Course Roadmap

Rasterization Pipeline

Core Concepts
- Sampling
- Antialiasing
- Transforms

Geometric Modeling

Lighting & Materials

Cameras & Imaging

Intro

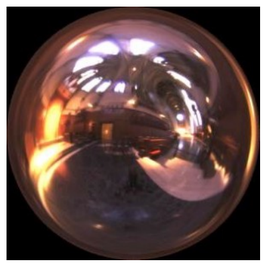
Rasterization

Transforms & Projection

Texture Mapping

Visibility, Shading, Overall Pipeline

← Starting today



Lecture 7:

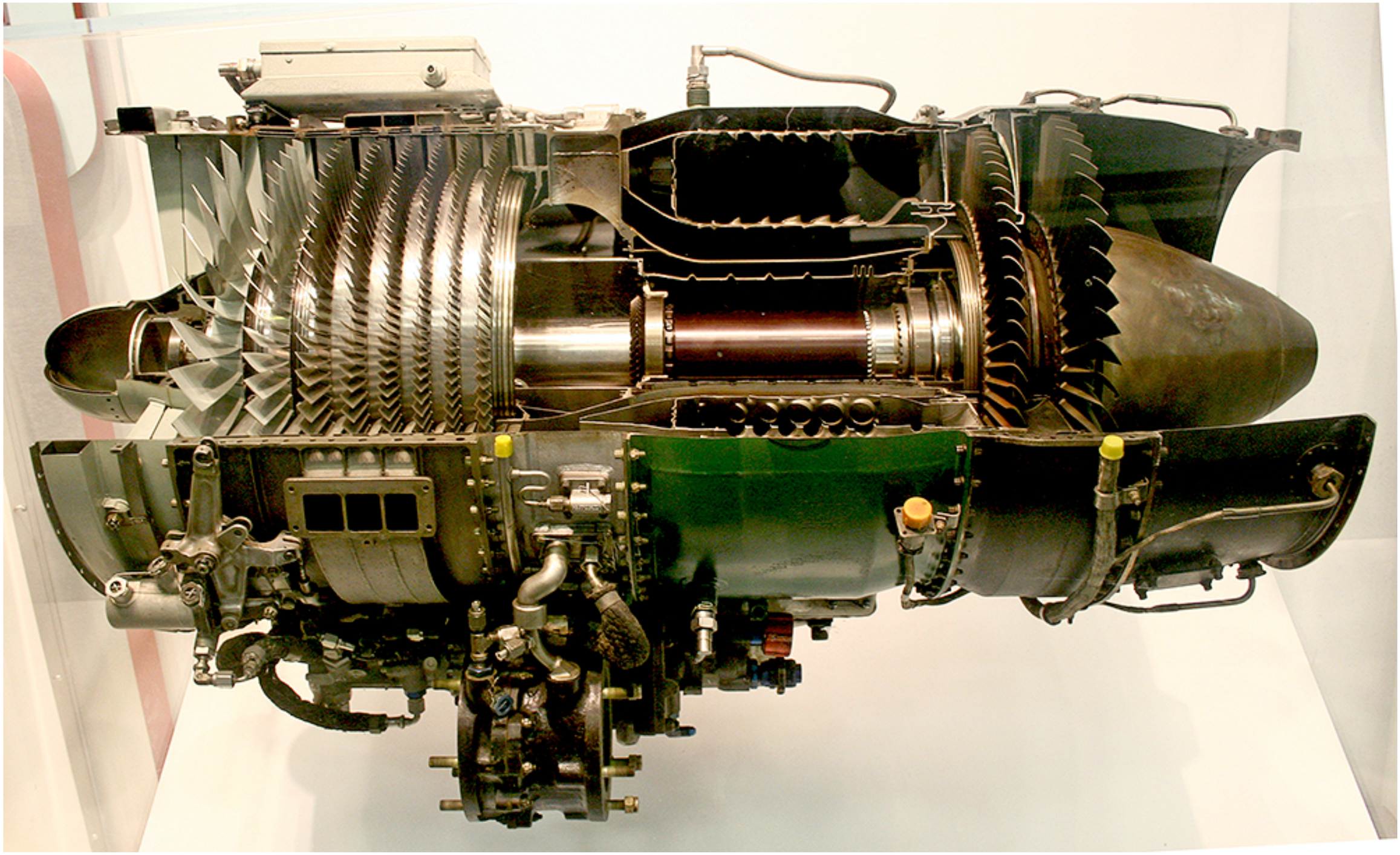
Introduction to Geometry, Splines and Bezier Curves

Computer Graphics and Imaging
UC Berkeley CS184/284A

Examples of Geometry

Examples of Geometry

Examples of Geometry



Examples of Geometry

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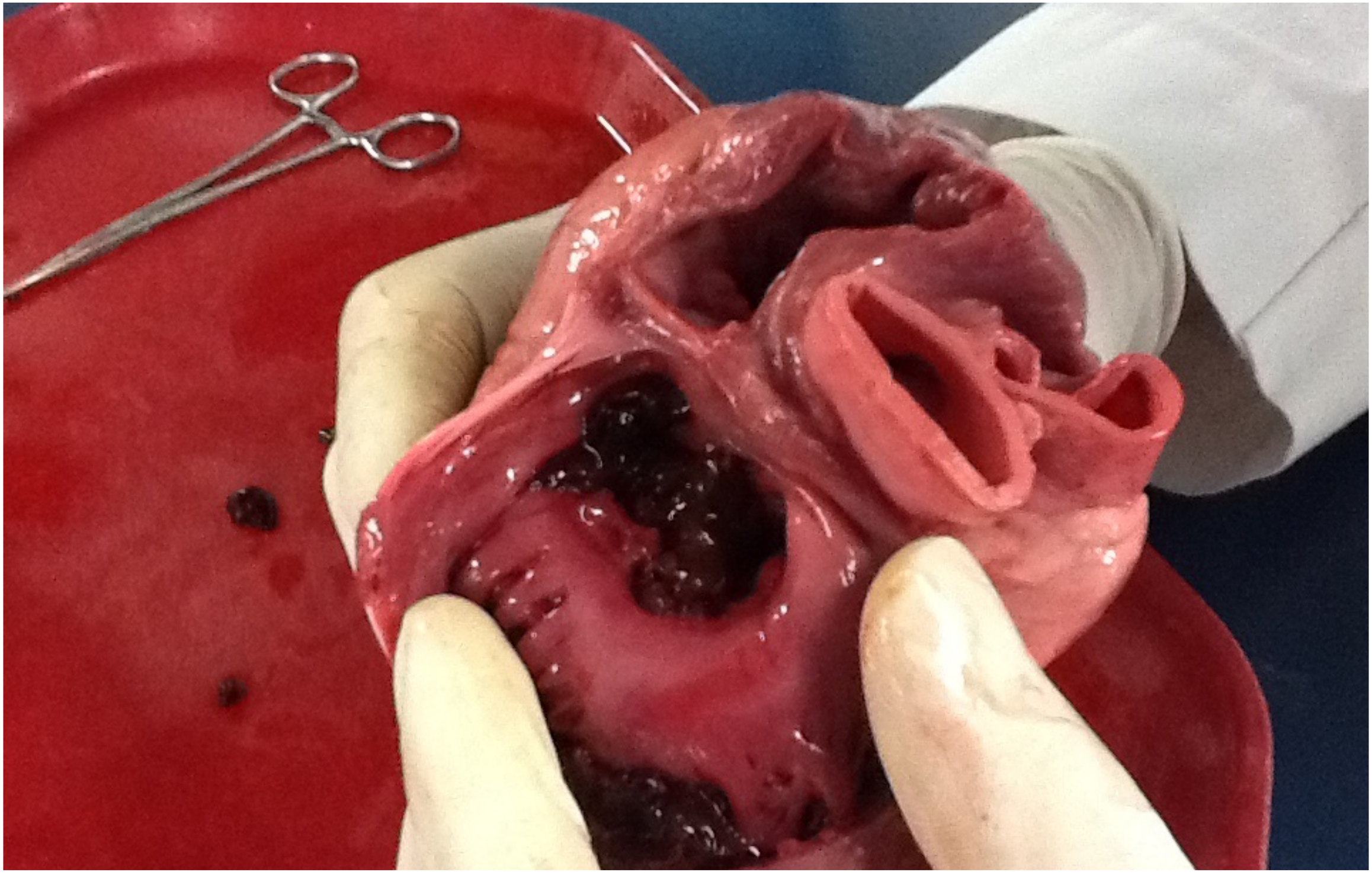


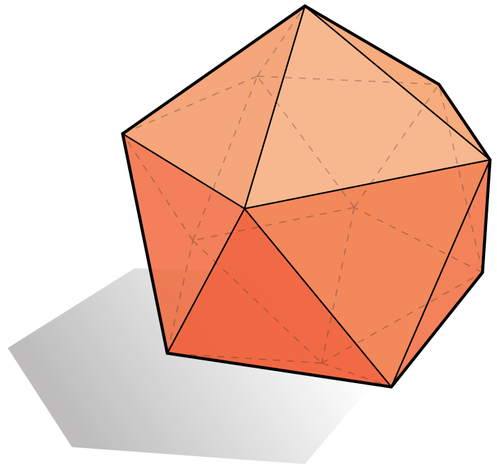
Many Ways to Represent Geometry

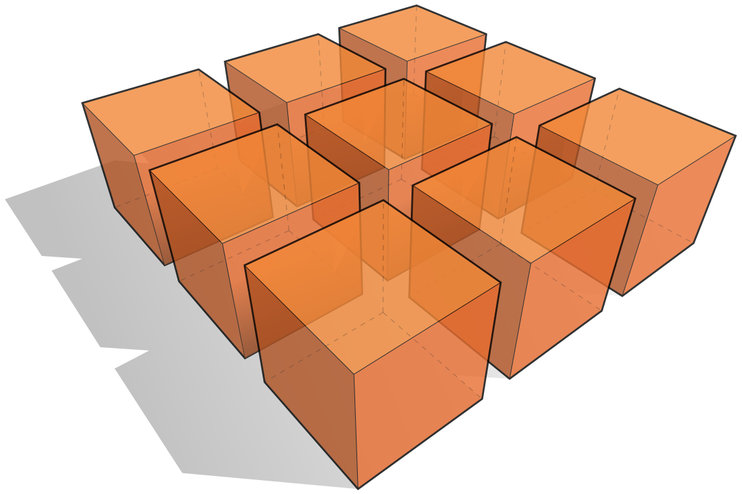
Explicit

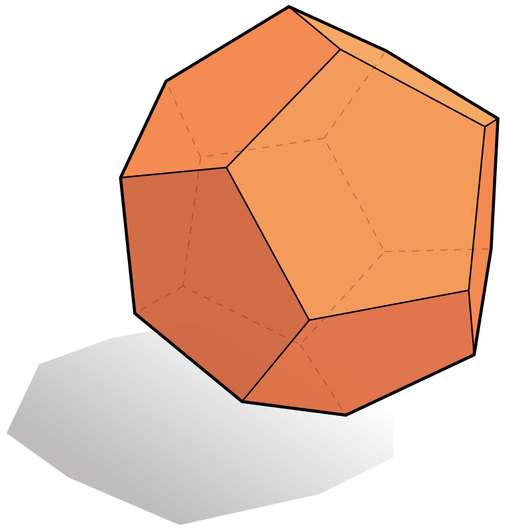
- point cloud
- polygon mesh
- subdivision, NURBS
- ...

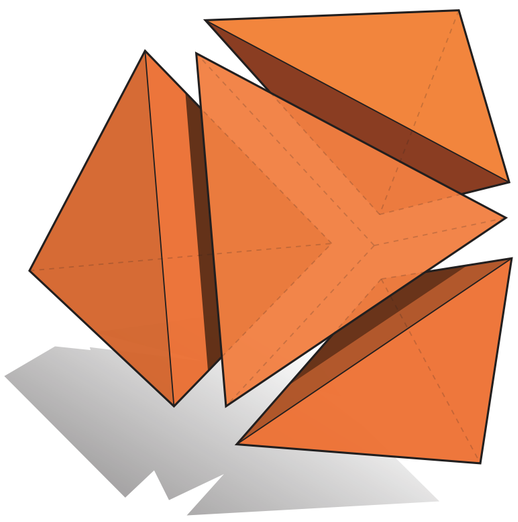
Implicit

- level sets
- algebraic surface
- distance functions
- ...









Each choice best suited to a different task/type of geometry

Smooth Curves

Smooth Curves and Surfaces

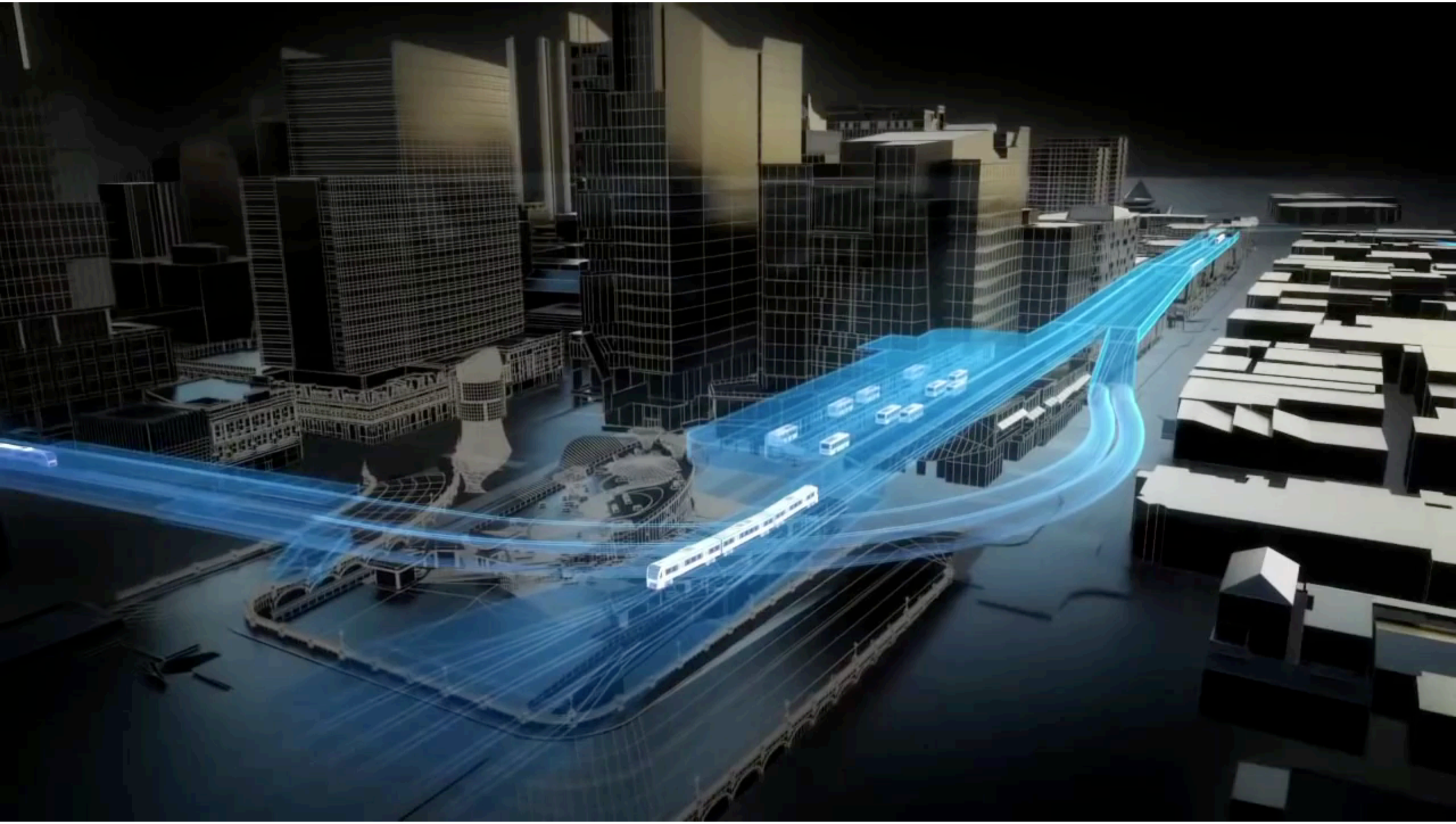
So far we can make:

- Things with corners (lines, triangles, squares, …)
- Specialty shapes (circles, ellipses, …)

