Triangles and Sampling 2

CS 184: Foundations of Computer Graphics

1 Rasterization Potpourri

1. What is the connection between rasterizing a polygon and discretizing a continuous function?

   Solution: A polygon in vector graphics format can be defined by an indicator function on continuous two-dimensional space. (From lecture, **inside**(tri, x, y).) A raster display, often a grid of pixels, is a discrete two-dimensional space.

   To rasterize a polygon is to approximate the indicator function in discrete space, i.e. to discretize the conitnuous function.

2. Why does aliasing occur? Describe or draw an example of aliasing of any kind.

   Solution: Aliasing occurs when high-frequency signals are undersampled (sampled at an insufficient frequency). The sampled function will incorrectly resemble a lower-frequency signal.

   Examples: jaggies, wagon-wheel effect, moiré pattern, etc.

3. What is the connection between applying a box blur to an image and supersampling each pixel within the image?

   Solution: One way to avoid aliasing is to filter out high-frequency signals before sampling.

   A box blur is a low-pass filter with a discrete, uniform kernel. Applied to an image, a box blur filters out high-frequency signals. It's an approximation of a filtered continuous signal.

   Supersampling (then downsampling) approximates sampling a low-pass filtered continuous signal. Per pixel, the average value within the pixel is approximated by taking multiple samples, then taking their average.

   Both the box blur and supersampling approximate the anti-aliasing technique of filtering out high-frequency signals, then sampling.

2 Tying Shoelaces

The cross product between two vectors, **a** and **b**, in three-dimensional space is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

1. Given two vectors, $\mathbf{a} = \begin{pmatrix} 10 \\ 7 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ -11 \\ 0 \end{pmatrix}$, calculate $\mathbf{a} \times \mathbf{b}$. Next, calculate $\mathbf{b} \times \mathbf{a}$.

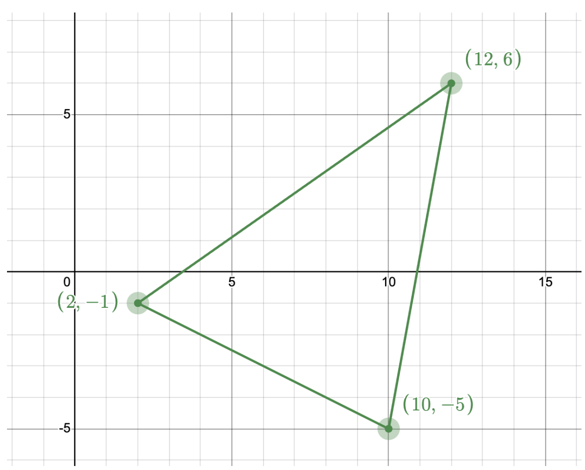
Solution: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 7 & 0 \\ -2 & -11 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -96 \end{pmatrix}$

$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -11 & 0 \\ 10 & 7 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 96 \end{pmatrix}$

Notice that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. So, the magnitude of both cross-product vectors is the same. However, $\mathbf{a} \times \mathbf{b}$ is on the negative z-axis (going into the page), whereas $\mathbf{b} \times \mathbf{a}$ is on the positive z-axis (coming out of the page).

2. Draw the triangle given by points $(2, -1)$, $(12, 6)$, $(10, -5)$. What is the *winding order* of the triangle? In other words, are the points given in clockwise or counter-clockwise order?

Solution: The points are in clockwise order.



3. What are the three vectors defined by the edges of this triangle? Assume the triangle is lying in the xy-plane. Select two of the three vectors. Calculate their cross product.

Solution: $\mathbf{a} = \begin{pmatrix} 10 \\ 7 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ -11 \\ 0 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -8 \\ 4 \\ 0 \end{pmatrix}$.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -11 & 0 \\ -8 & 4 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -96 \end{pmatrix} = -\mathbf{c} \times \mathbf{b}$$

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 4 & 0 \\ 10 & 7 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -96 \end{pmatrix} = -\mathbf{a} \times \mathbf{c}$$

4. In general, given triangle vertices $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, how can winding order be determined?

