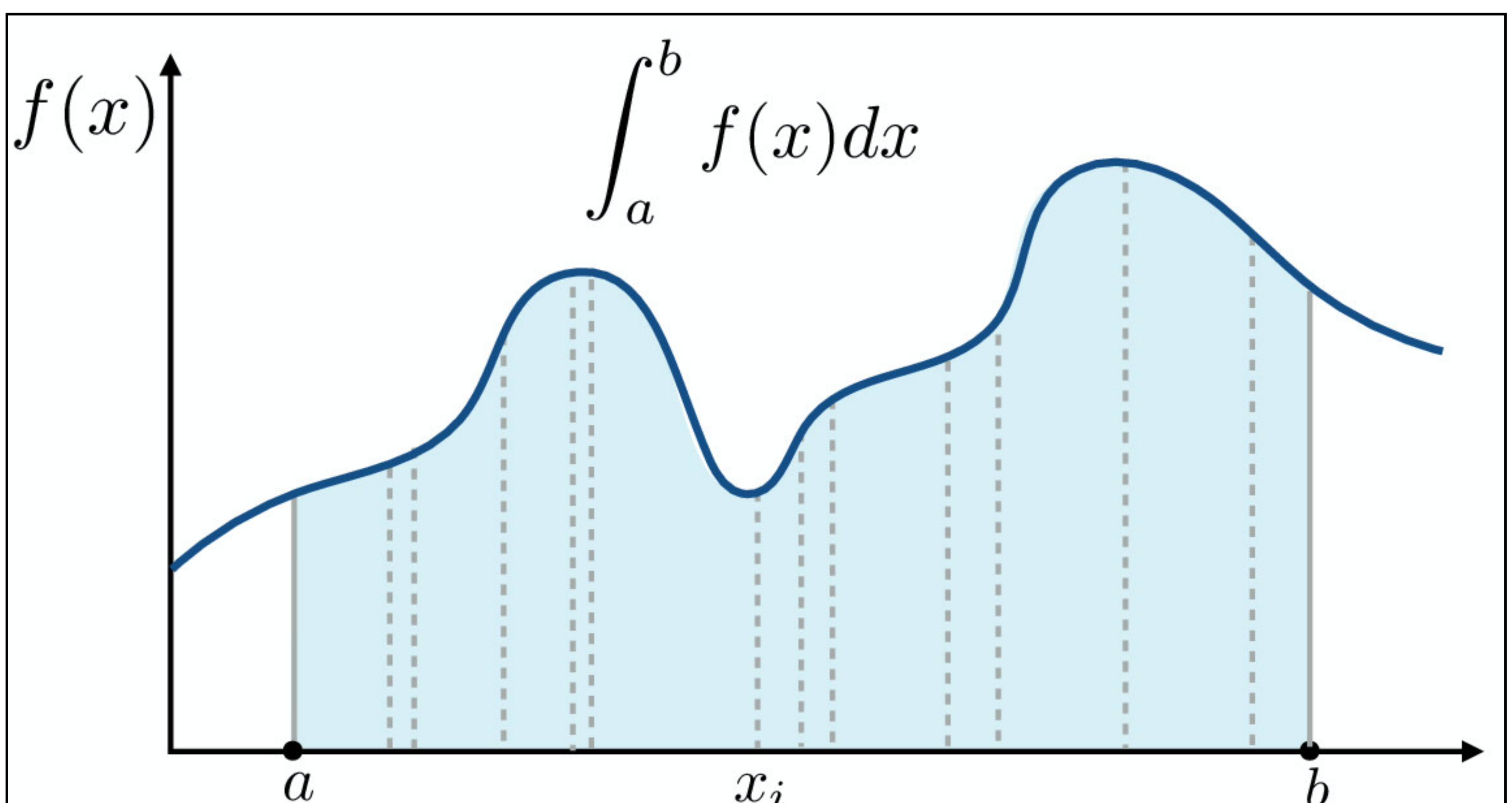


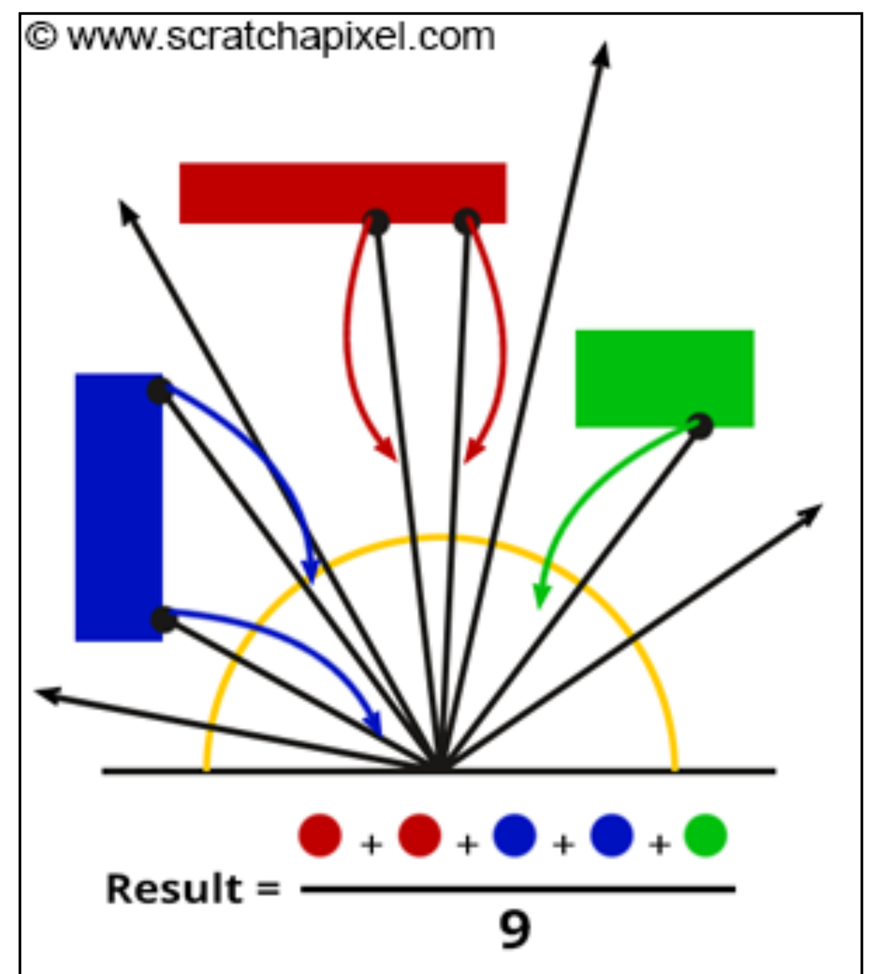
Example 2

Computing Radiance

The practical way

- Scenes Are Usually Too Complicated Too Be Analytically Represented
- We Can Only Compute an Approximation to: $\int_{\Omega} L$