Many applications require designed, smooth shapes

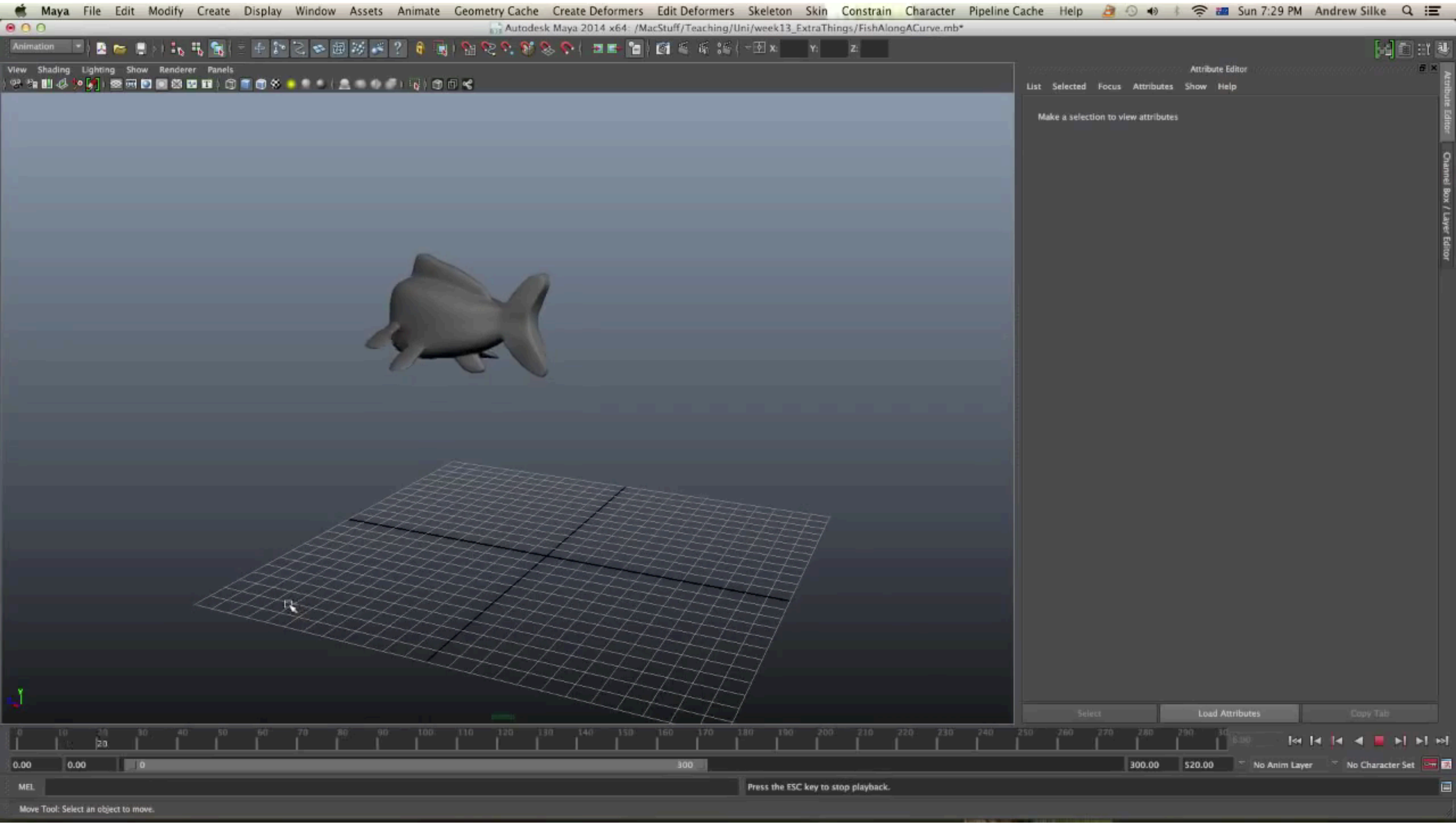
- Camera paths, vector fonts, …
- Resampling filter functions
- CAD design, object modeling, …

Camera Paths



Flythrough of proposed Perth Citylink subway, https://youtu.be/rIJMuQPwr3E

Animation Curves

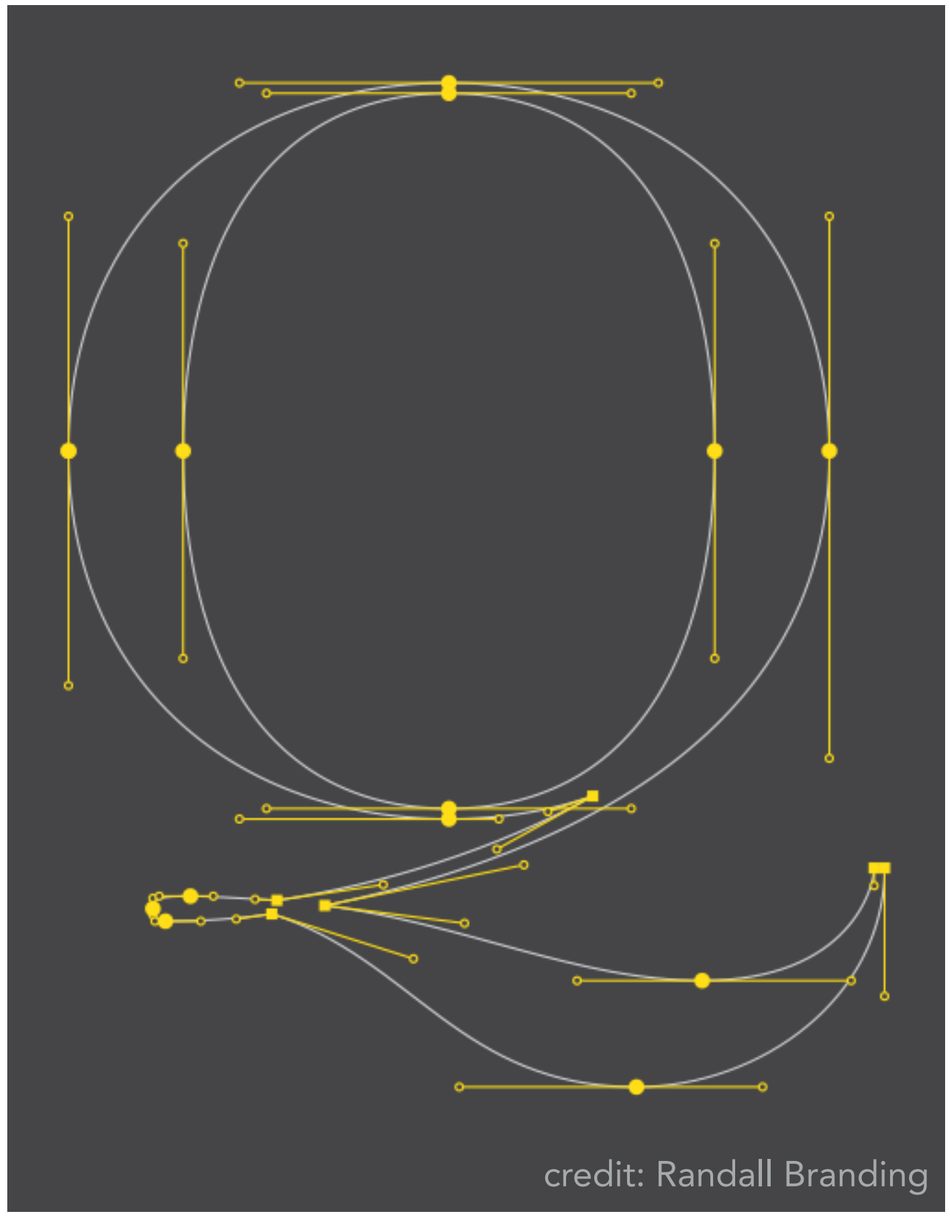


Maya Animation Tutorial: https://youtu.be/b-o5wtZlJPc

Vector Fonts

The Quick Brown Fox Jumps Over The Lazy Dog

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 0123456789



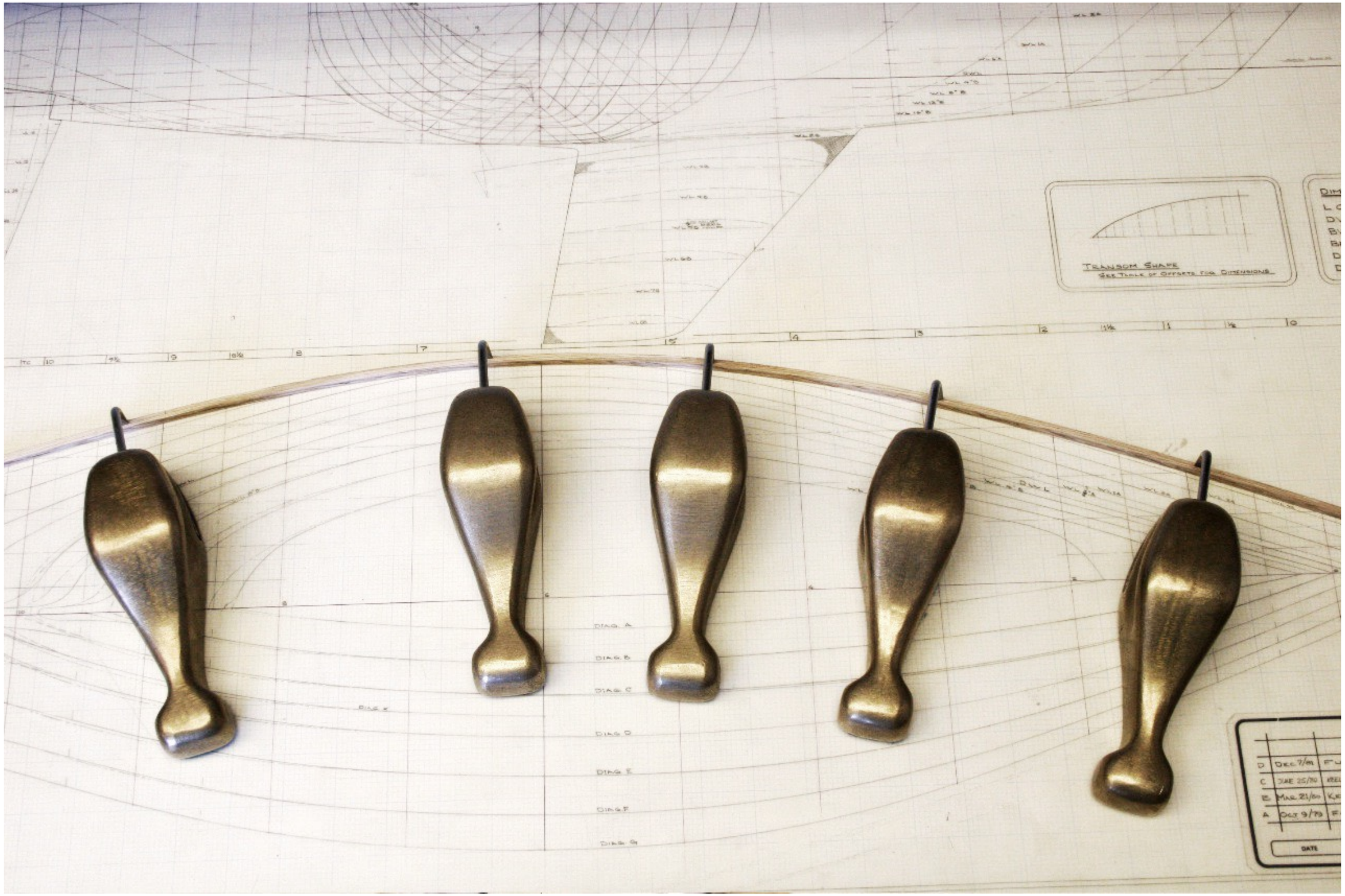
Baskerville font - represented as cubic Bézier splines

CAD Design

3D Car Modeling with Rhinoceros

Splines

A Physical Spline for Hand-Drafting



http://www.alatown.com/spline-history-architecture/

Spline Topics

Interpolation

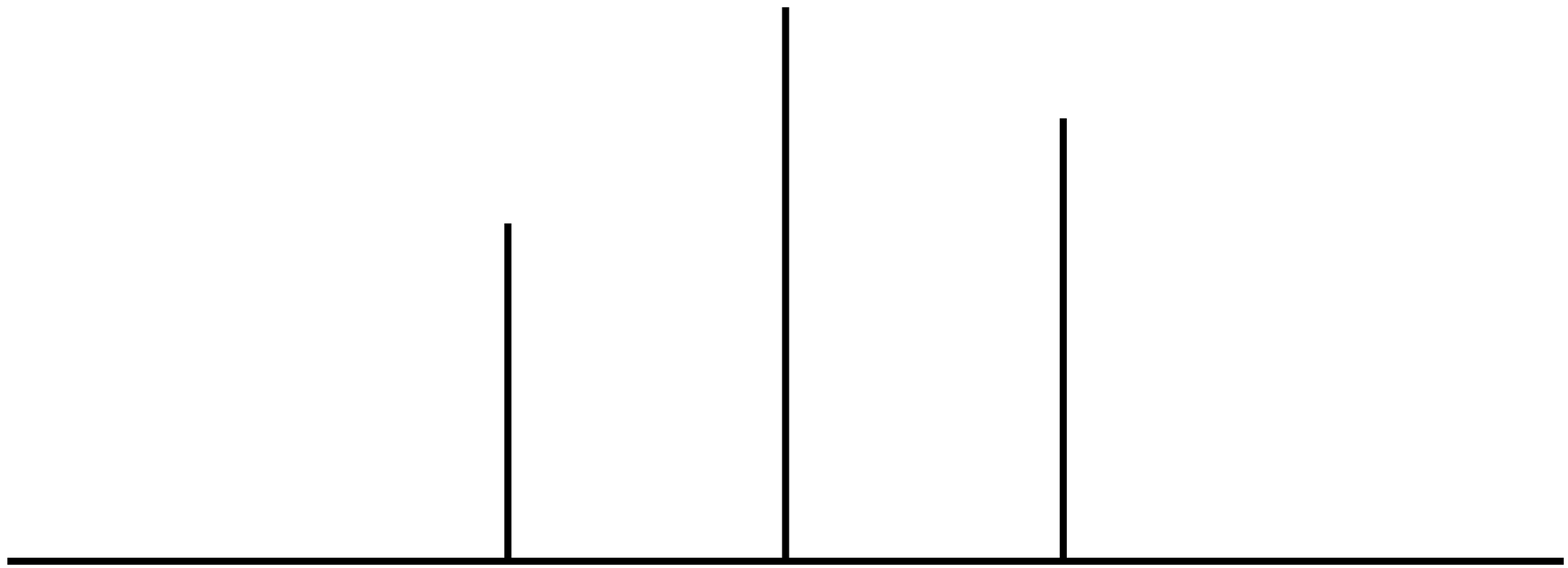
- Cubic Hermite interpolation
- Catmull-Rom interpolation