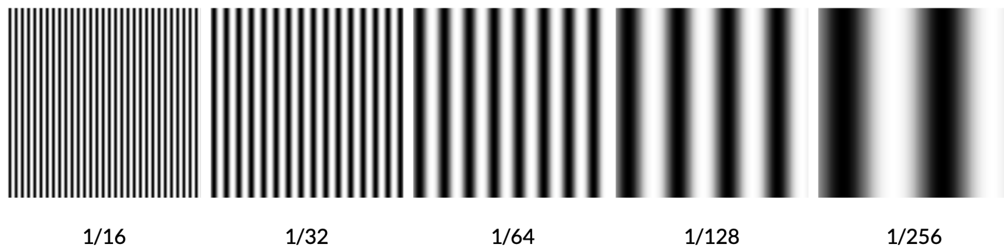
Solution:

One method is to calculate the cross product of two consecutive edges. For example, $(\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_1)$. This is equivalent to calculating $((x_1 - x_0)(y_2 - y_1) - (x_2 - x_1)(y_1 - y_0))\mathbf{k}$. If the component on the z-axis is negative, winding order is clockwise. If the component on the z-axis is positive, winding order is counterclockwise.

There are other approaches that can be taken. For example, $(\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0) = (\mathbf{p}_1 - \mathbf{p}_0) \times -(\mathbf{p}_0 - \mathbf{p}_2) = (\mathbf{p}_0 - \mathbf{p}_2) \times -(\mathbf{p}_1 - \mathbf{p}_0)$. This is equivalent to the first approach — that is, calculating the cross-product in the order that the vertices are presented in.

3 Nyquist Theorem

1. We have five examples of spatial frequencies in cycles per pixel (given by the number below the image). What frequency should each image be sampled at to avoid aliasing?



Solution: By the Nyquist theorem, we need to sample at a rate greater than double the highest signal frequency. Thus, in order, the images should be sampled at $> \frac{1}{8}$, $> \frac{1}{16}$, $> \frac{1}{32}$, $> \frac{1}{64}$, $> \frac{1}{128}$.

2. We have two wheels that rotate at different speeds with different numbers of spokes that we want to record with our camera.

- **Wheel A** has 4 spokes and rotates at a rate of 6 rotations per second.
- **Wheel B** has 6 spokes and rotates at a rate of 5 rotations per second.

What frame rate (in frames per second) would our camera need to avoid aliasing effects?

Solution: In 1 rotation, a wheel with n spokes shows the same "image" n times, so if that rotation took 1 second, it would have a frequency of n hz. Knowing that:
Wheel A has a signal frequency of $\frac{4}{rot} * 6\frac{rot}{s} = 24hz$.
Wheel B has a signal frequency of $\frac{6}{rot} * 5\frac{rot}{s} = 30hz$.
To avoid aliasing, using the Nyquist theorem, we need to sample at a rate more than double the highest signal frequency - B's in this case, so we need a rate faster than 60 fps.

4 Ashley vs. Helicopter

1. Ashley's video camera records at 128 frames per second. The helicopter she is filming has eight blades that rotate. At what rotation rate(s) of the helicopter rotor will Ashley's video suffer from aliasing effects? Give your answer in rotations per second.

   Solution: The Nyquist frequency of Ashley's video camera is 64 frames per second (half the sampling frequency). So long as the helicoper rotates at less than 64 cycles per second, Ashley will not see any aliasing. At 64 cycles per second and greater, Ashley will see aliasing.

   The helicopter has 8 cycles/rotation, so rotating at 8 rotations/second achieves the Nyquist frequency of 64 cycles per second. Any rotation frequency greater than 8 rotations/second will also cause the video to suffer from aliasing.