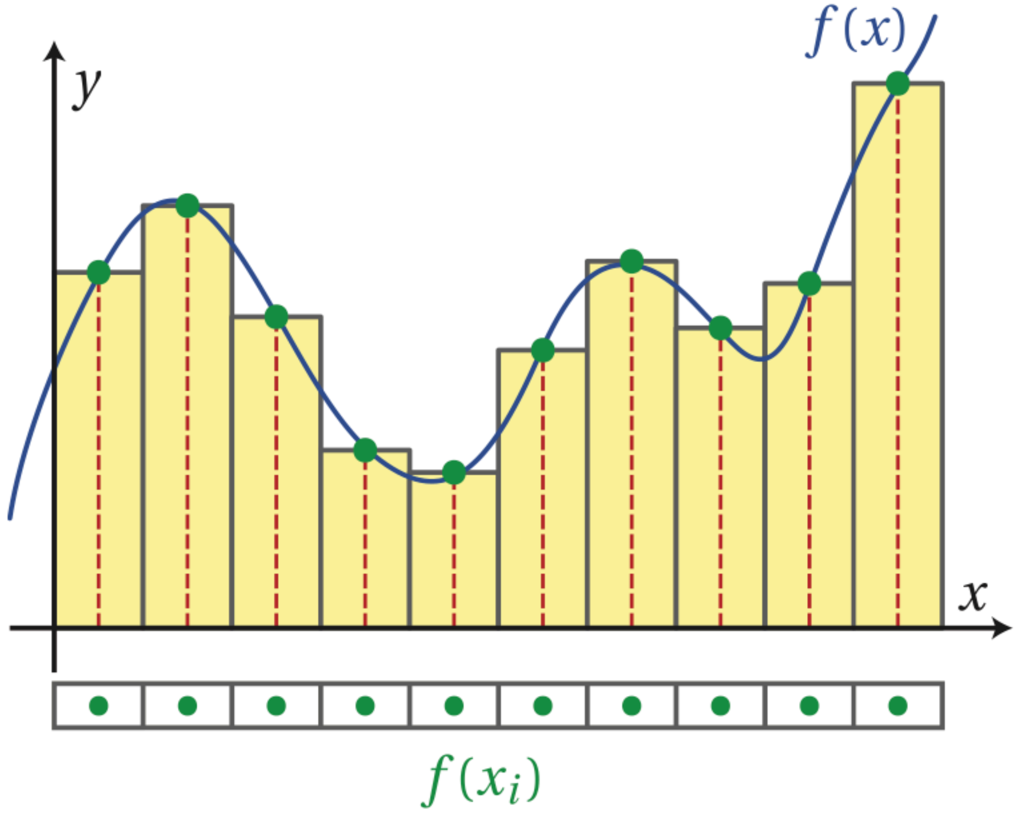




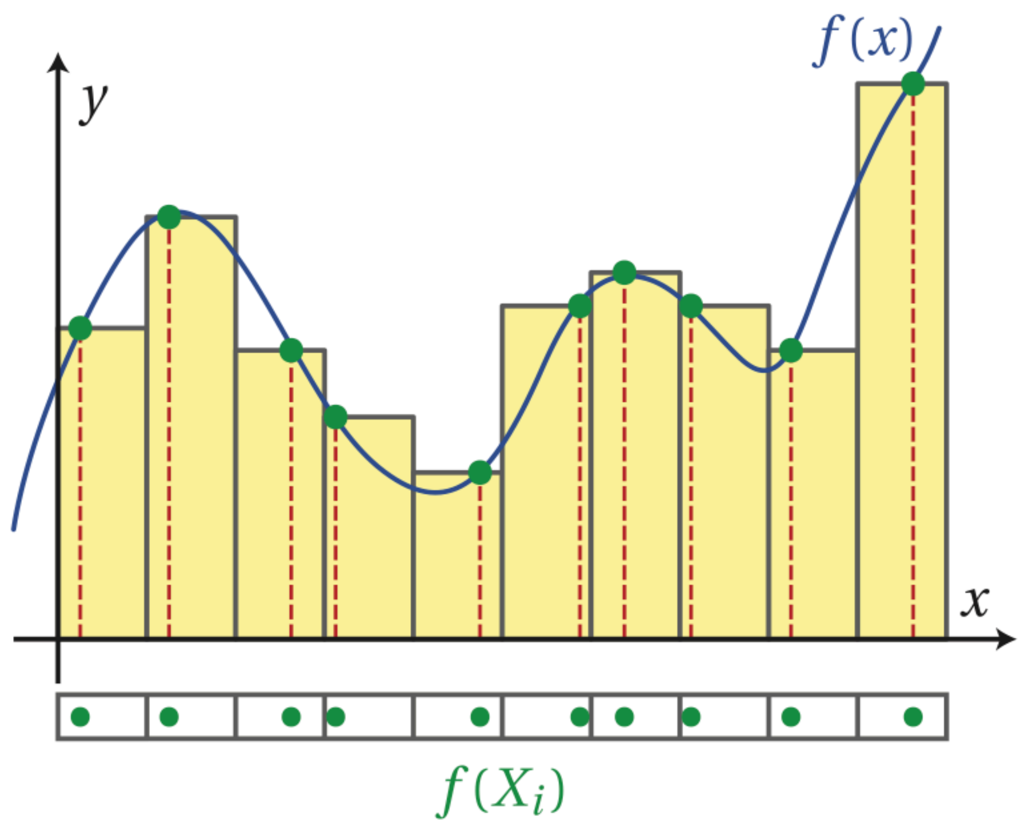
Monte Carlo Methods

A numerical method of solving problems by random sampling

- Comparison to Deterministic Method — Recall Riemann Sum



Riemann sum

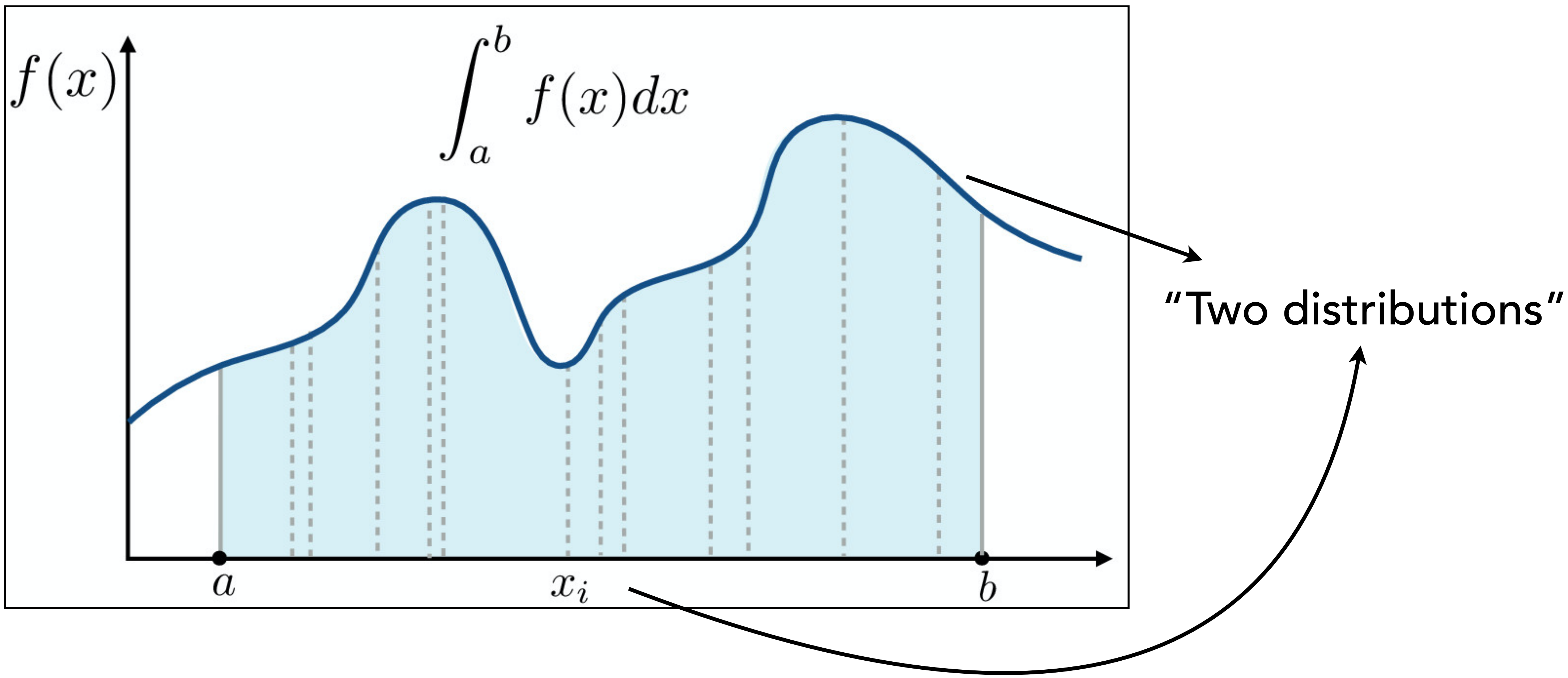


Stratified Monte Carlo integration

Monte Carlo Estimator