Bezier curves

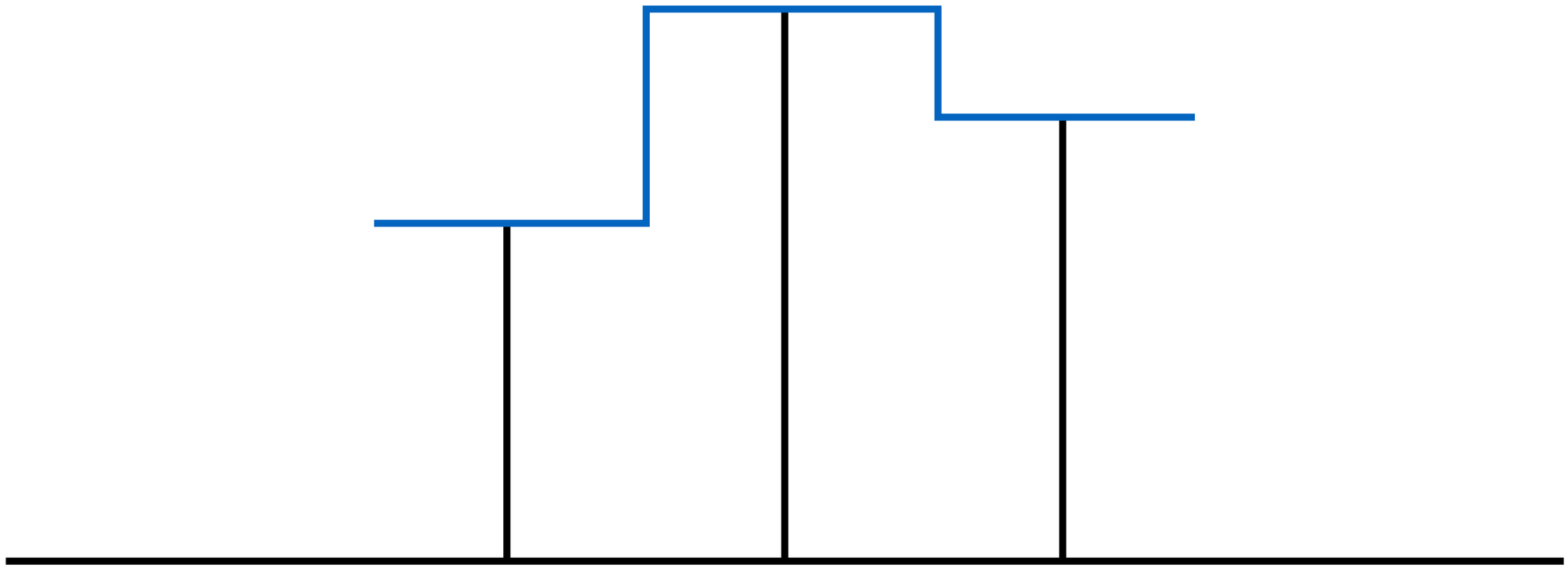
Bezier surfaces

Cubic Hermite Interpolation

Goal: Interpolate Values

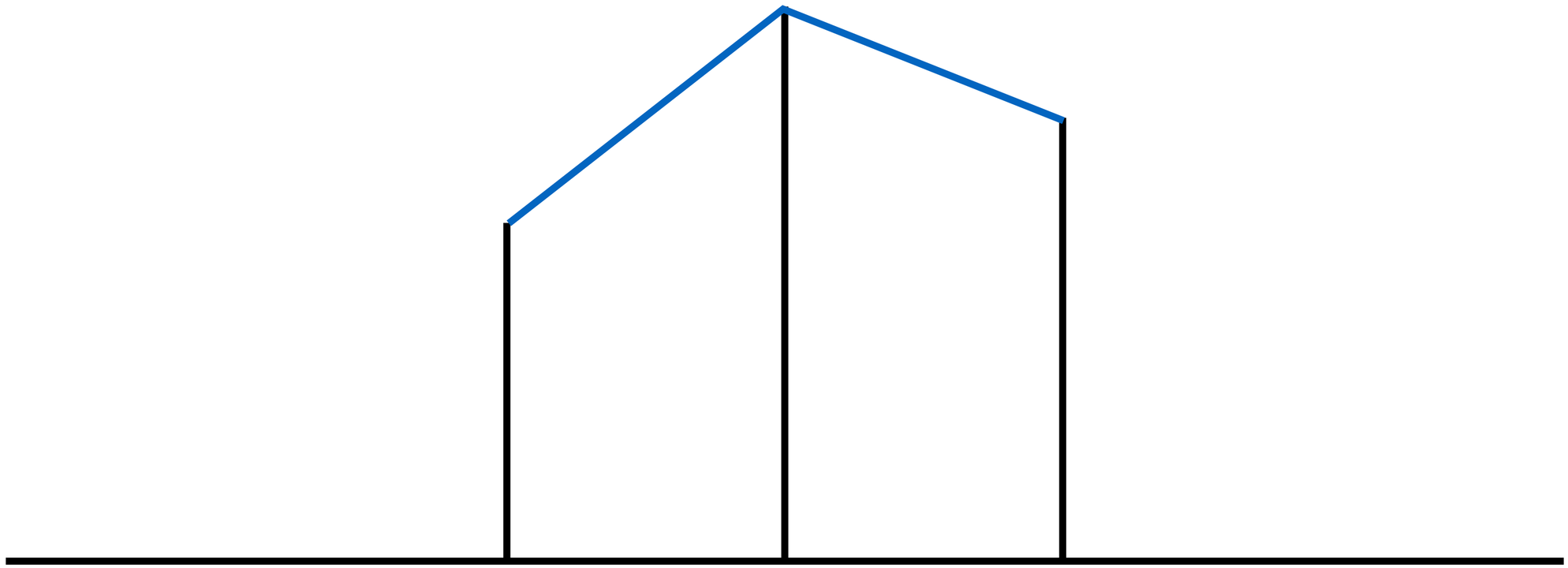


Nearest Neighbor Interpolation



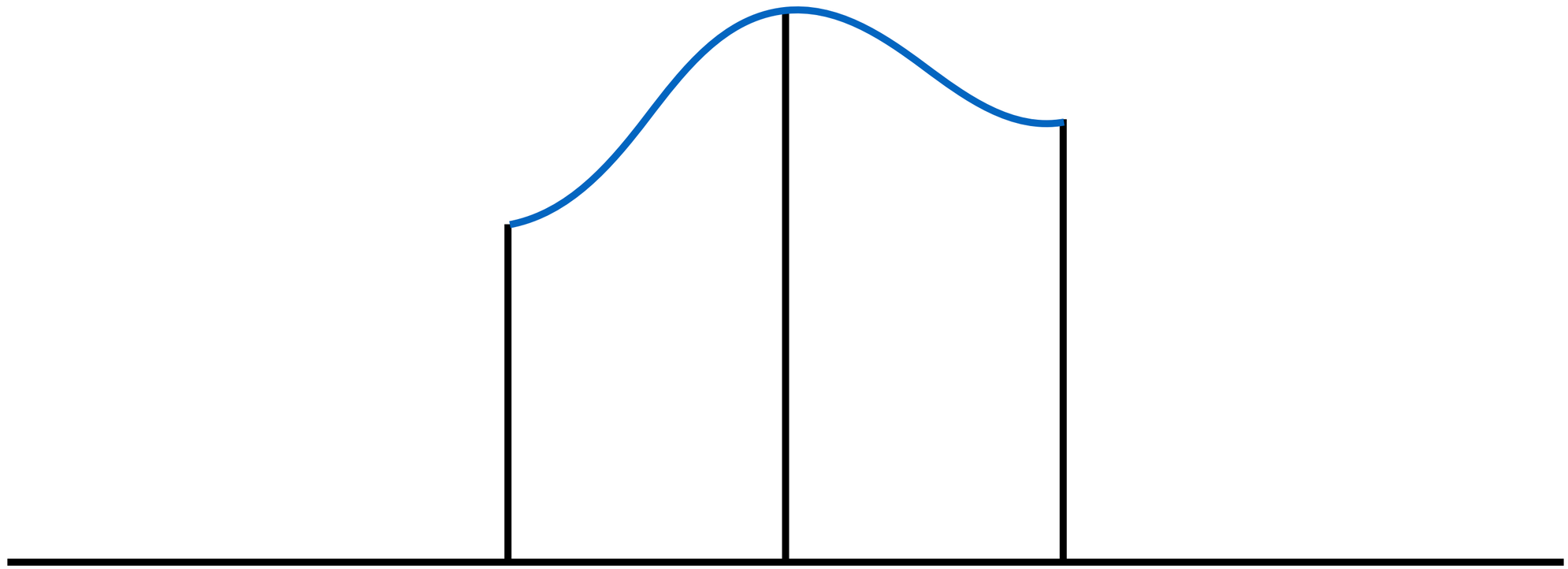
Problem: values not continuous

Linear Interpolation

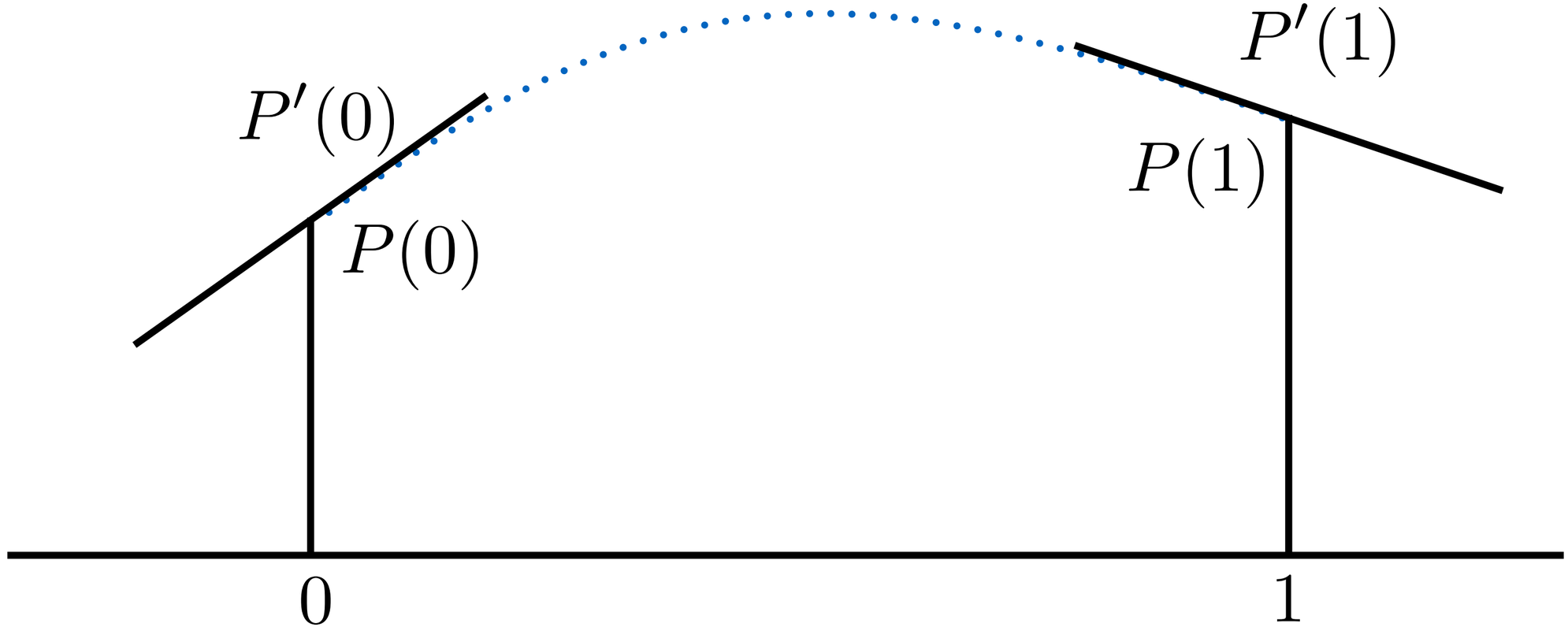


Problem: derivatives not continuous

Smooth Interpolation?



Cubic Hermite Interpolation



Inputs: values and derivatives at endpoints

Cubic Polynomial Interpolation

Cubic polynomial

$$P(t) = a\ t^3 + b\ t^2 + c\ t + d$$

Why cubic?

4 input constraints – need 4 degrees of freedom

$$P(0) = h_0$$
$$P(1) = h_1$$
$$P'(0) = h_2$$
$$P'(1) = h_3$$

Cubic Polynomial Interpolation

Cubic polynomial

$$P(t) = a\ t^3 + b\ t^2 + c\ t + d$$

$$P'(t) = 3a\ t^2 + 2b\ t + c$$

Set up constraint equations

$$P(0) = h_0 = d$$

$$P(1) = h_1 = a + b + c + d$$

$$P'(0) = h_2 = c$$

$$P'(1) = h_3 = 3a + 2b + c$$

Solve for Polynomial Coefficients

$$
\begin{aligned}
h_0 &= d \\
h_1 &= a + b + c + d \\
h_2 &= c \\
h_3 &= 3a + 2b + c
\end{aligned}
$$

$$
\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
$$

Solve for Polynomial Coefficients

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

(Check that these matrices are inverses)

Matrix Form of Hermite Function

$$P(t) = a\ t^3 + b\ t^2 + c\ t + d$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

Interpretation 1: Matrix Rows = Coefficient Formulas

$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

Interpretation 2: Matrix Columns = ?

$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2t^3 - 3t^2 + 1 \\ -2t^3 + 3t^2 \\ t^3 - 2t^2 + t \\ t^3 - t^2 \end{bmatrix}^T \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

Hermite Basis Functions

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} H_0(t) & H_1(t) & H_2(t) & H_3(t) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$t^3 \qquad H_0(t) = 2t^3 - 3t^2 + 1$$

$$t^2 \qquad H_1(t) = -2t^3 + 3t^2$$

$$t \qquad H_2(t) = t^3 - 2t^2 + t$$

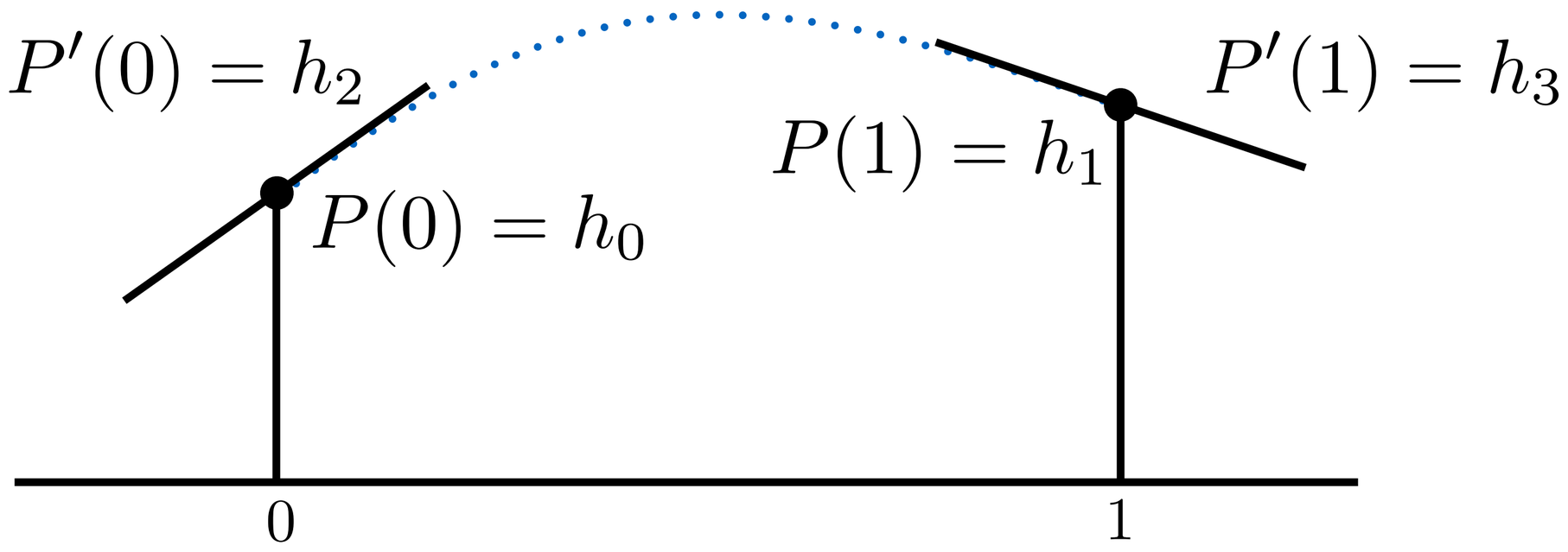
$$1 \qquad H_3(t) = t^3 - t^2$$

Basis functions for cubic polynomials

Hermite basis functions for cubic polynomials

Either basis can represent any cubic polynomial through linear combination

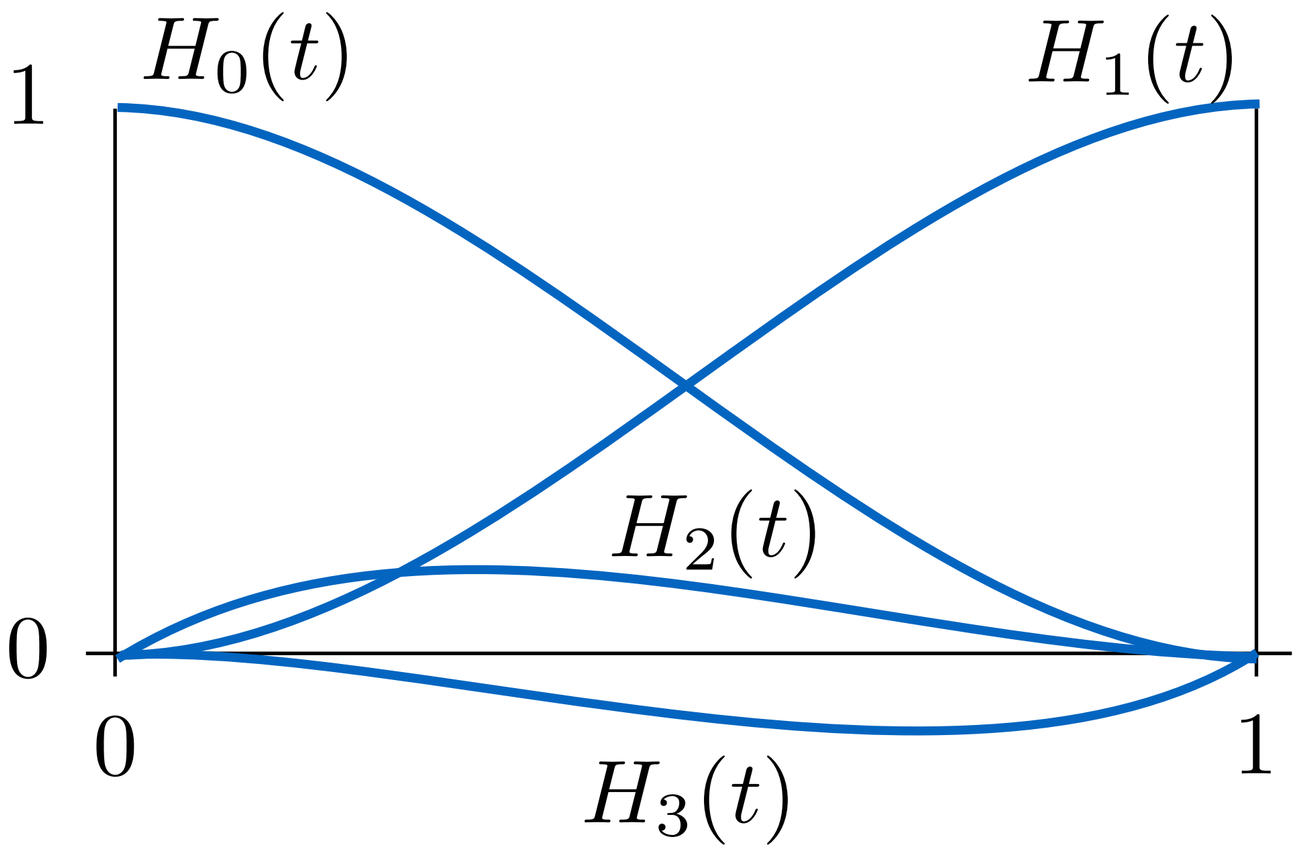
Recap: Cubic Hermite Interpolation



Inputs: values and derivatives at endpoints
Output: cubic polynomial that interpolates
Solution: weighted sum of Hermite basis functions

$$P(t) = h_0\, H_0(t) + h_1\, H_1(t) + h_2\, H_2(t) + h_3\, H_3(t)$$

Hermite Basis Functions



$$H_0(t) = 2t^3 - 3t^2 + 1$$

$$H_1(t) = -2t^3 + 3t^2$$

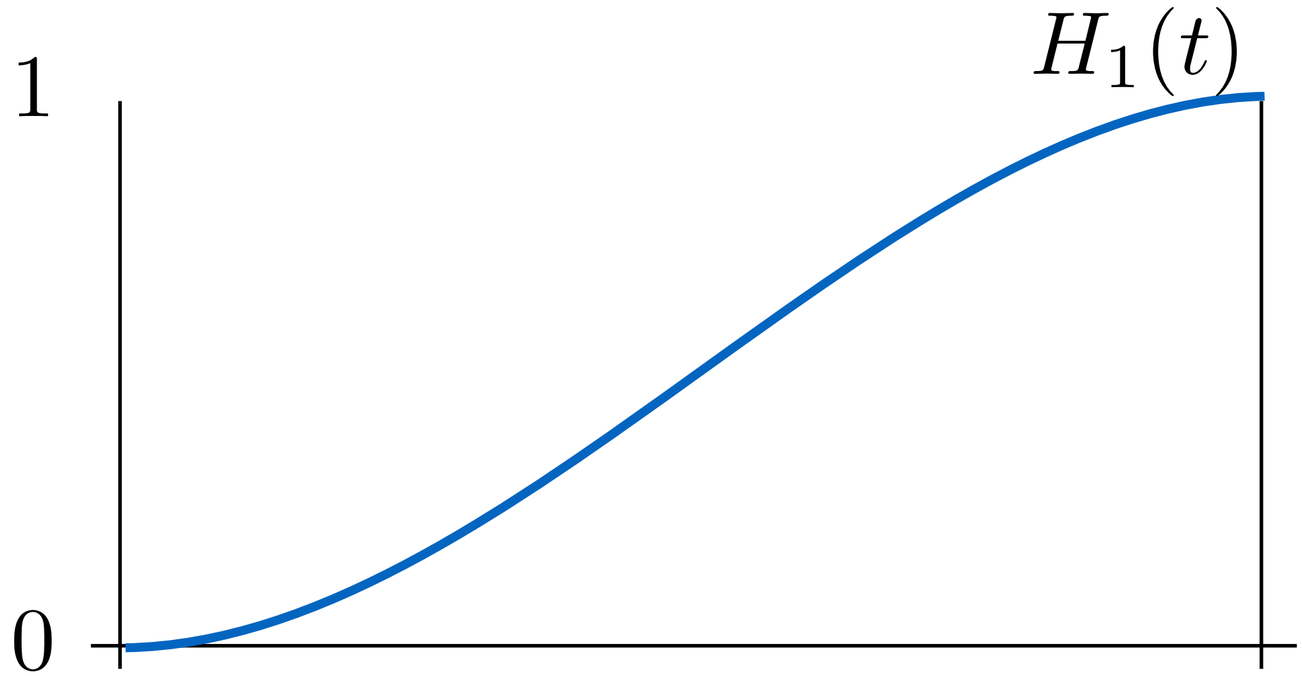
$$H_2(t) = t^3 - 2t^2 + t$$

$$H_3(t) = t^3 - t^2$$

Ease Function

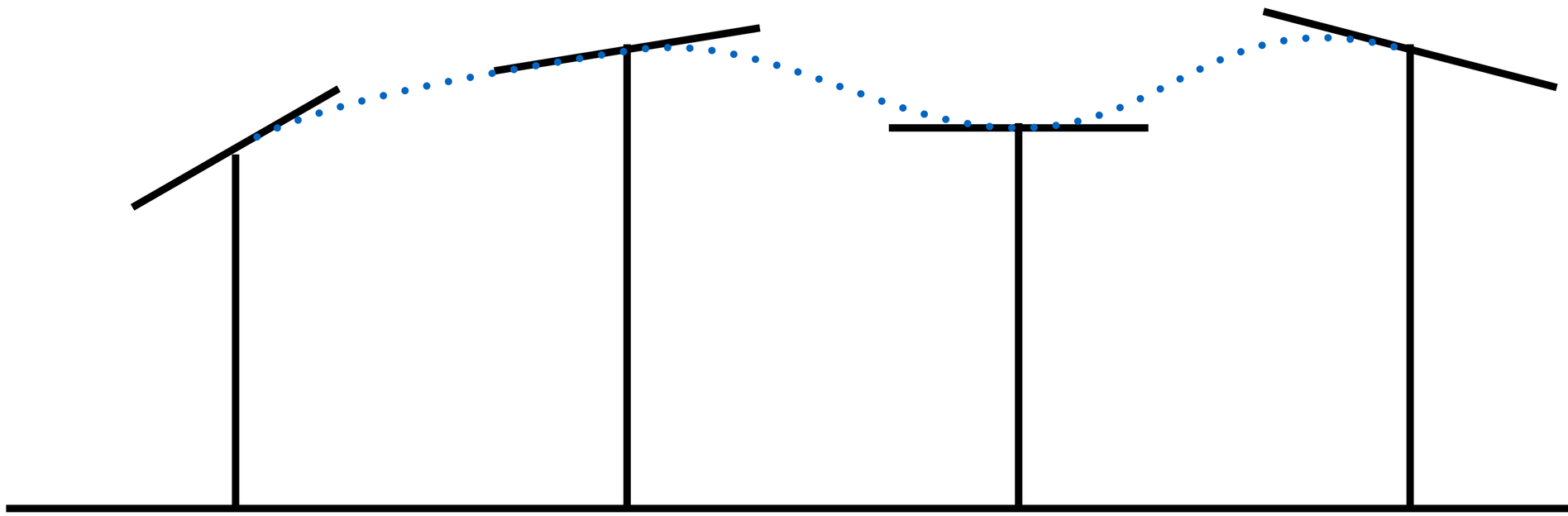
A very useful function

In animation, start and stop gently (zero velocity)



$$H_1(t) = -2t^3 + 3t^2 = t^2(3 - 2t)$$

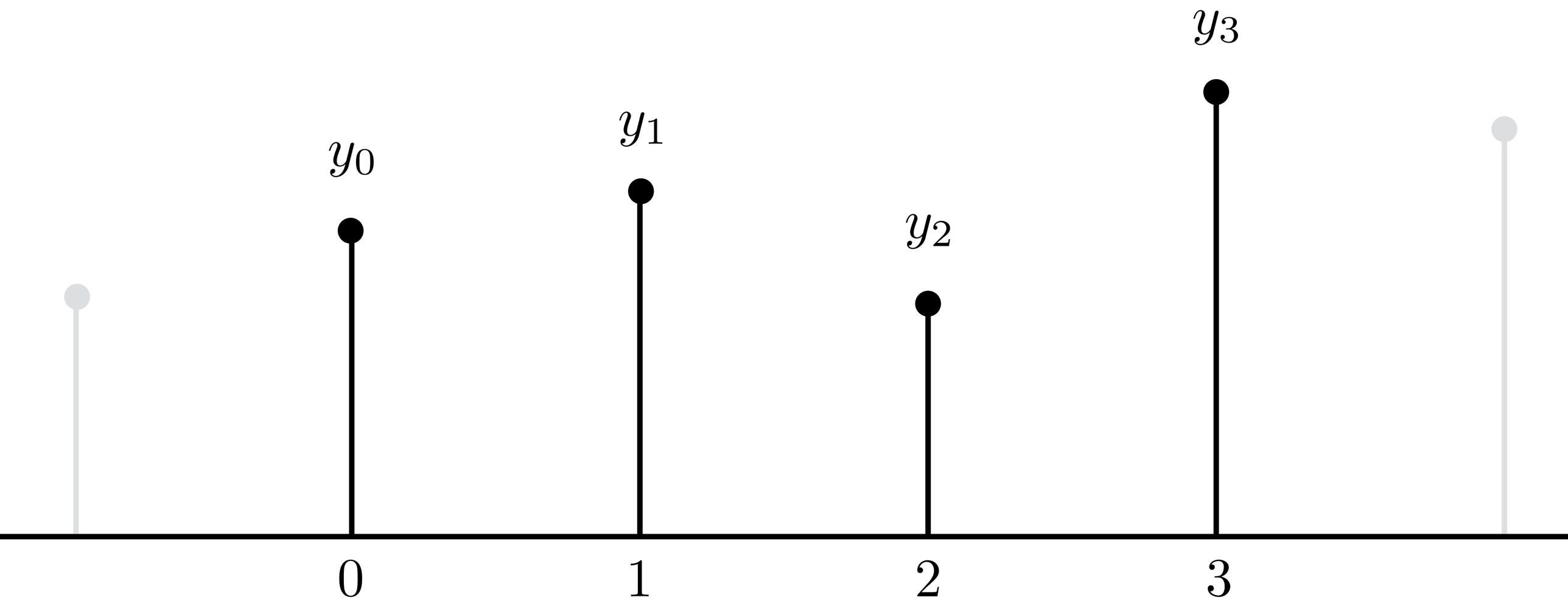
Hermite Spline Interpolation



Inputs: sequence of values and derivatives

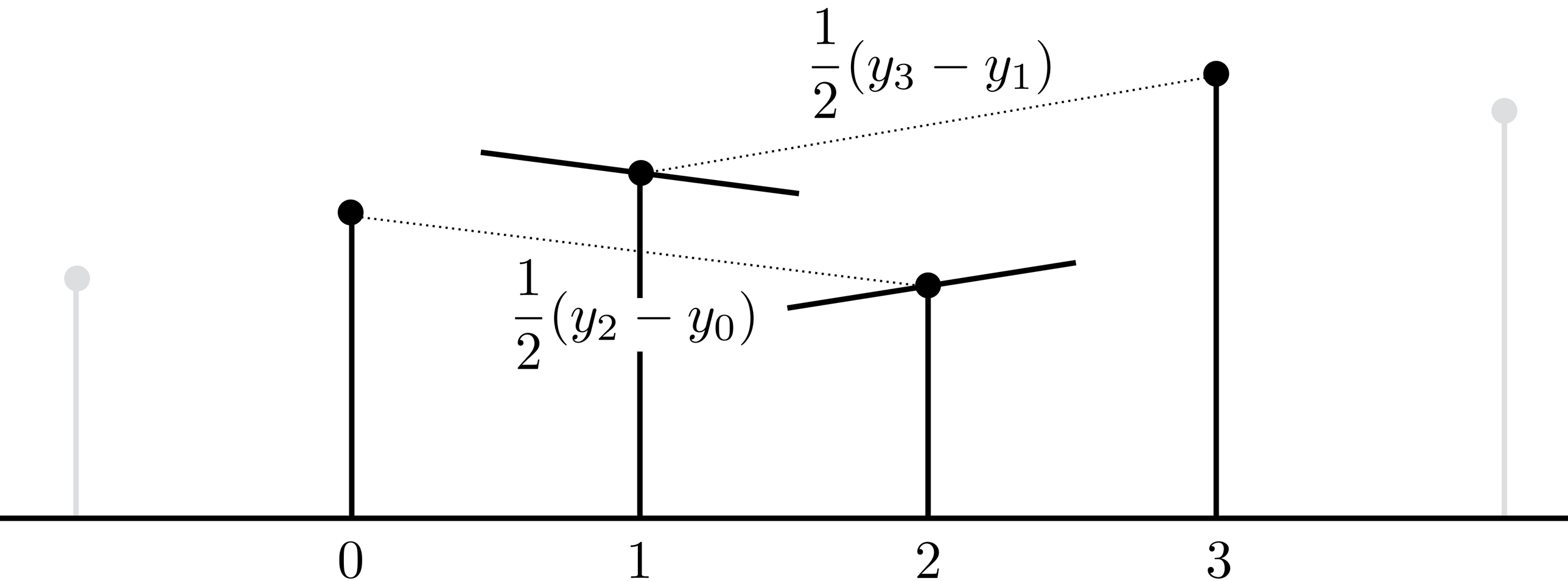
Catmull-Rom Interpolation

Catmull-Rom Interpolation



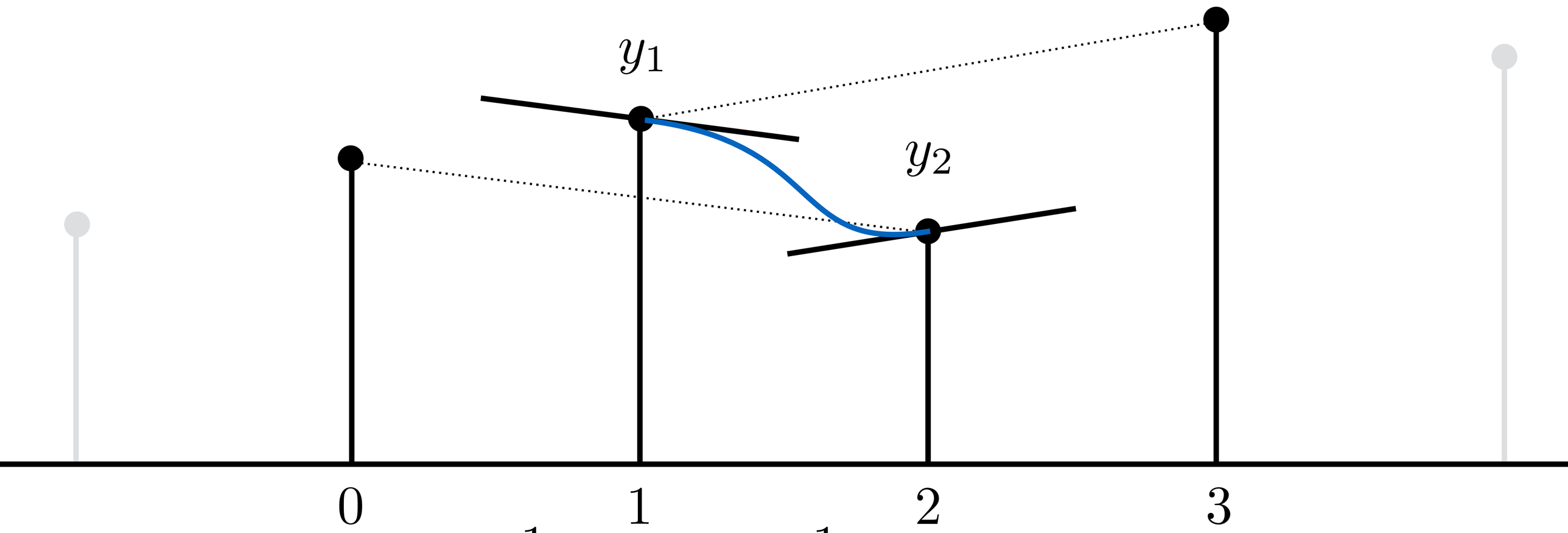
Inputs: sequence of values

Catmull-Rom Interpolation



Rule for derivatives:
Match slope between previous and next values

Catmull-Rom Interpolation

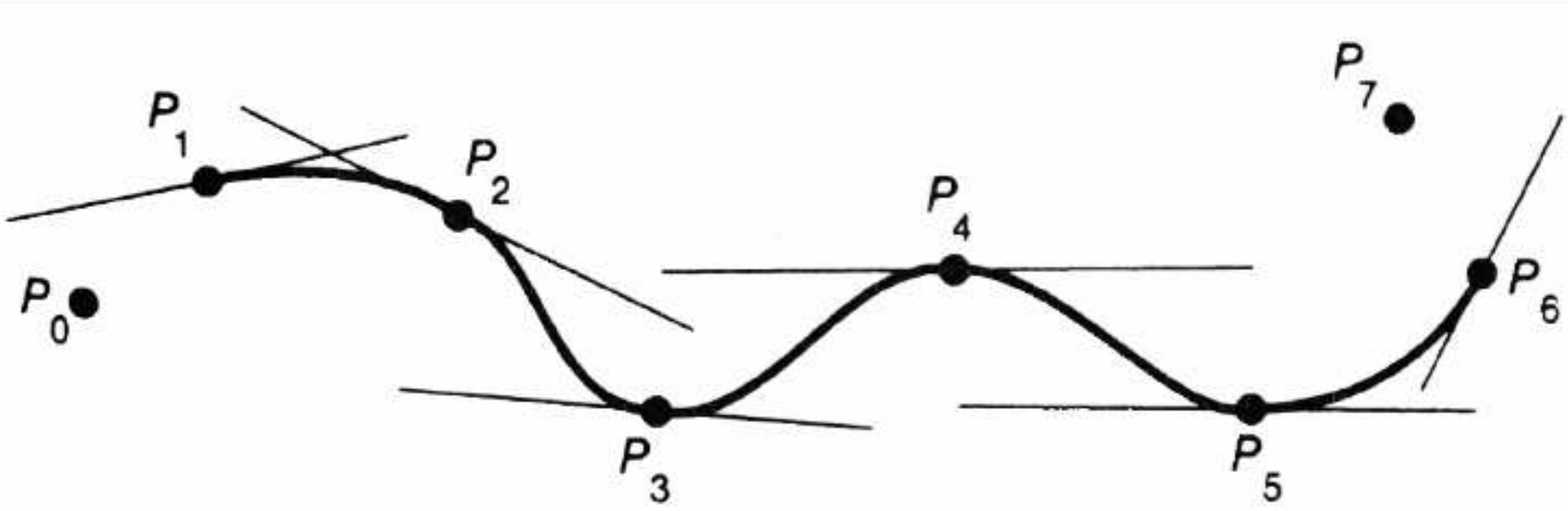


Then use Hermite interpolation

Catmull-Rom Spline

Input: sequence of points

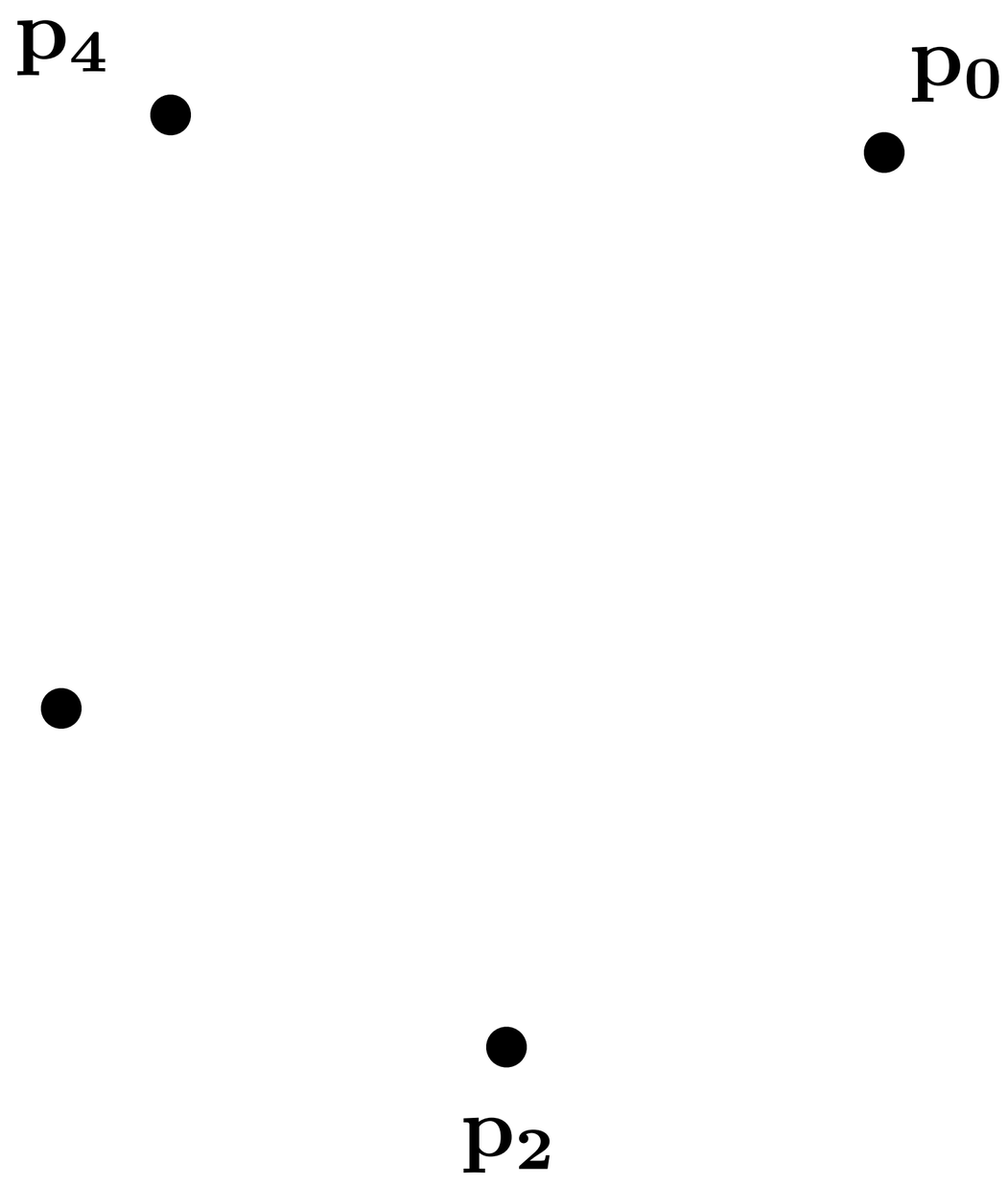
Output: spline that interpolates all points with C1 continuity