A numerical method of solving problems by random sampling

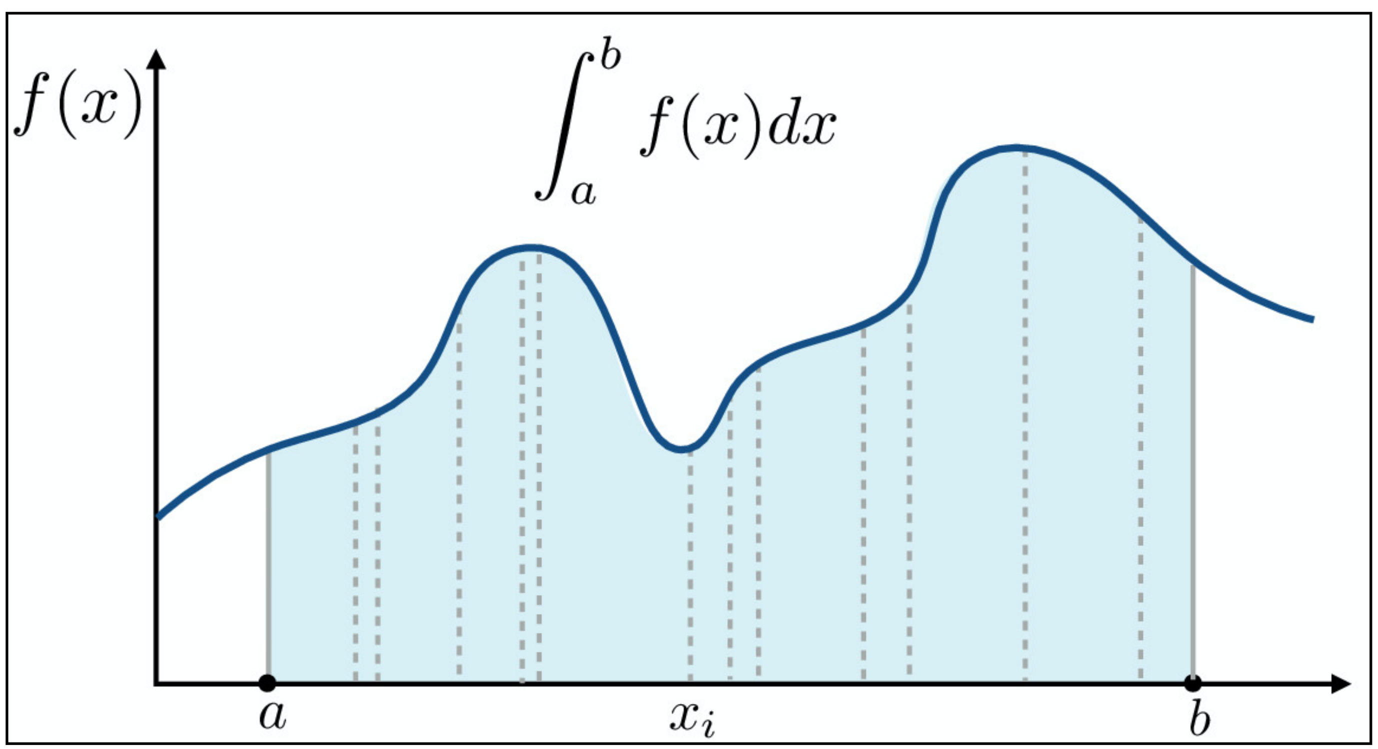
- Error Not Dependent on the Dimension of the Integral
- Approximation Error Not Dependent on the Smoothness of the Functions

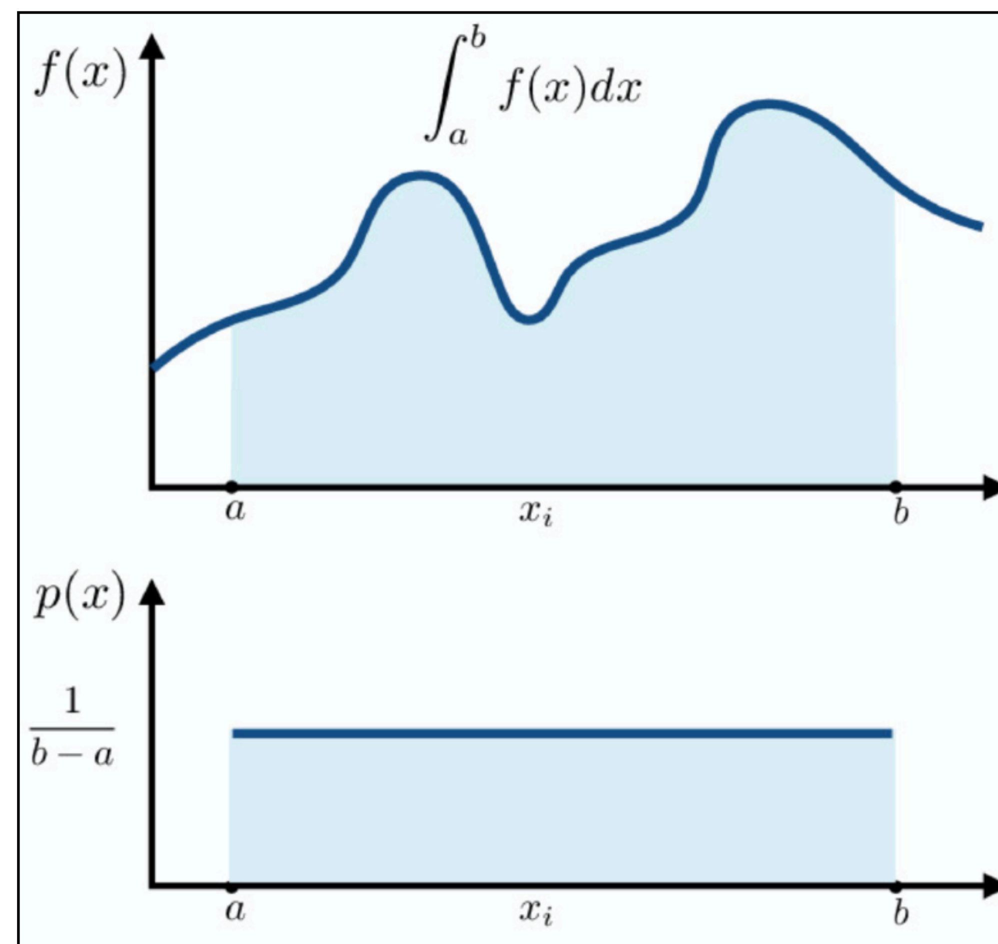


Monte Carlo Estimator

- The Distribution of the Function
- The Distribution of Random Variable for Sampling