Interpolating Points & Vectors

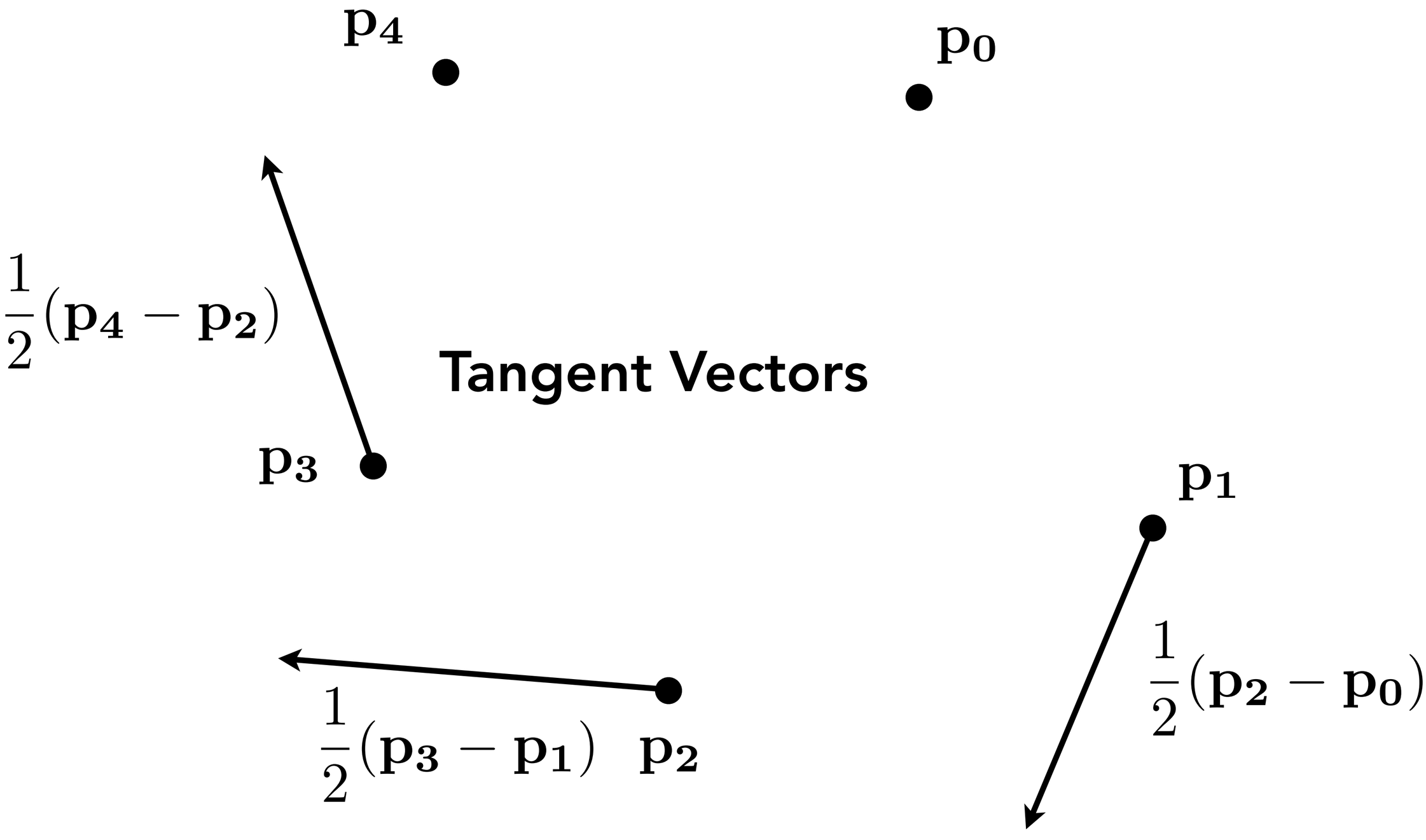
Can Interpolate Points As Easily As Values

E.g. point (0,1,3) in 3D space, or even a general vector in N dimensions



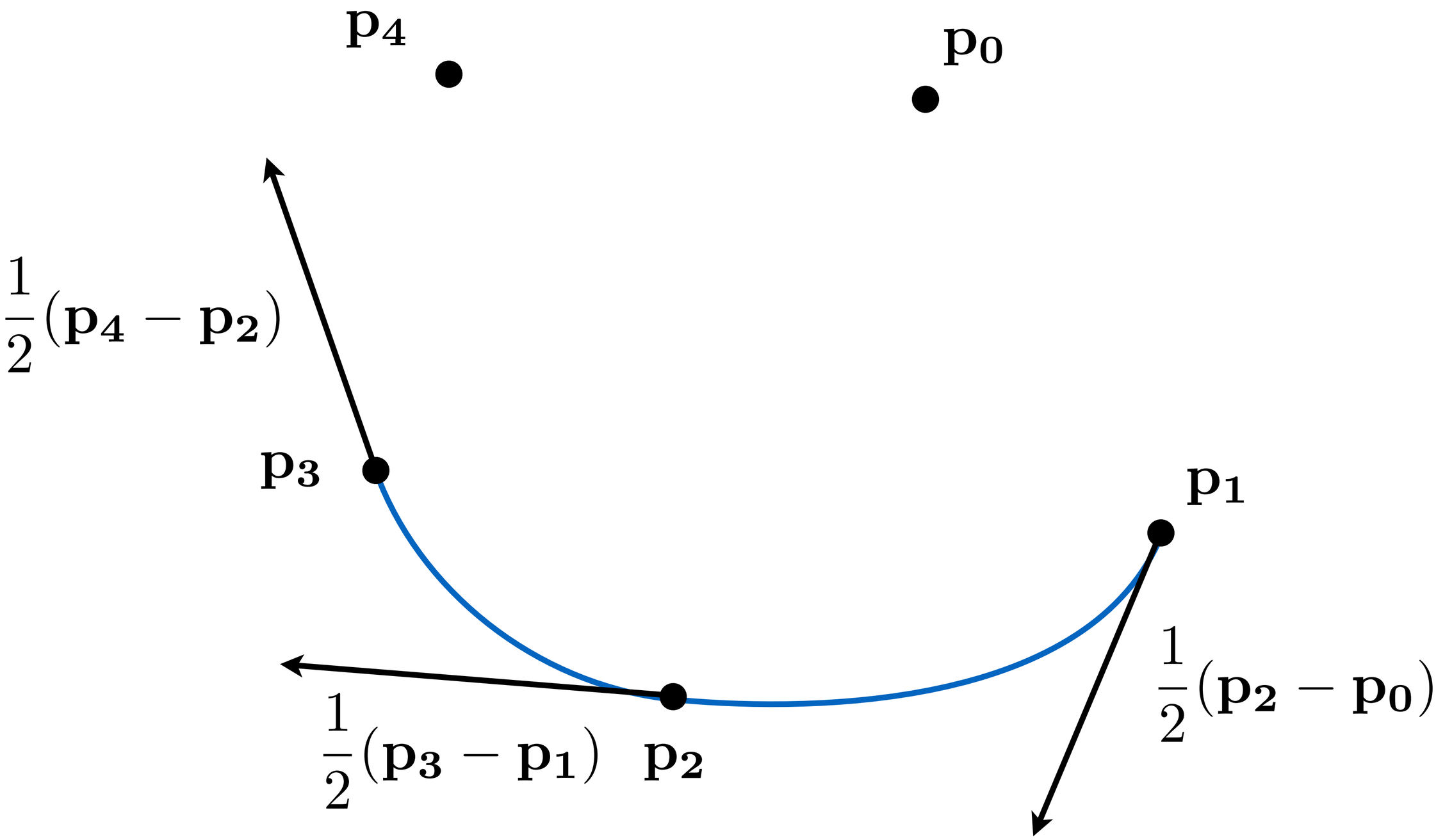
Catmull-Rom 3D spline control points

Can Interpolate Points As Easily As Values



Catmull-Rom 3D tangent vectors

Can Interpolate Points As Easily As Values



Catmull-Rom 3D space curve

Use Basis Functions to Define Curves

General formula for a particular interpolation scheme: $\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{p}_i F_i(t)$

$$x(t) = \sum_{i=0}^{n} x_i F_i(t) \qquad y(t) = \sum_{i=0}^{n} y_i F_i(t) \qquad z(t) = \sum_{i=0}^{n} z_i F_i(t)$$

Coefficients $\mathbf{p}_i$ can be points & vectors, not just values

$F_i(t)$ are basis functions for the interpolation scheme.

Saw $H_i(t)$ for Hermite interpolation earlier. Will see $C_i(t)$ for Catmull-Rom shortly, and $B_i(t)$ for Bézier scheme later. The basis functions are properties of the interpolation scheme.

Matrix Form of Catmull-Rom Space Curve?

Use Hermite matrix form

- Points & tangents given by Catmull-Rom rules

Hermite points

$$\mathbf{h}_0 = \mathbf{p}_1$$
$$\mathbf{h}_1 = \mathbf{p}_2$$

Hermite tangents

$$\mathbf{h}_2 = \tfrac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$$
$$\mathbf{h}_3 = \tfrac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$$

$$P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Hermite matrix

Convert Catmull-Rom inputs to Hermite inputs

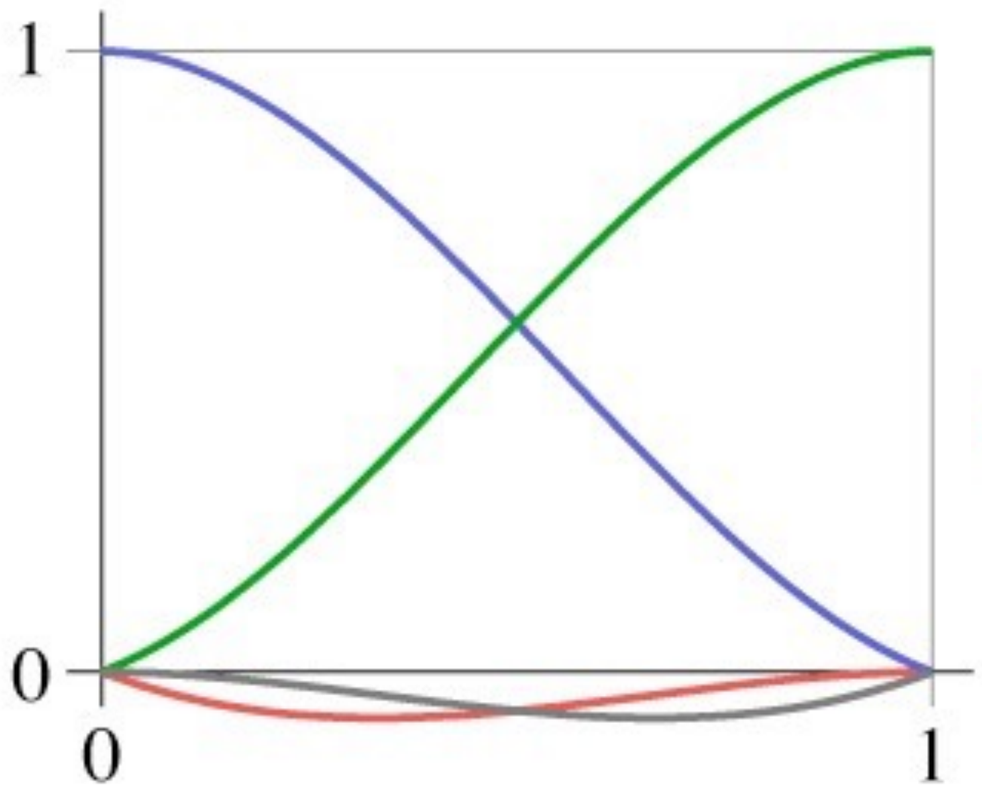
Matrix Form of Catmull-Rom Space Curve

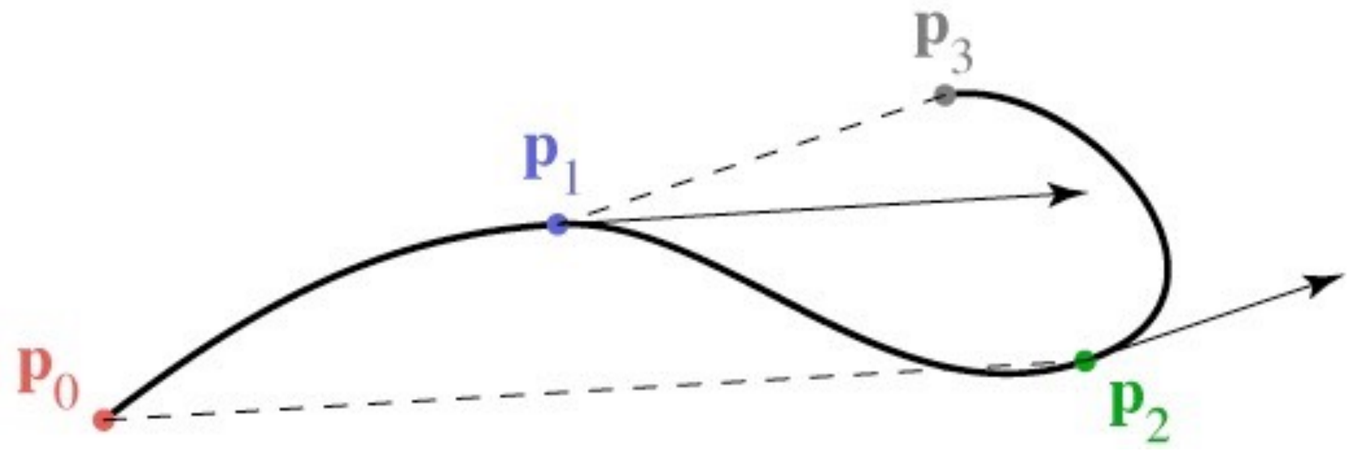
$$P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \\ 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$= C_0(t)\ \mathbf{p}_0 + C_1(t)\ \mathbf{p}_1 + C_2(t)\ \mathbf{p}_2 + C_3(t)\ \mathbf{p}_3$$

Matrix columns = Catmull-Rom basis functions

Catmull-Rom Basis Functions

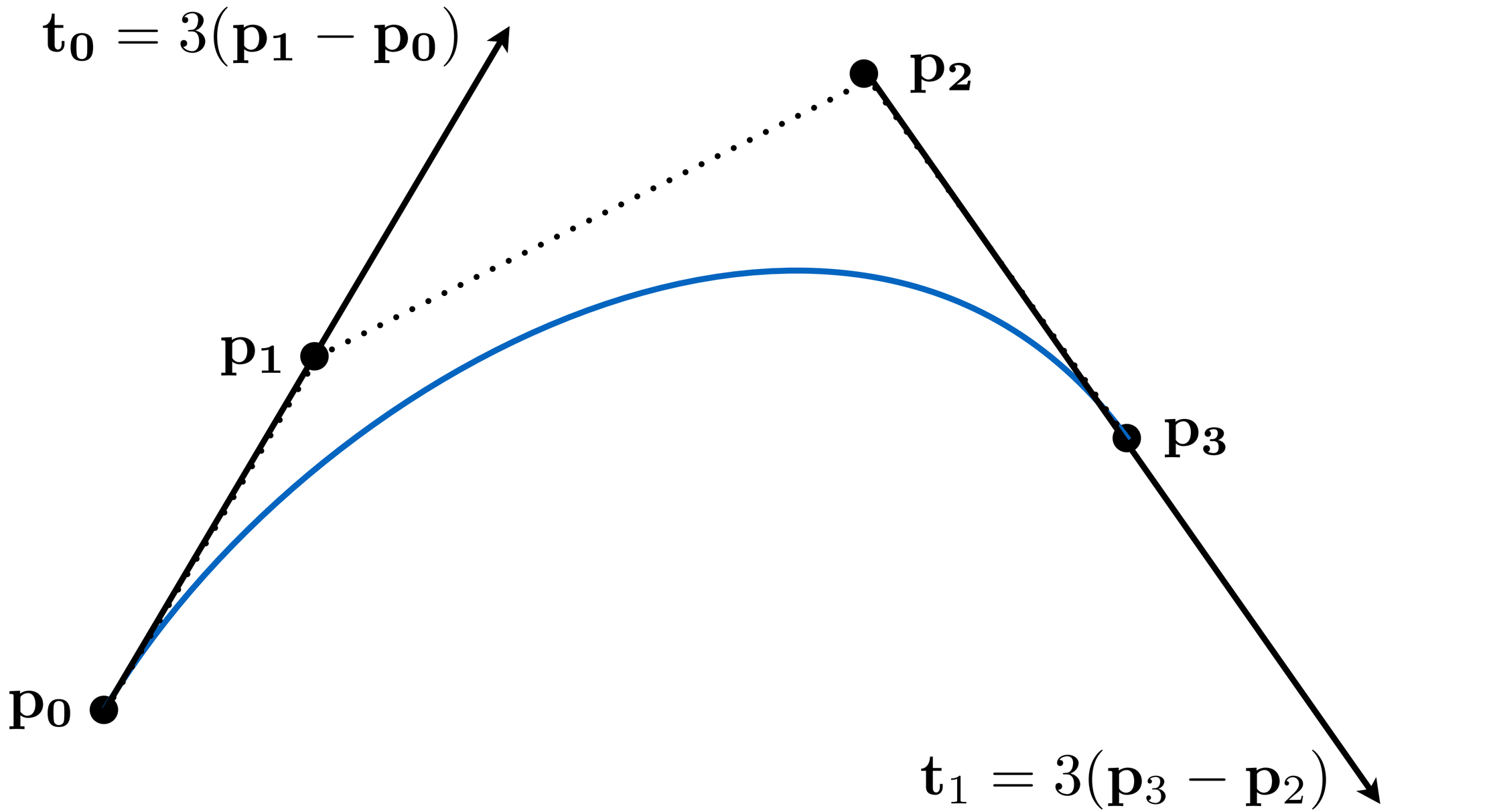




Bézier Curves

Examples of Geometry

Defining Cubic Bézier Curve With Tangents



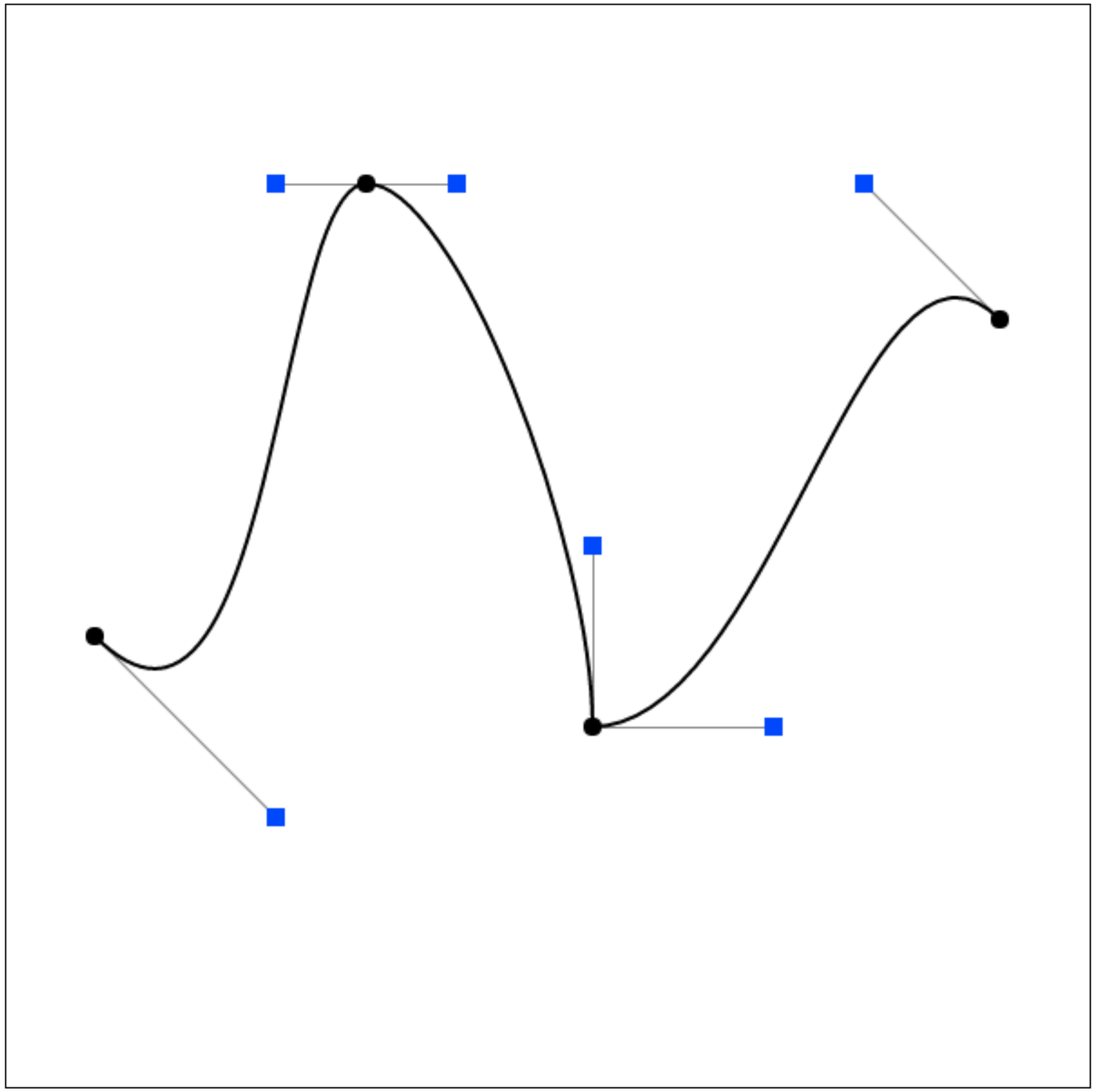
Matrix Form of Cubic Bézier Curve?

$$P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$= B_0^3(t)\ \mathbf{p}_0 + B_1^3(t)\ \mathbf{p}_1 + B_2^3(t)\ \mathbf{p}_2 + B_3^3(t)\ \mathbf{p}_3$$

Good exercise to derive this matrix yourself.
One way: use Hermite matrix equation again.
What are the points and tangents?

Demo – Piecewise Cubic Bézier Curve



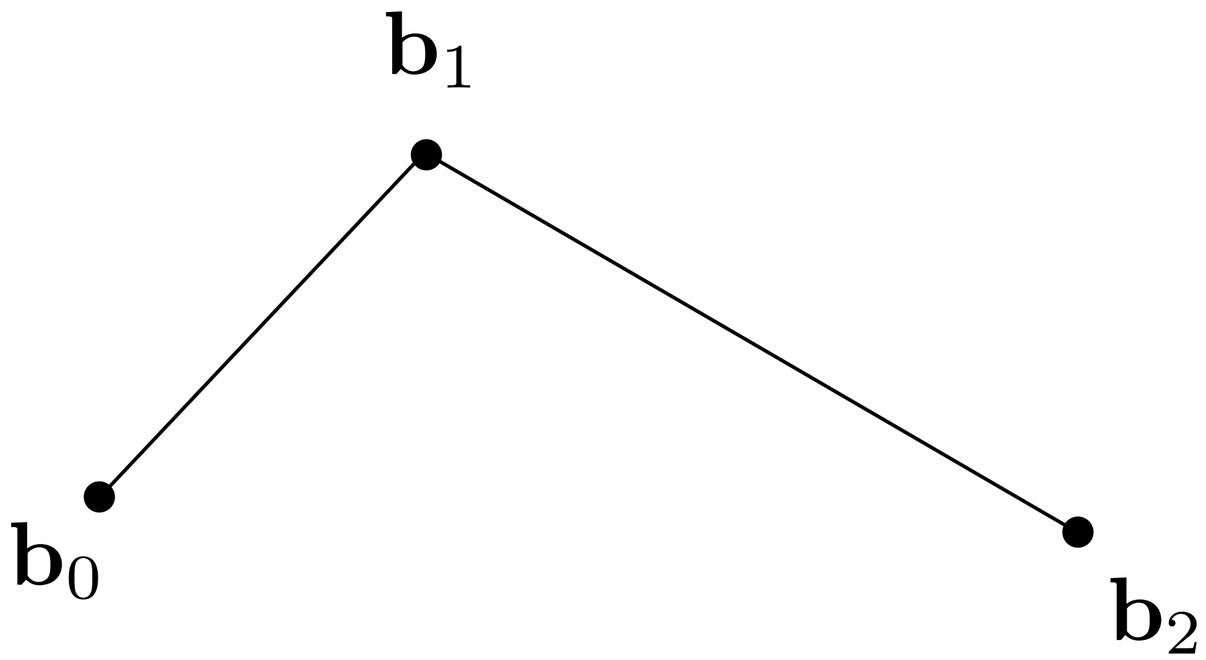
David Eck, http://math.hws.edu/eck/cs424/notes2013/canvas/bezier.html

Evaluating Bézier Curves

De Casteljau Algorithm

Bézier Curves – de Casteljau Algorithm

Consider three points (quadratic Bezier)

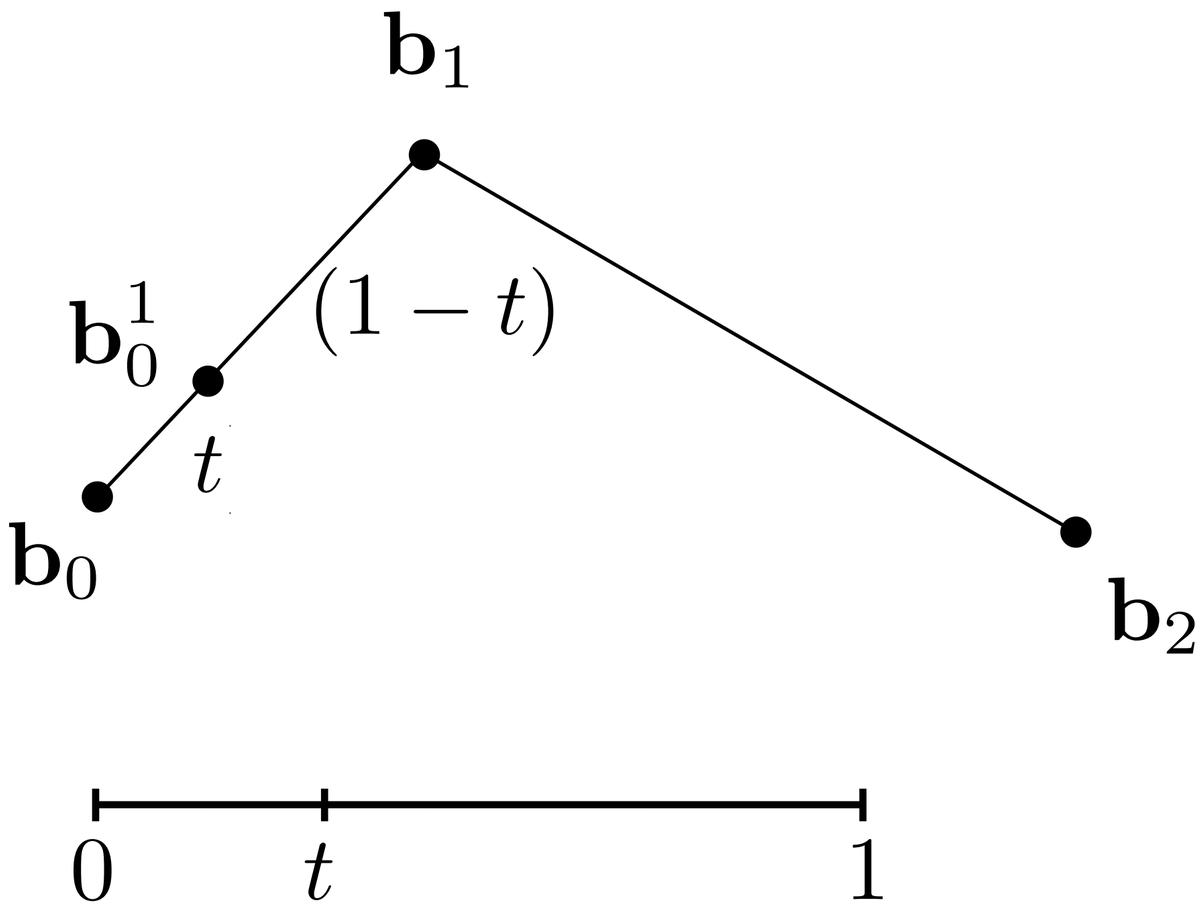


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Insert a point using linear interpolation