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$





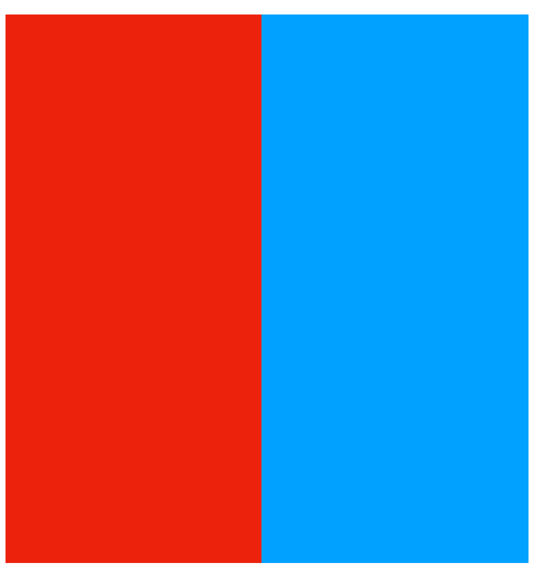
Monte Carlo Estimator

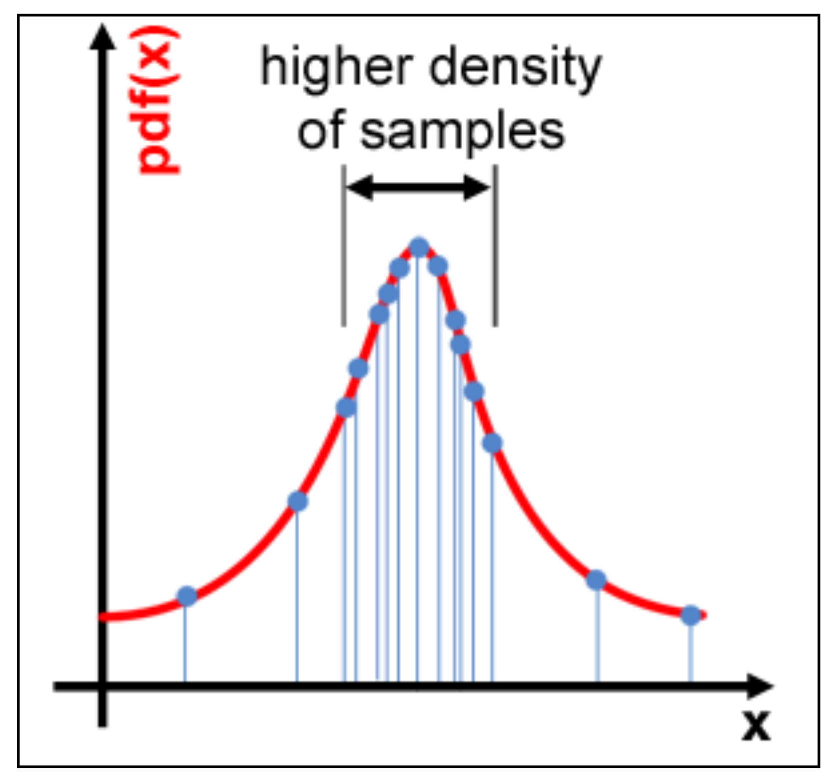
$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$

Or

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{pdf(X_i)}$$

Why dividing by pdf ?





Bias and Consistency

- For any given estimator, is it consistent? Is it biased?
- Consistency: “converges to the correct answer”
- Unbiased: “estimate is correct on average”

Biased? Consistent?

- We try to estimate the integral of an image by:
  - Taking m by m samples at fixed grid position
  - Sum each sample together

m=4

m=16

m=64

m=∞

Properties of Monte Carlo Estimator

$$\hat{F}_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{pdf(X_i)}$$

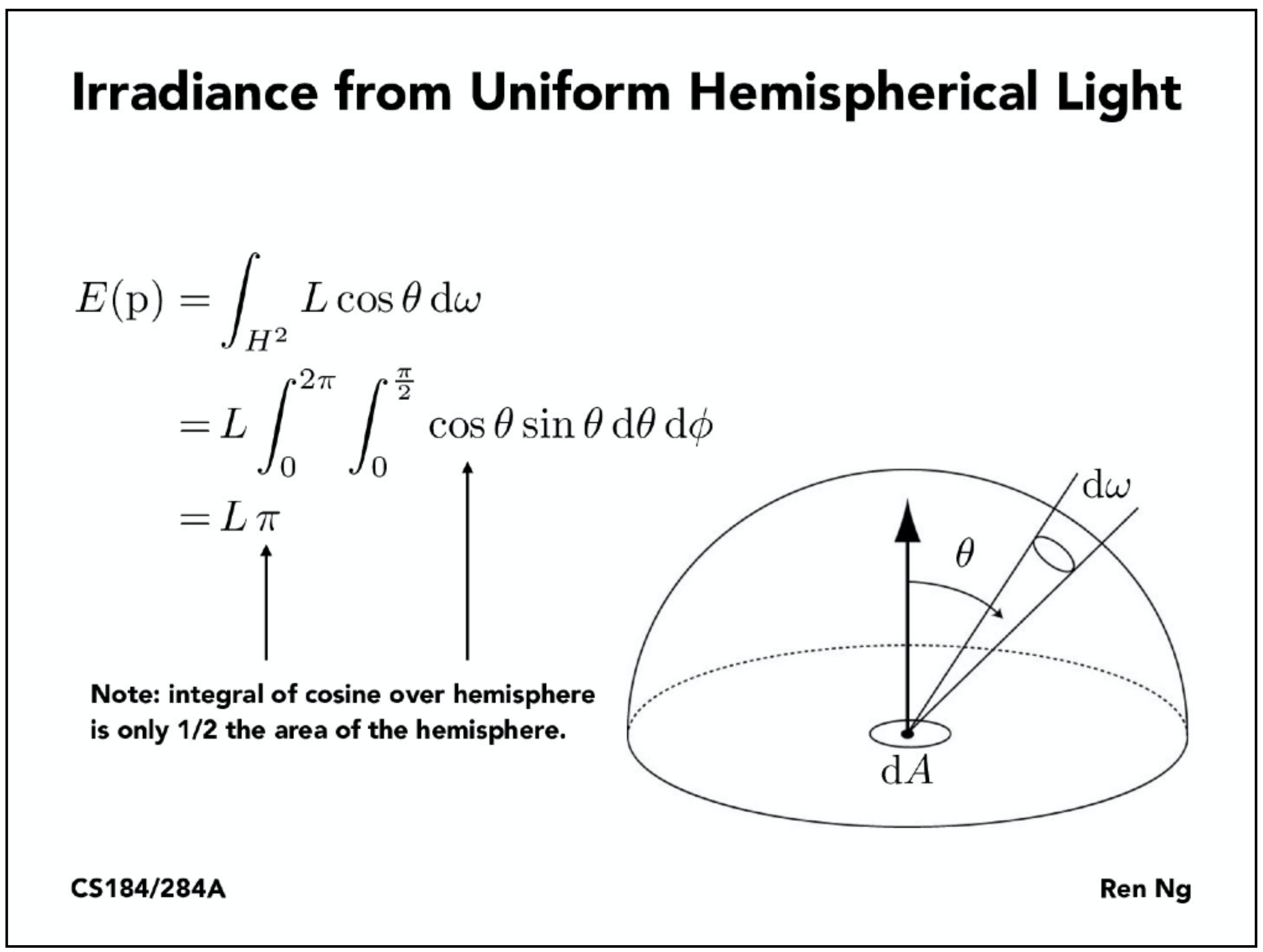
Monte Carlo Estimate

$$F = \int_{\Omega} f(x)dx$$

True integral

- It's stochastic —> reduce variance!
- A Monte Carlo estimator is unbiased and consistent
- Monte Carlo estimation converges to the function expected value, as the sample size approaches infinity "the law of large number"
- Easily extended to multi-dimension integral
- Parallel Nature: each processor of a parallel computer can be assigned the task of making a random trial

Monte Carlo Estimator — Integration over a Hemisphere



$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{pdf(X_i)}$$

Uniform sampling over hemisphere:

$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{L(p, \omega_i) \cos\theta_i}{?}$$

Monte Carlo Estimator
— Integration over a Hemisphere

$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{L(p, \omega_i) \cos \theta_i}{\frac{1}{2\pi}}$$

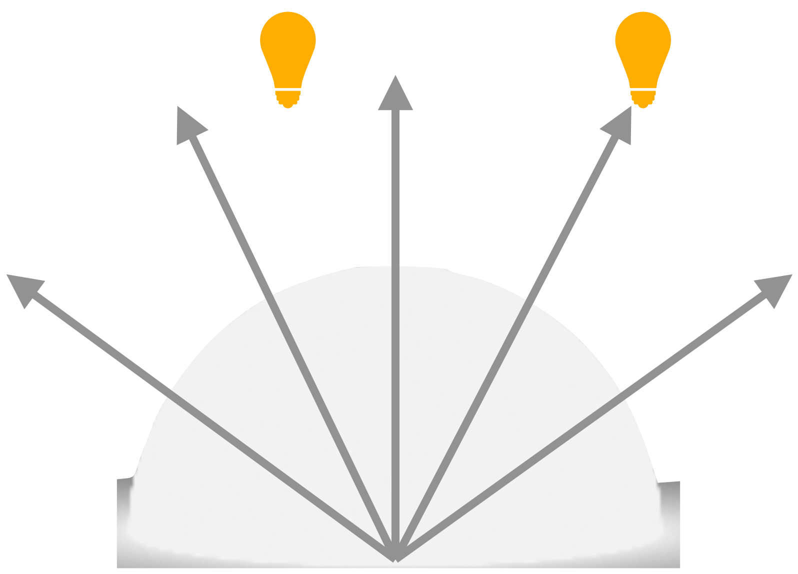
```
for i in 0...N:
  w_i = get_sample_from_hemisphere
  cos_theta = f(w_i, normal)
  L_i = L(p, w_i)
  E += cos_theta * L_i / N * pdf_of_sample
```

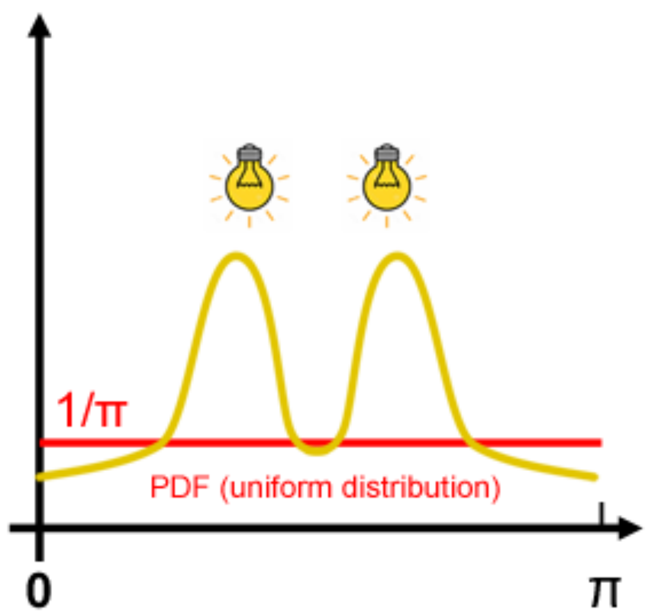
Where to sample more?

Hall of Mammals (by Oliksiy Yakovlyev)

Importance Sampling

- We easily missed some portion of light that contributes a lot



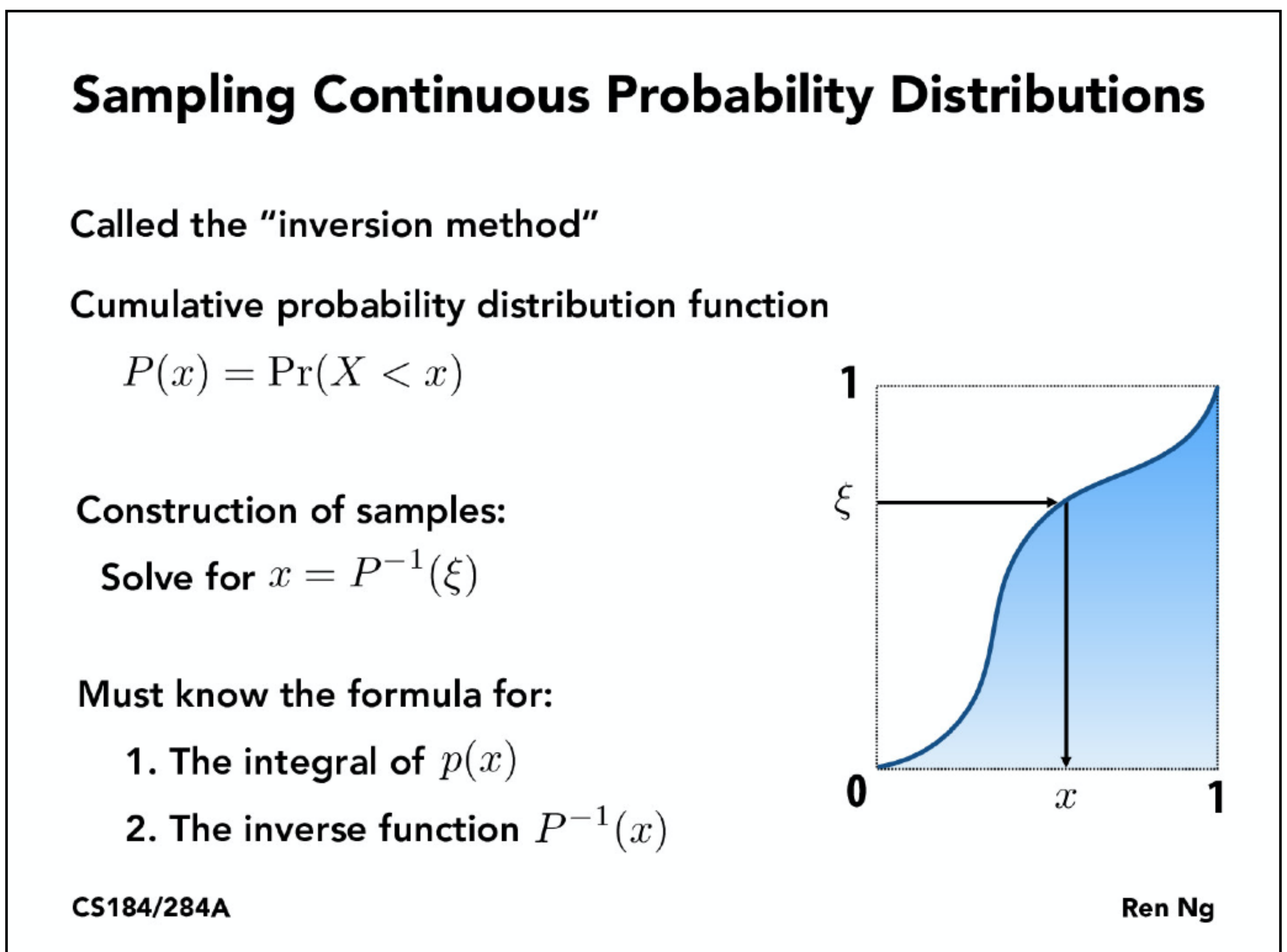


- The goal of Monte Carlo Estimator is to have ___ variance
- If $pdf(x)$ is a function proportional to $f(x)$, then the variance of the Monte Carlo integral will be ___