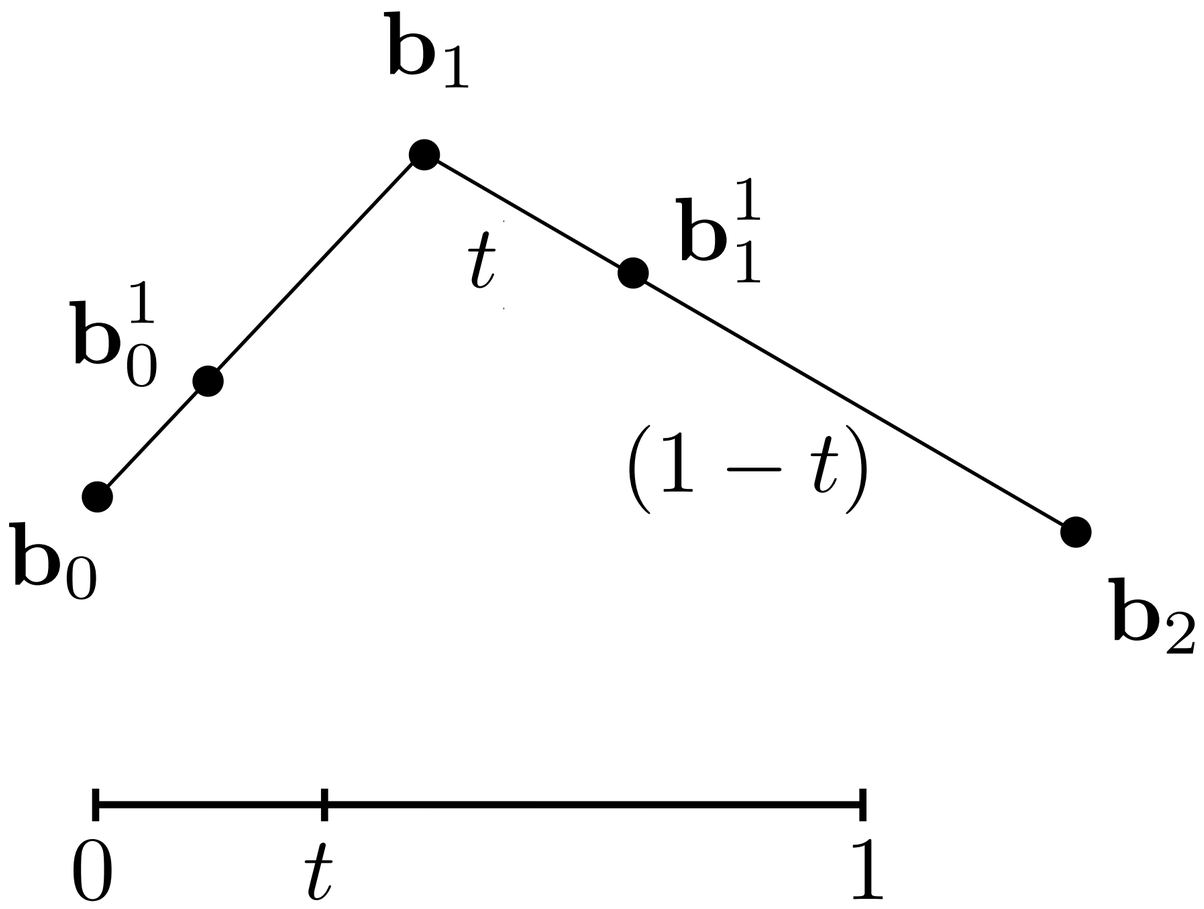


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Insert on both edges

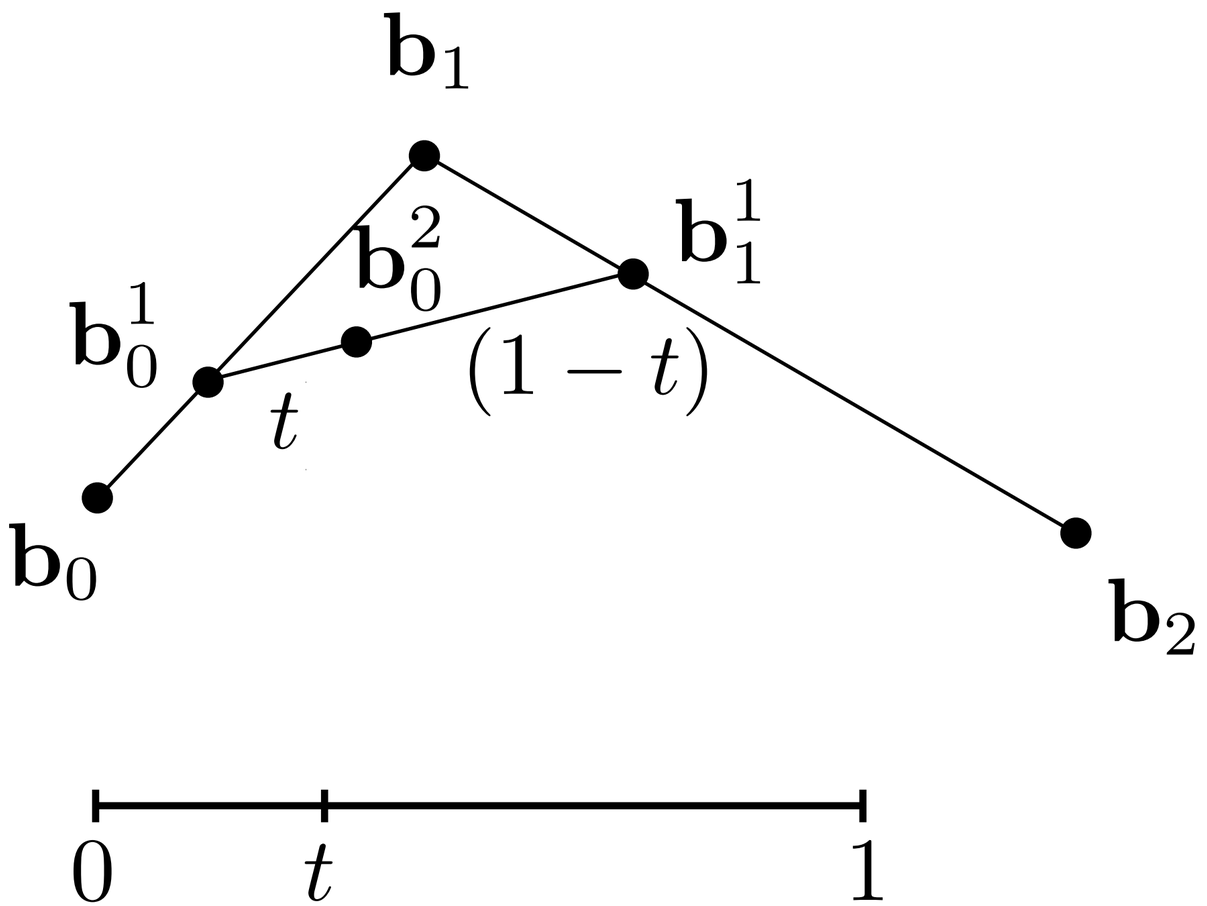


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Repeat recursively

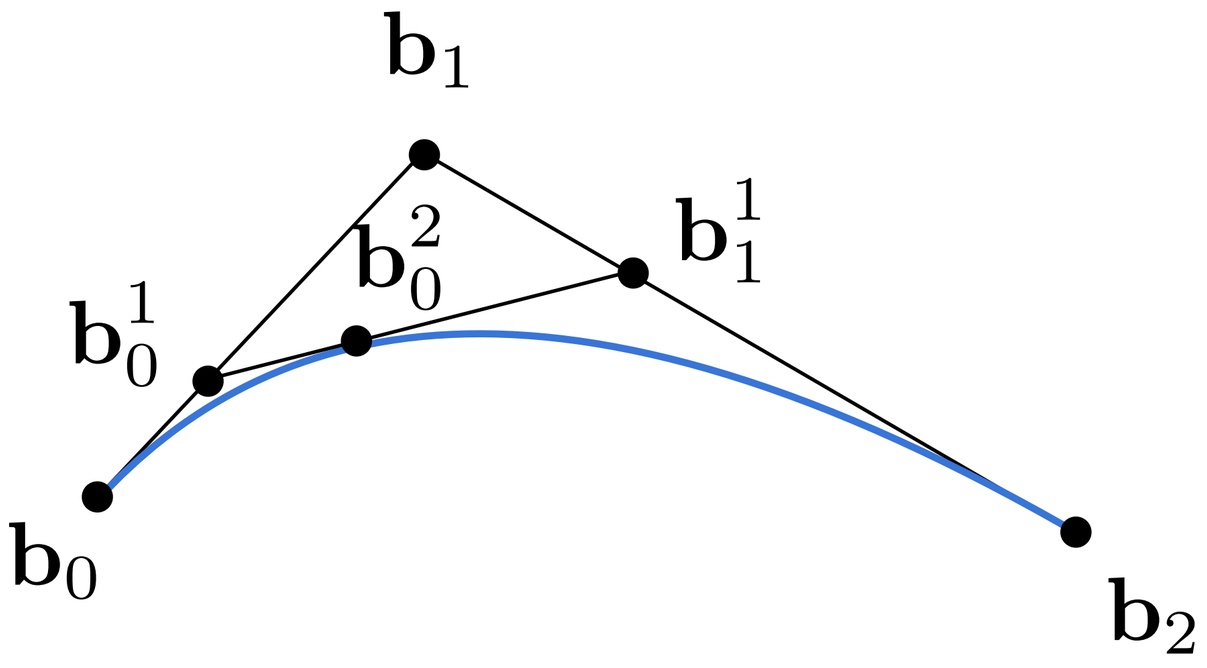


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Algorithm defines the curve



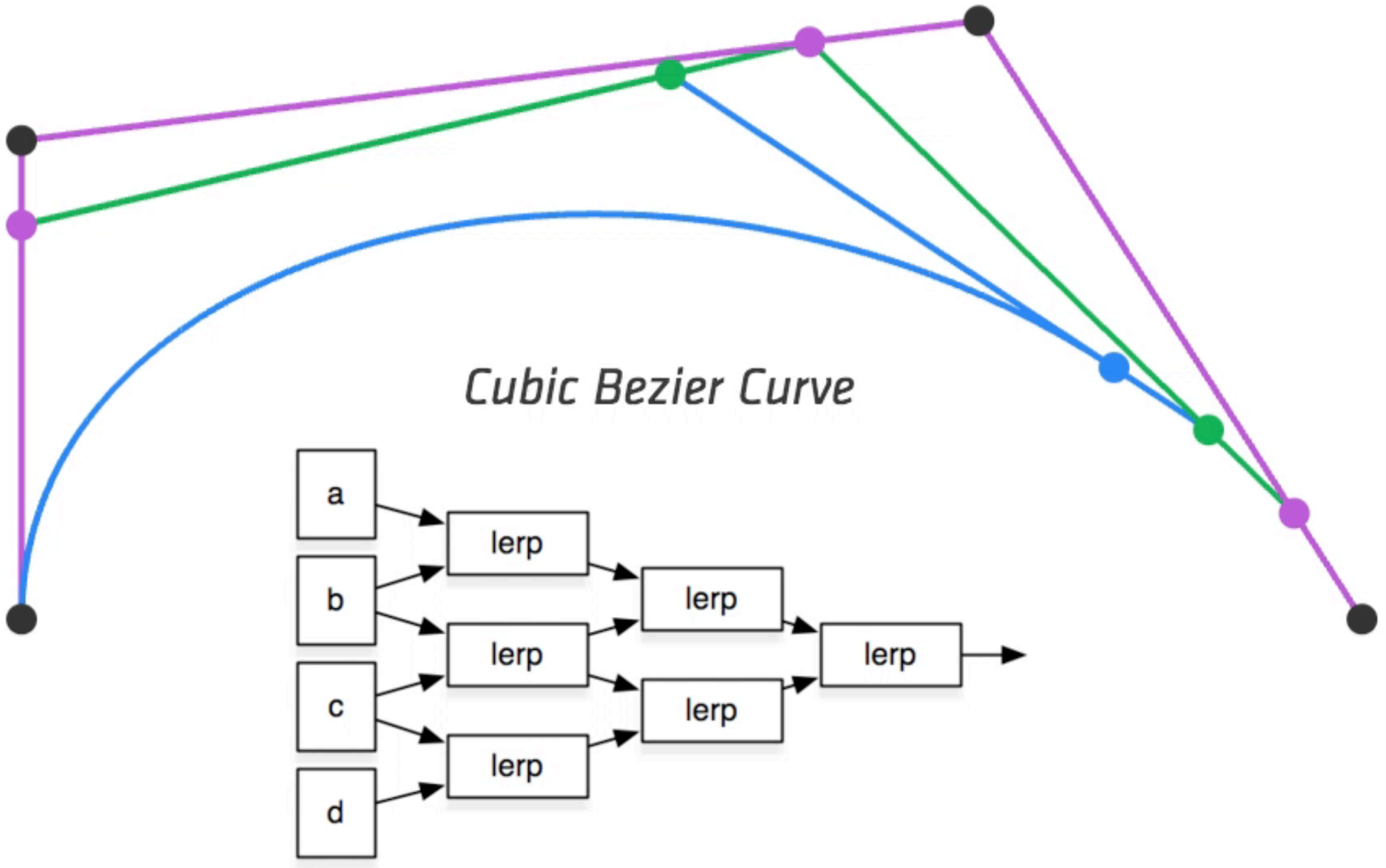
Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

"Corner cutting" recursive subdivision

Successive linear interpolation

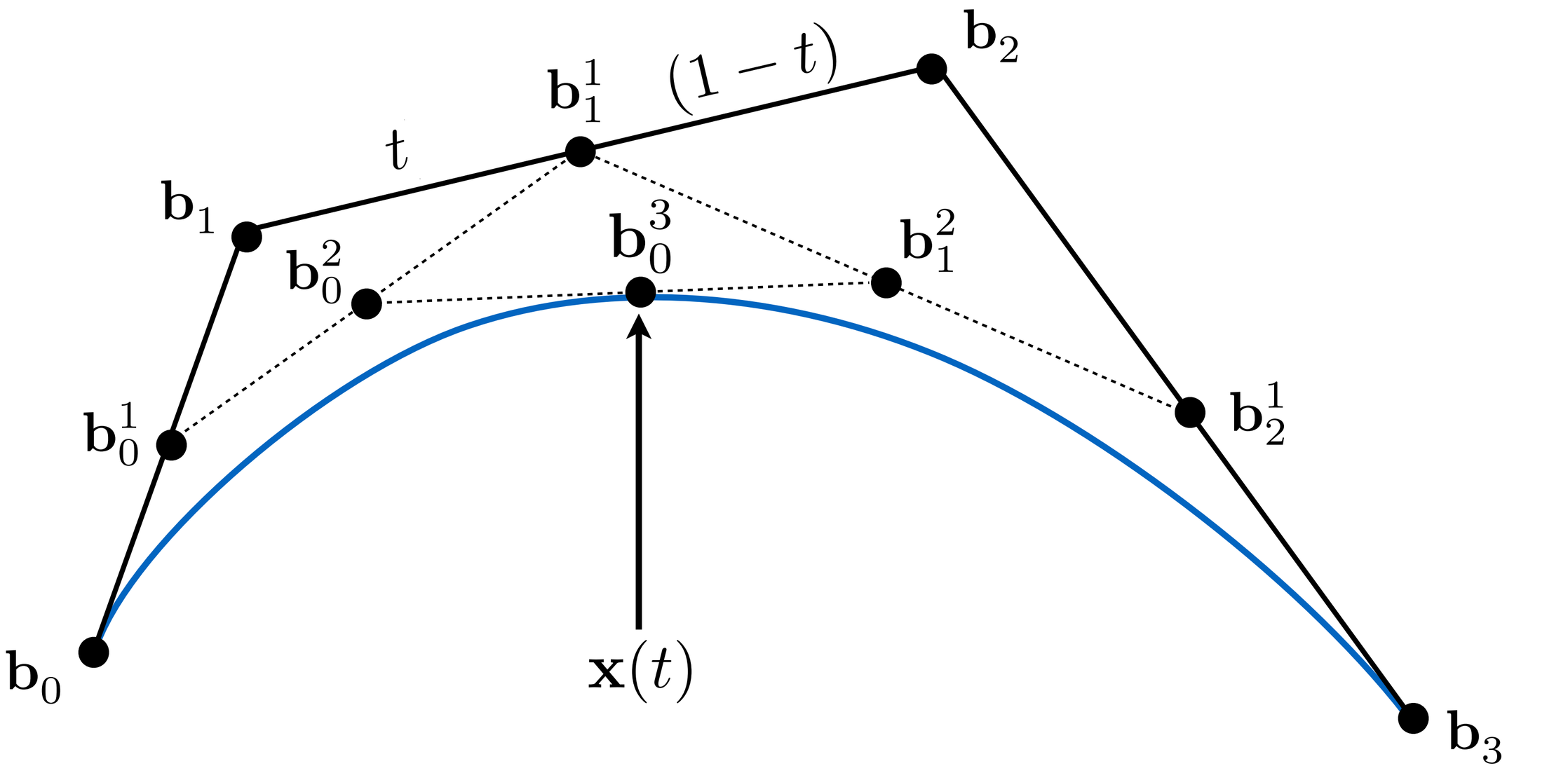
Visualizing de Casteljau Algorithm



Animation: Steven Wittens, Making Things with Maths, http://acko.net

Cubic Bézier Curve – de Casteljau

Consider four points
Same recursive linear interpolations

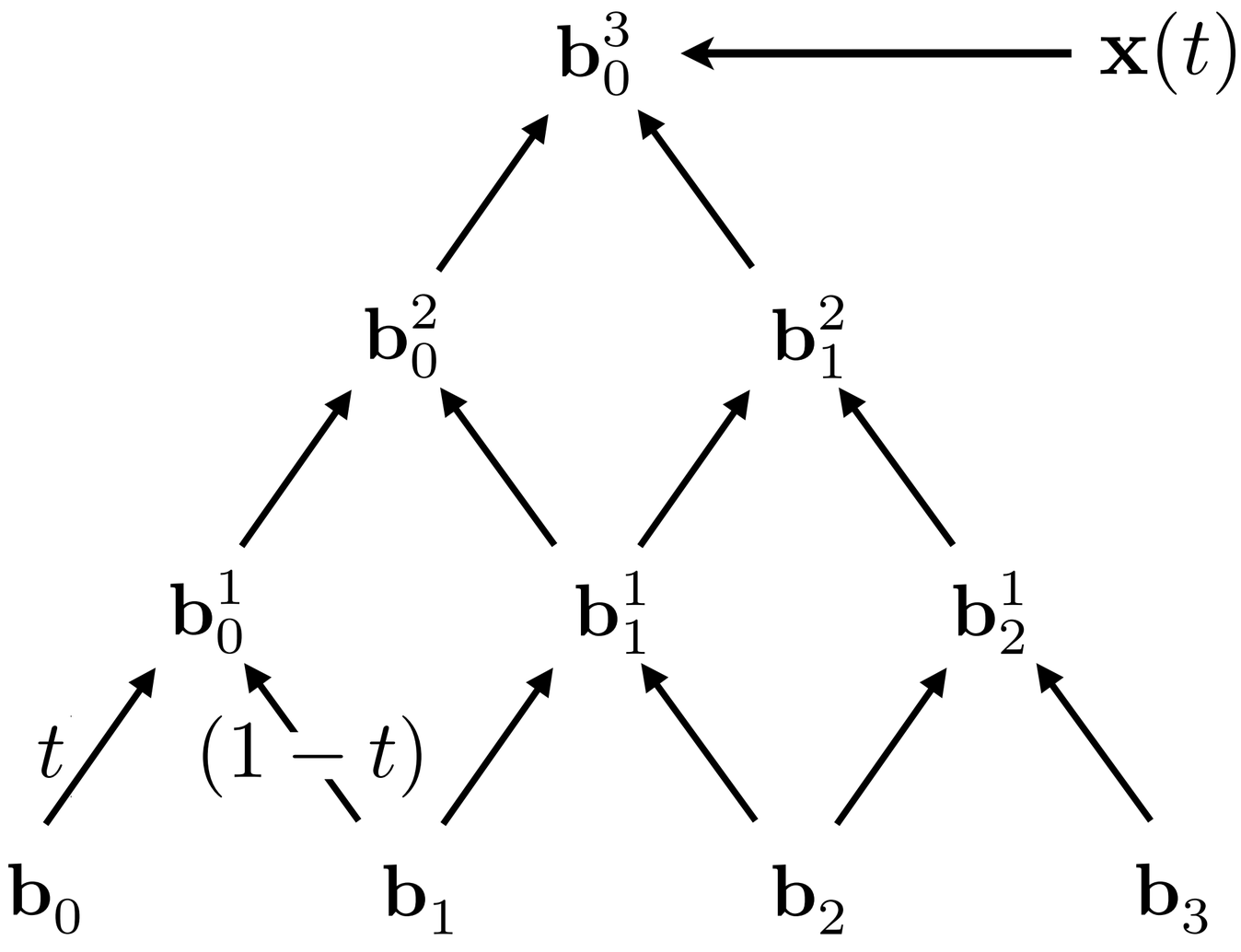


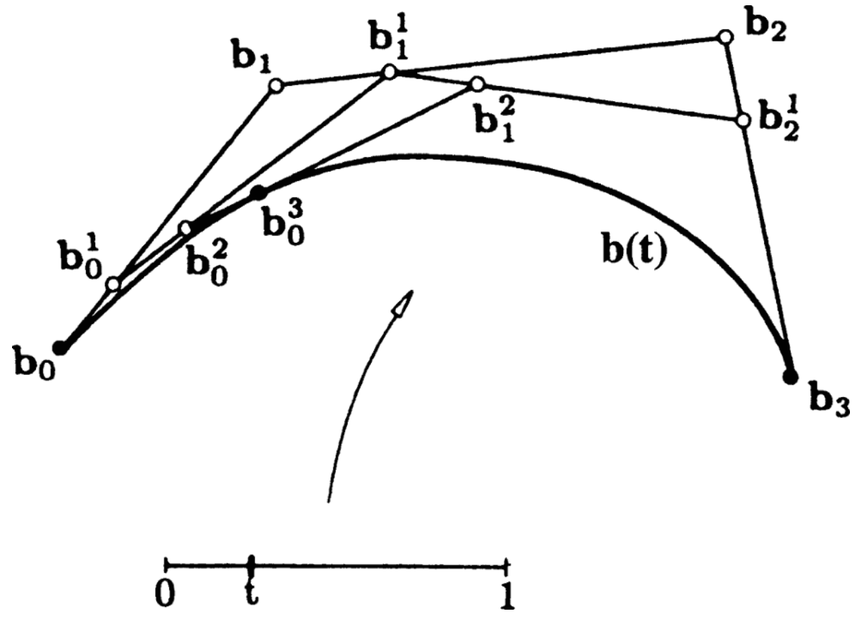
Evaluating Bézier Curves

Algebraic Formula

Bézier Curve – Algebraic Formula

de Casteljau algorithm gives a pyramid of coefficients

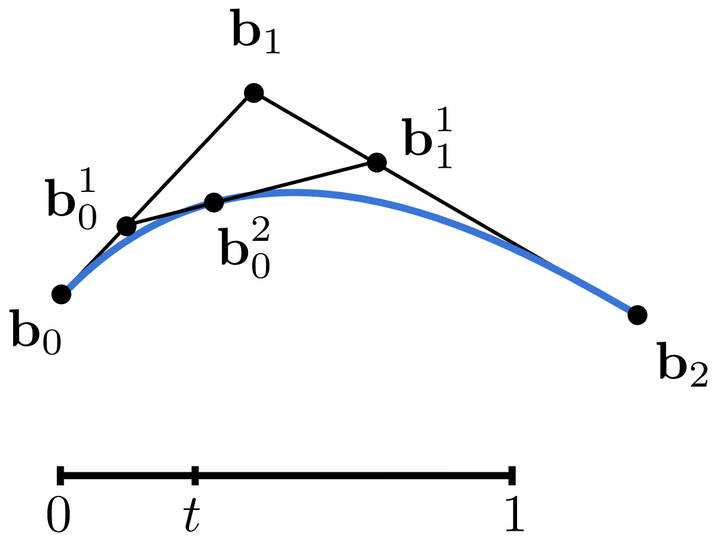




Every rightward arrow is multiplication by t,
Every leftward arrow by (1–t)

Bézier Curve – Algebraic Formula

Example: quadratic Bézier curve from three points



$$\mathbf{b}_0^1(t) = (1-t)\mathbf{b}_0 + t\mathbf{b}_1$$

$$\mathbf{b}_1^1(t) = (1-t)\mathbf{b}_1 + t\mathbf{b}_2$$

$$\mathbf{b}_0^2(t) = (1-t)\mathbf{b}_0^1 + t\mathbf{b}_1^1$$

$$\mathbf{b}_0^2(t) = (1-t)^2\mathbf{b}_0 + 2t(1-t)\mathbf{b}_1 + t^2\mathbf{b}_2$$