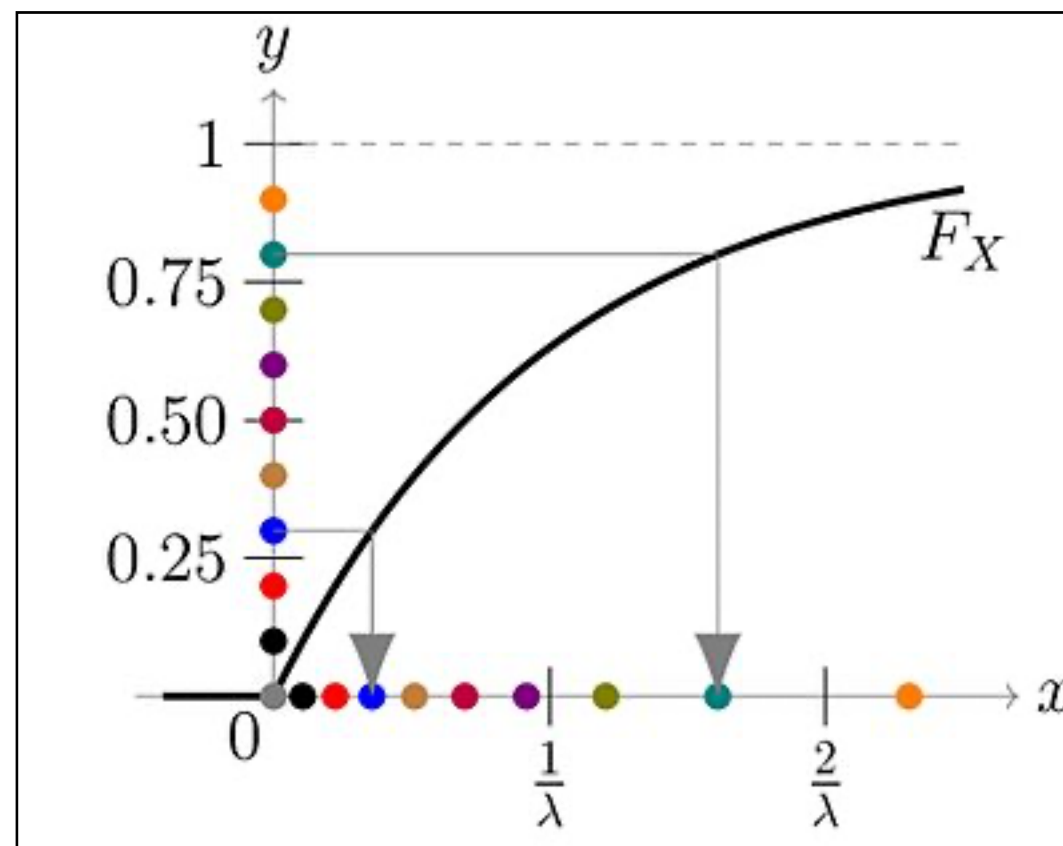
$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{pdf(X_i)}$$

Importance Sampling

- Inversion method — generate X from $U(0,1)$
- Probability Density Function: PDF
- Cumulative Density Function: CDF



What's the shape of the pdf given this cdf?



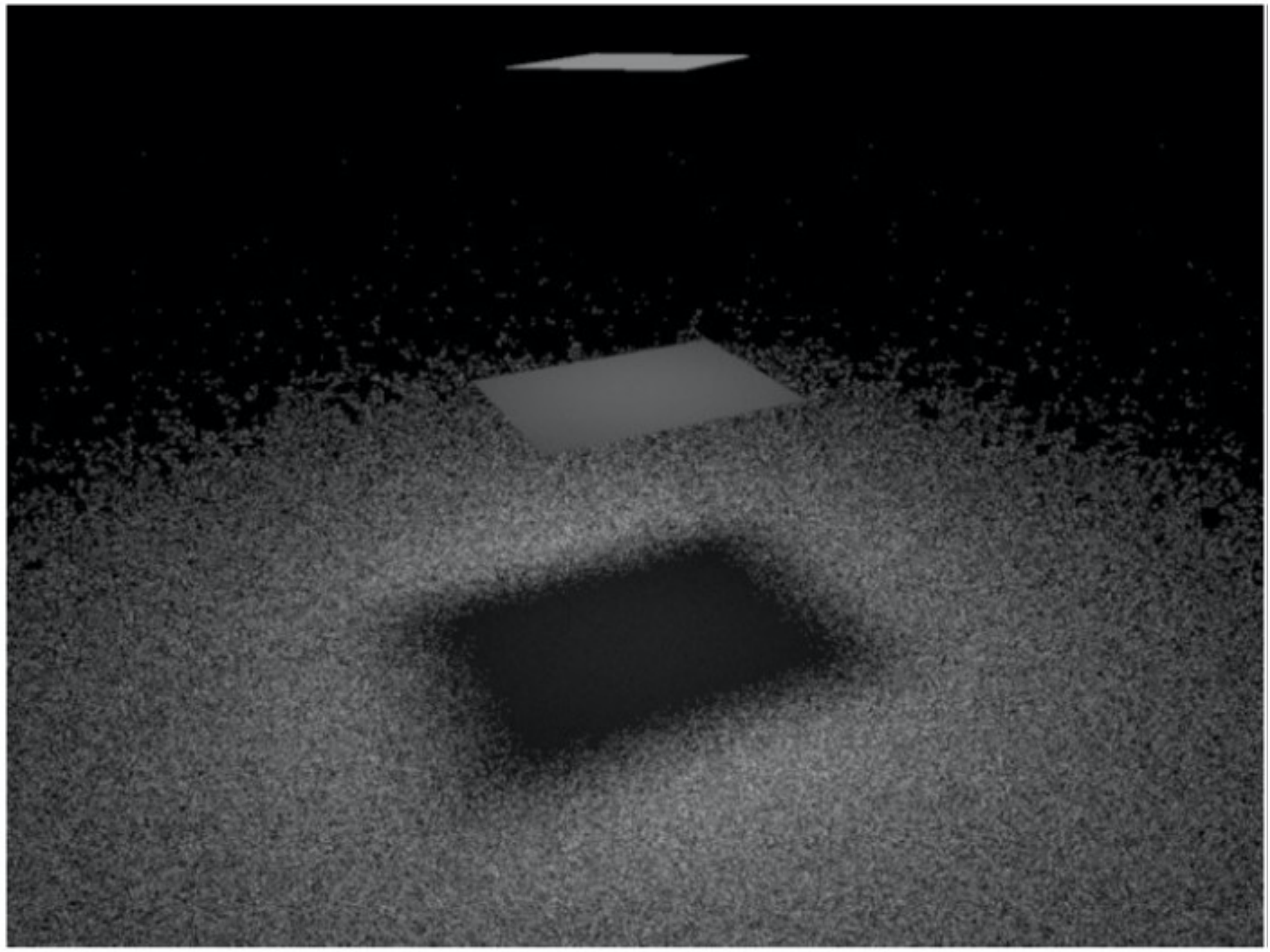
Importance Sampling

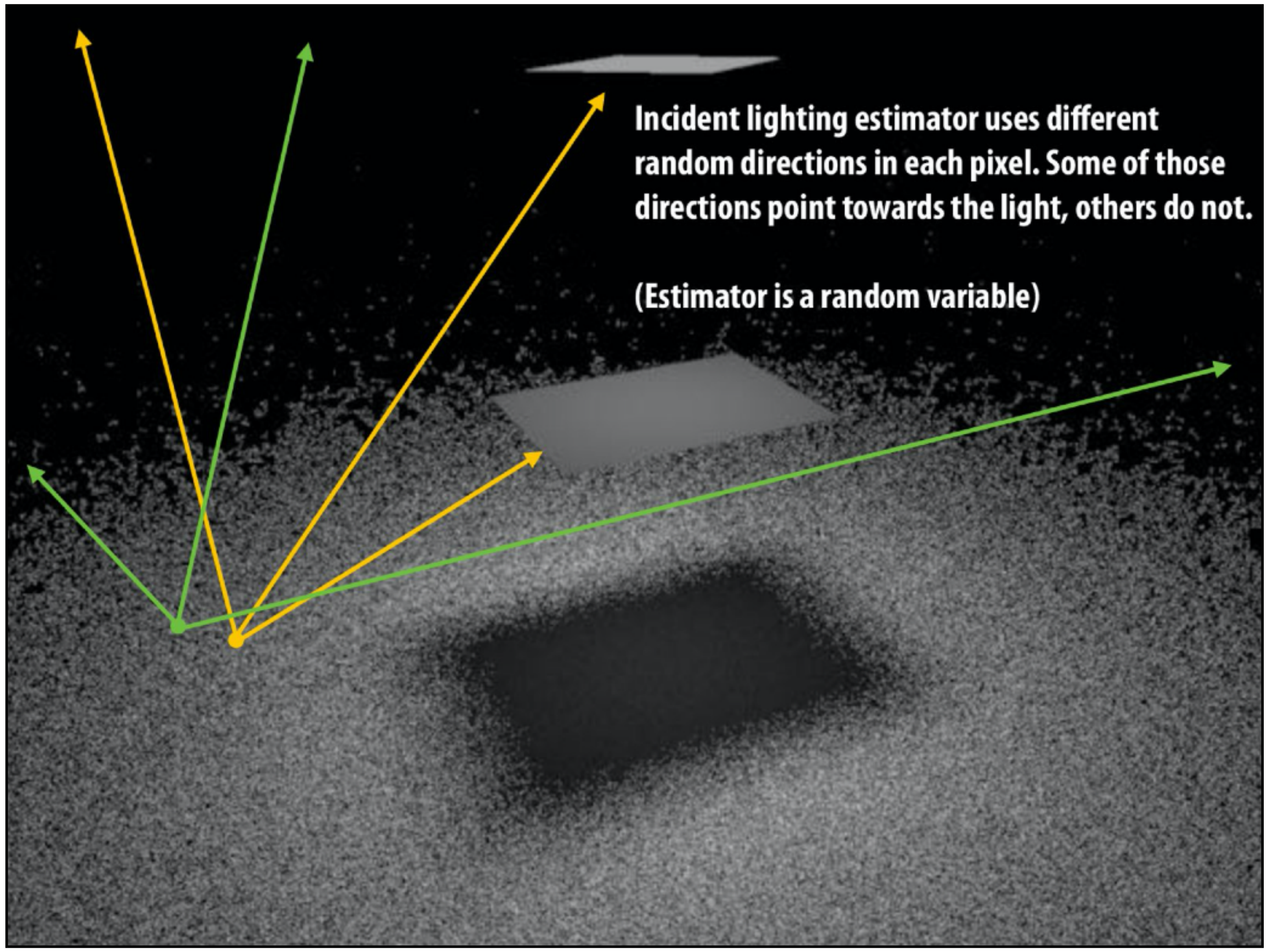
- Where in this code is different if sampling from different distributions

```
for i in 0...N:
  w_i = get_sample_from_hemisphere
  cos_theta = f(w_i, normal)
  L_i = L(p, w_i)
  E += cos_theta * L_i / N * pdf_of_sample
```

Demo

Without Importance Sampling

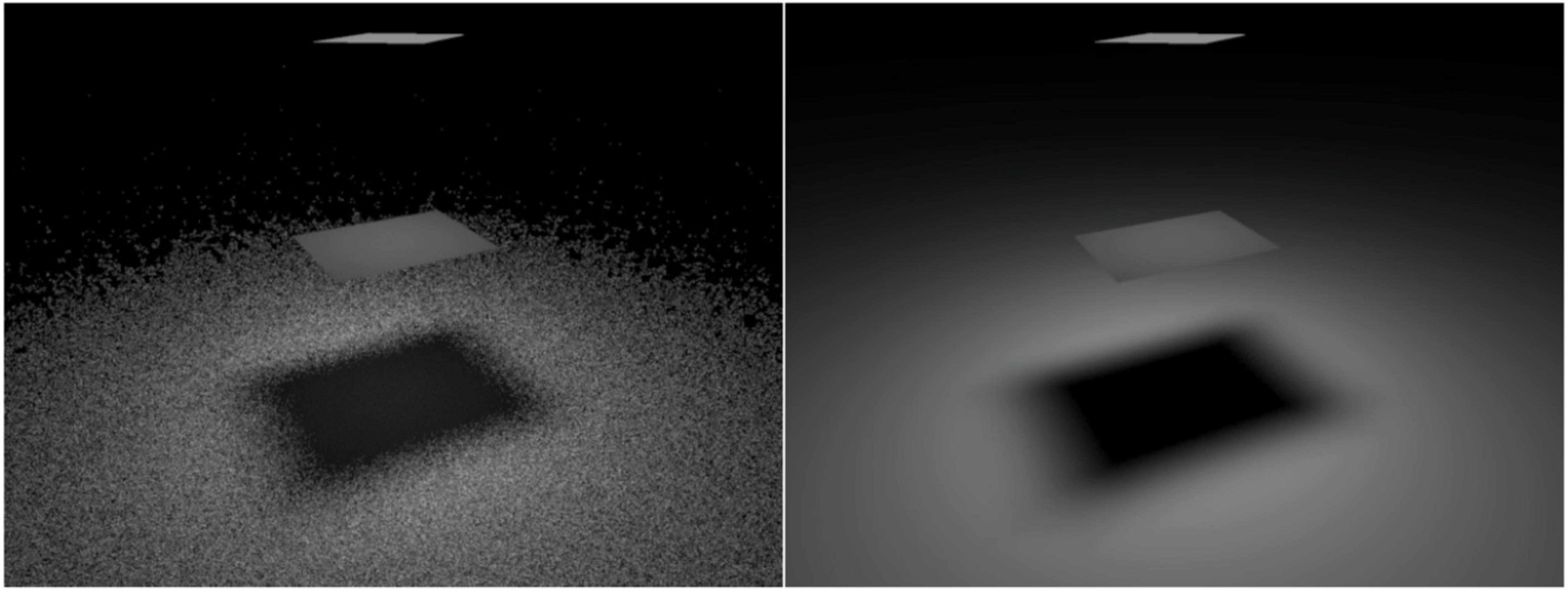




Sampling solid angle

100 random directions on hemisphere

With Importance Sampling



Sampling solid angle

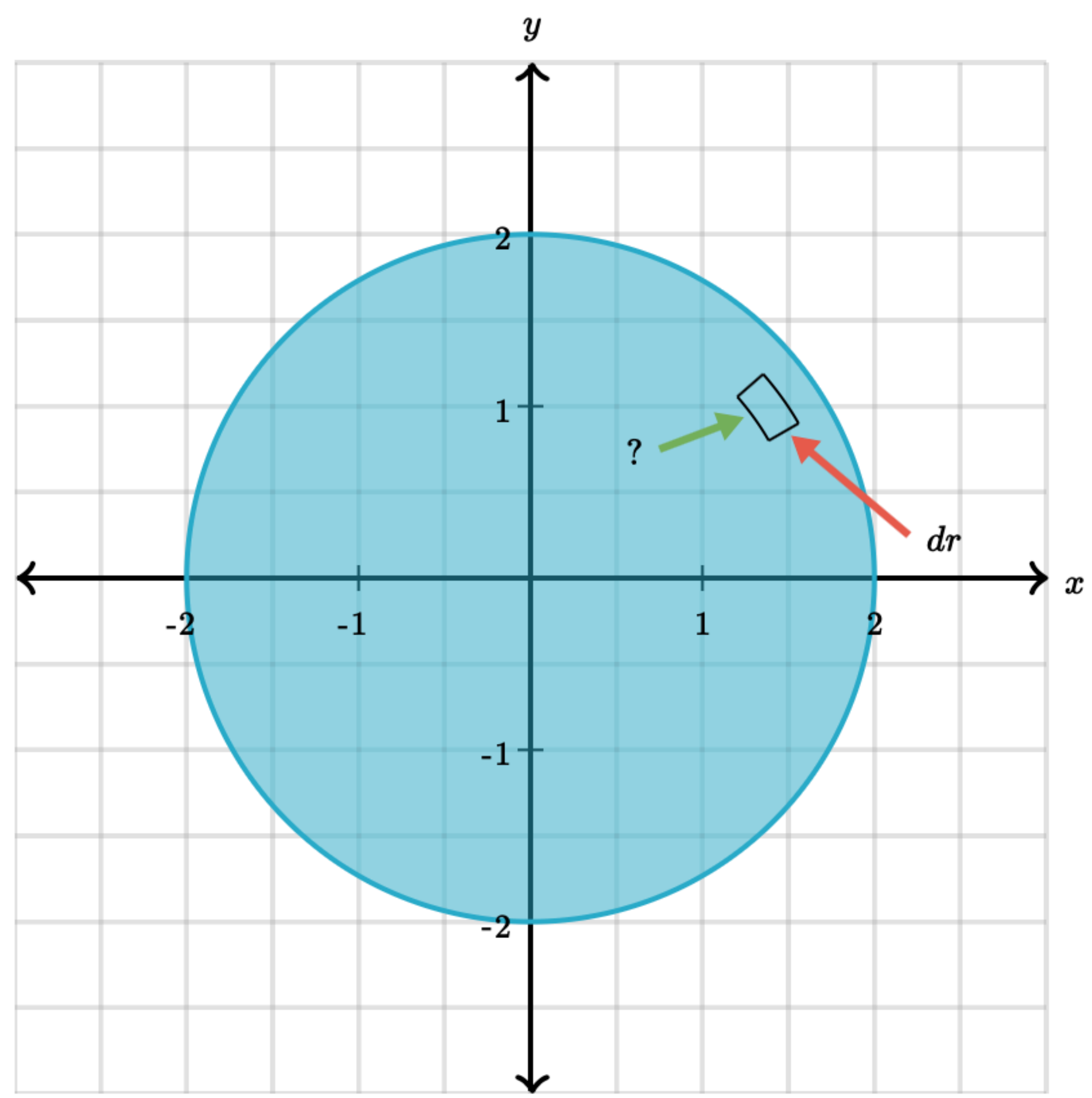
100 random directions on hemisphere

Sampling light source area

100 random points on area of light source

Sampling Random Variables

Uniform Sampling of a Unit Disk



Sampling Random Variables

Uniform Sampling of a Unit Disk

1. (requirement) PDF is constant with respect to area:

$$p(x, y) = \frac{1}{\pi} \quad \Rightarrow \quad p(r, \theta) = r \cdot p(x, y) = \frac{r}{\pi}$$

2. Marginal Density $\quad p(r) = \int_0^{2\pi} p(r, \theta) d\theta = 2r \quad \Rightarrow \quad P(r) = r^2$

3. Conditional Density $\quad p_\theta(\theta) = \frac{p(r, \theta)}{p(r)} = \frac{1}{2\pi} \quad \Rightarrow \quad P_\theta(\theta \,|\, r) = \frac{\theta}{2\pi}$

What's the assumption of this step?
(r and θ independent)

4. Inversion

$$P(r) = r^2 \quad \Rightarrow \quad r = \sqrt{\xi_1}$$

$$P_\theta(\theta \,|\, r) = \frac{\theta}{2\pi} \quad \Rightarrow \quad \theta = 2\pi\xi_2$$

generate less samples for smaller radii

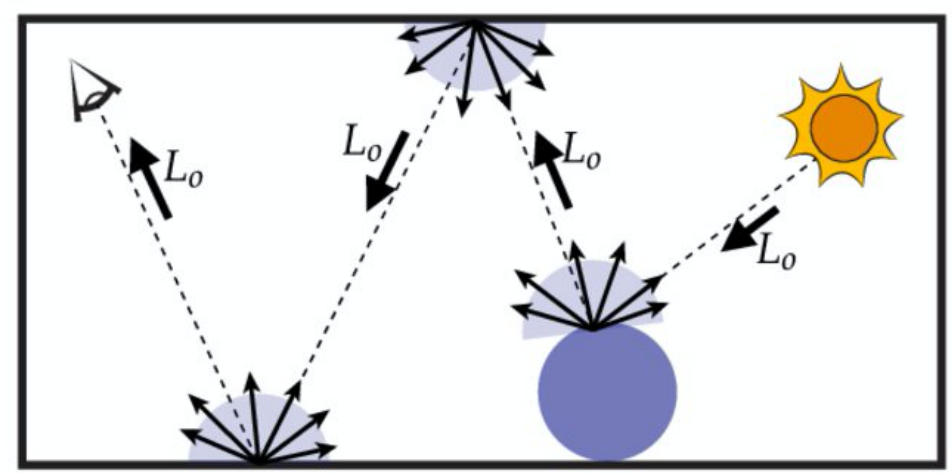
Transforming Between Distributions

Example (polar coordinates):

- sample (r, θ) with density $p(r, \theta)$
- $x = r\cos\theta$ and $y = r\sin\theta$
- $J(x) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ and its determinant $|J(x)| = r$

Monte Carlo Path Tracing

- Indirect illumination
- Forward and backward tracing



$$L_o(\mathbf{p}, \omega_o) \;=\; L_e(\mathbf{p}, \omega_o) + \int_{\mathcal{H}^2} f_r(\mathbf{p}, \omega_i \to \omega_o) L_i(\mathbf{p}, \omega_i) \cos\theta \, d\omega_i$$

Demo