Bézier Curve – General Algebraic Formula

Bernstein form of a Bézier curve of order n:

$$\mathbf{b}^n(t) = \mathbf{b}_0^n(t) = \sum_{j=0}^{n} \mathbf{b}_j B_j^n(t)$$

Bézier curve order n
(vector polynomial of degree n)

Bernstein polynomial
(scalar polynomial of degree n)

Bézier control points
(vector in R^N)

Bernstein polynomials:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Bézier Curve – General Algebraic Formula

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^0 = 1,$$

$$(x+y)^1 = x+y,$$

$$(x+y)^2 = x^2+2xy+y^2,$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3,$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4,$$

Wikipedia

Be … of order n:

$$B_i^n(t)$$

Bernstein polynomial

(scalar polynomial of degree n)

…trol points

in $\mathbb{R}^N$)

Be…

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Bézier Curve – Algebraic Formula: Example

Bernstein form of a Bézier curve of order n:

$$\mathbf{b}^n(t) = \sum_{j=0}^{n} \mathbf{b}_j B_j^n(t)$$

Example: assume n = 3 and we are in R^3

i.e. we could have control points in 3D such as:

$$\mathbf{b}_0 = (0, 2, 3),\ \mathbf{b}_1 = (2, 3, 5),\ \mathbf{b}_2 = (6, 7, 9),\ \mathbf{b}_3 = (3, 4, 5)$$

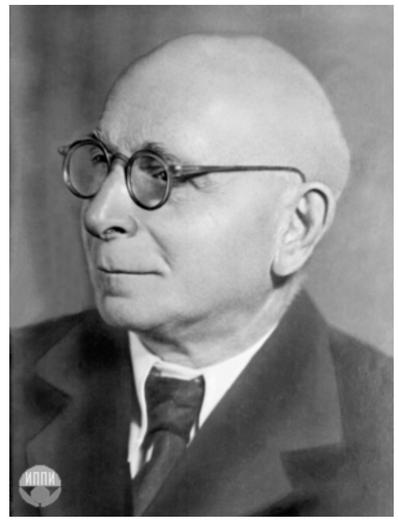
These points define a Bezier curve in 3D that is a cubic polynomial in t:

$$\mathbf{b}^n(t) = \mathbf{b}_0\,(1-t)^3 + \mathbf{b}_1\,3t(1-t)^2 + \mathbf{b}_2\,3t^2(1-t) + \mathbf{b}_3\,t^3$$

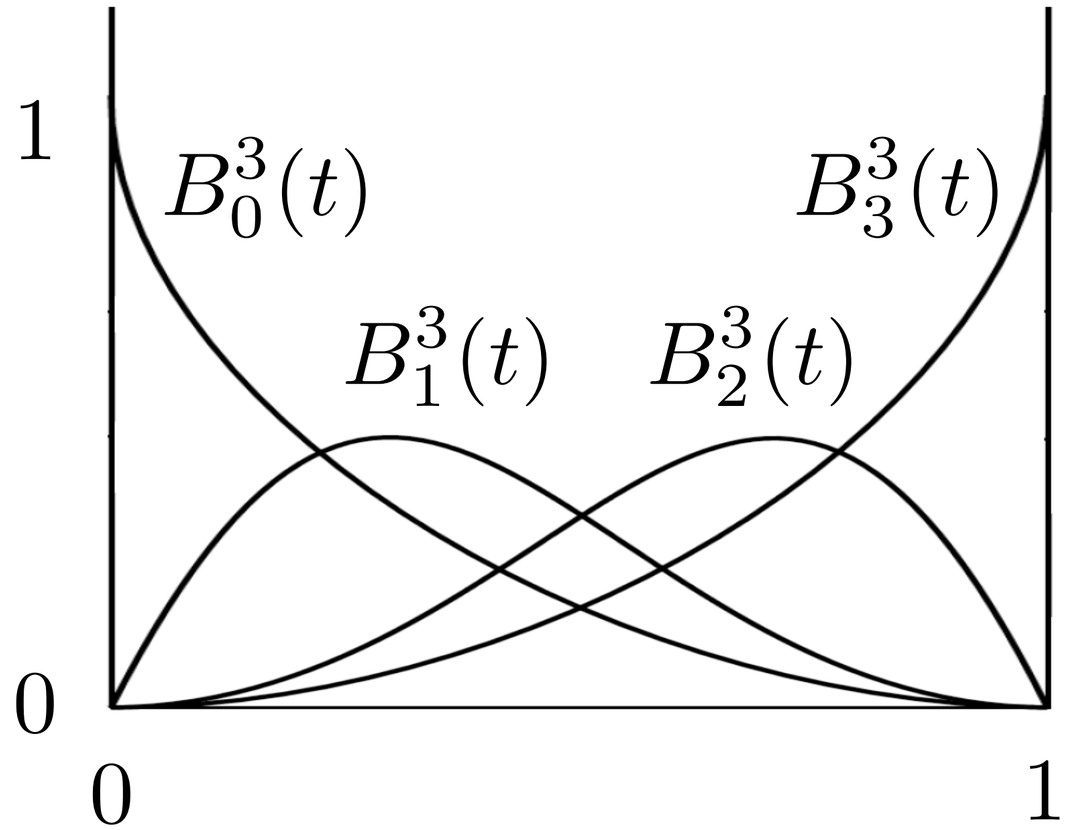
Cubic Bézier Basis Functions

Bernstein polynomials:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



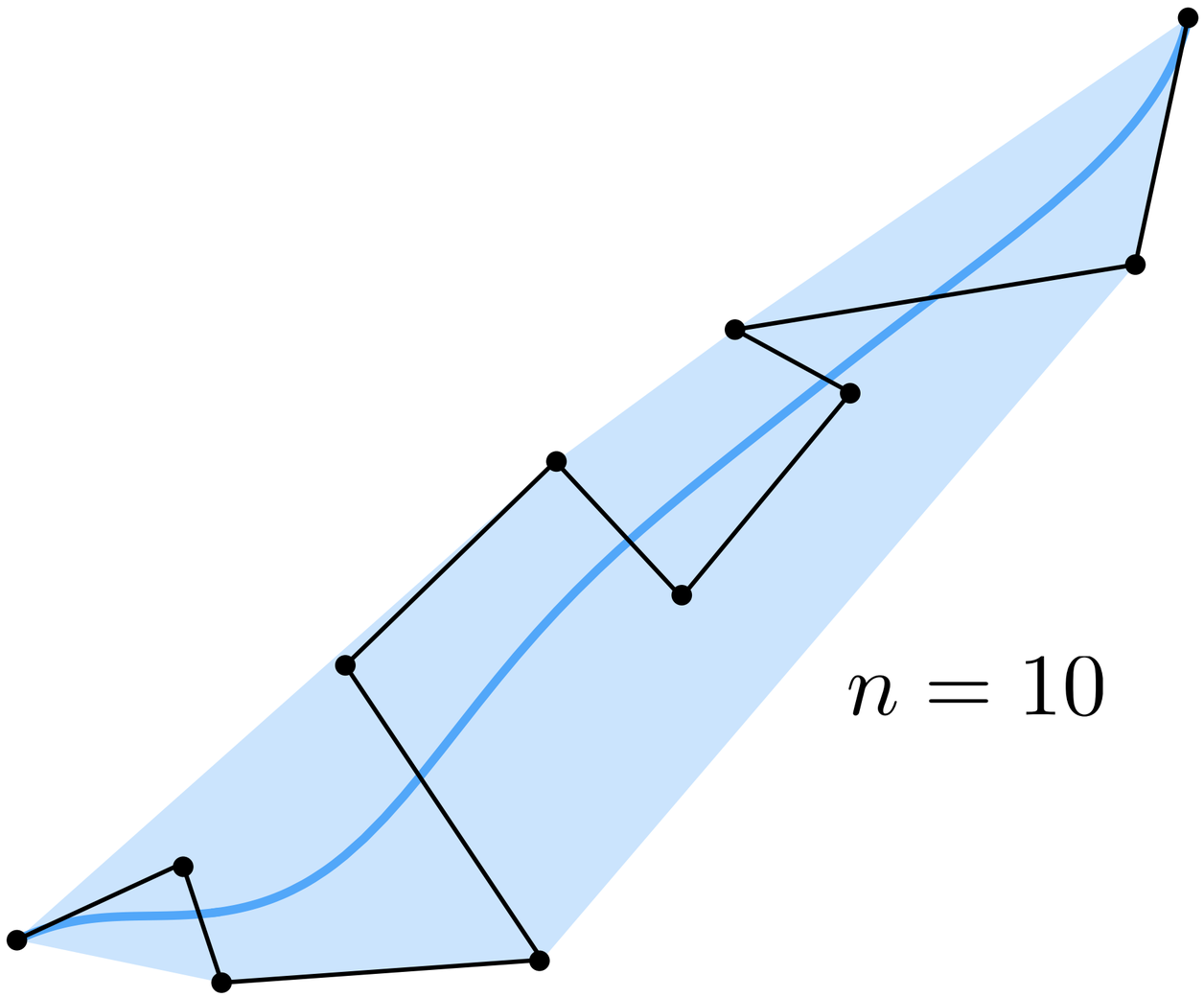
Sergei N. Bernstein
1880 – 1968



Piecewise Bézier Curves (Bézier Spline)

Higher-Order Bézier Curves?

High-degree Bernstein polynomials don’t interpolate well



$n = 10$

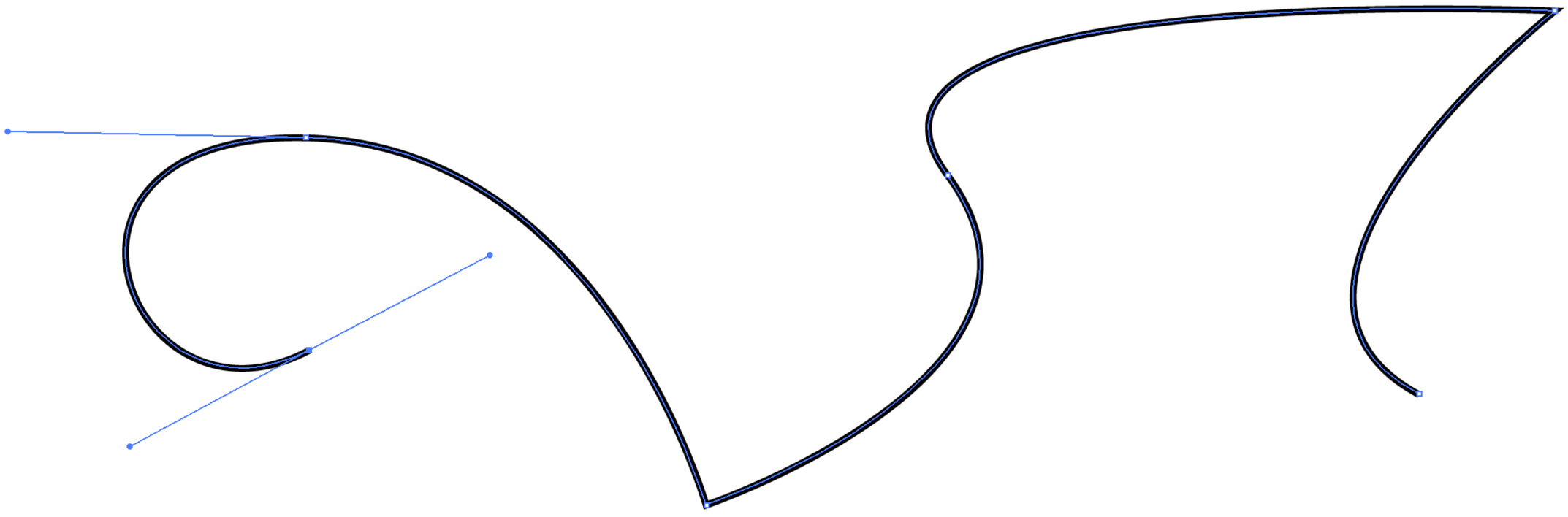
Very hard to control!

Uncommon

Piecewise Bézier Curves

Instead, chain many low-order Bézier curve

Piecewise cubic Bézier the most common technique



Widely used (fonts, paths, Illustrator, Keynote, …)

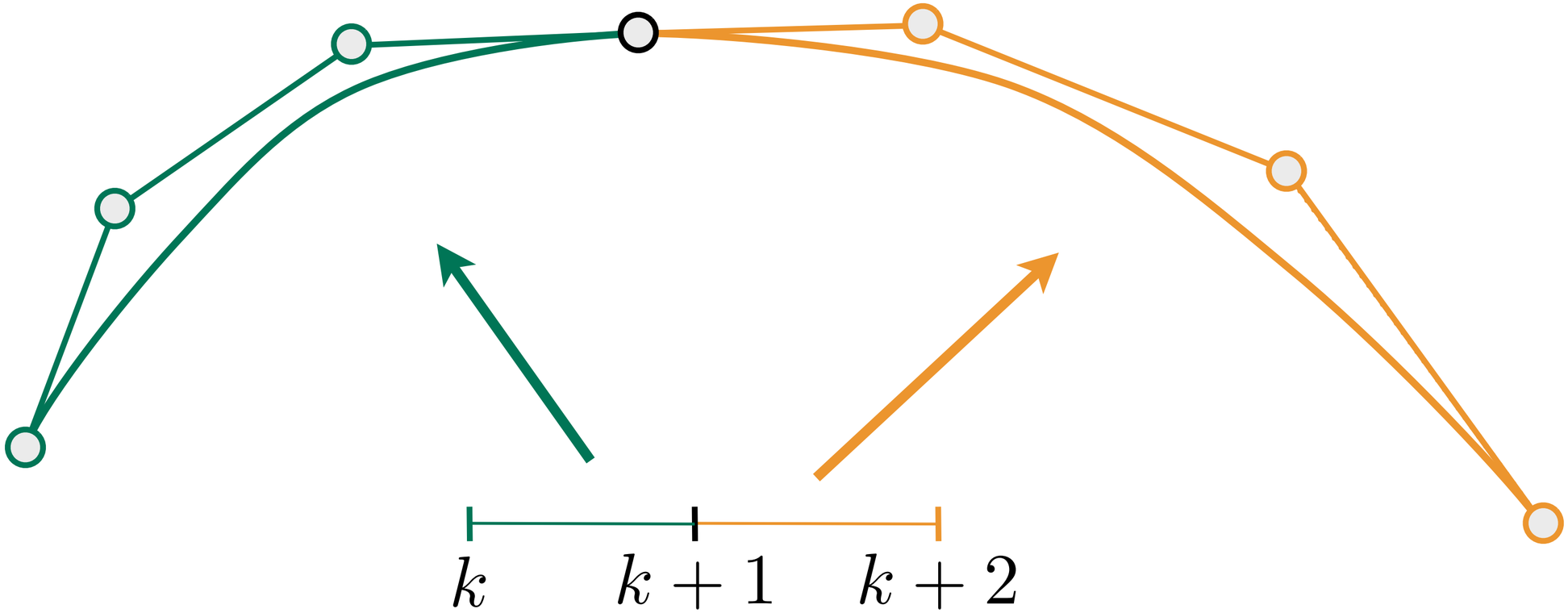
Piecewise Bézier Curve – Continuity

Two Bézier curves

$$\mathbf{a} : [k, k+1] \to \mathbb{R}^N$$

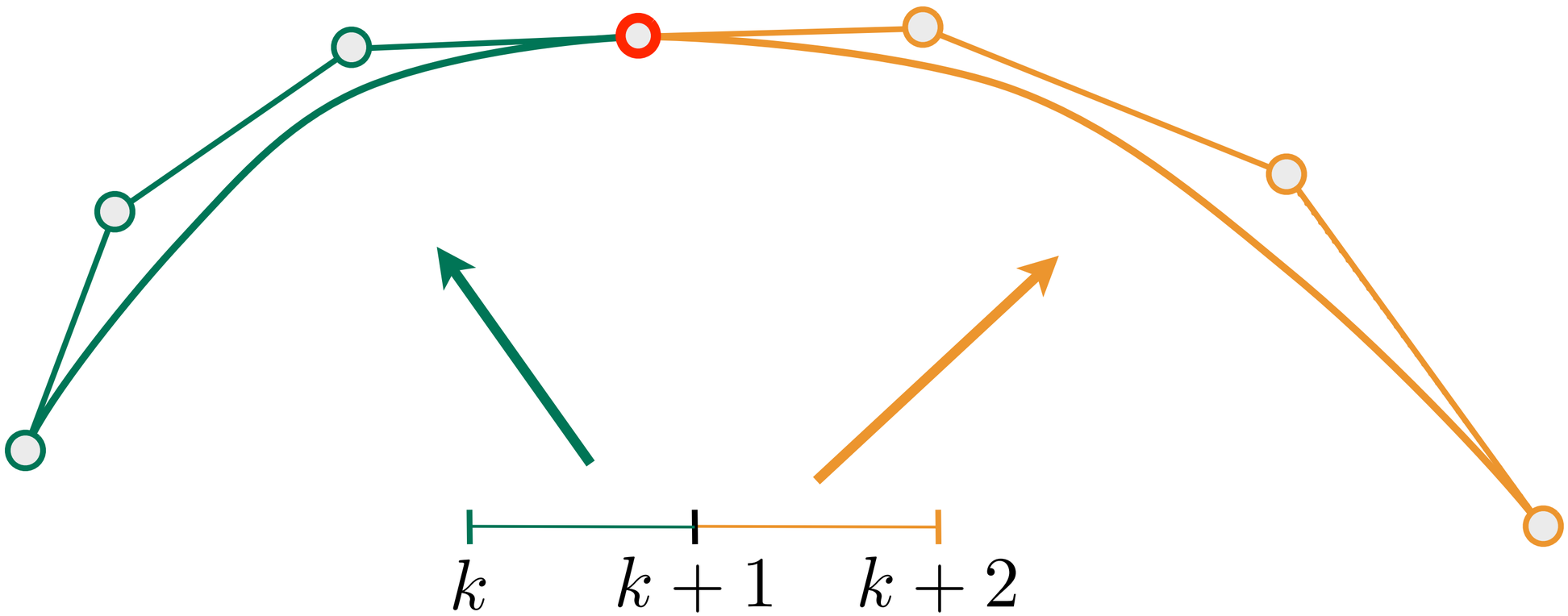
$$\mathbf{b} : [k+1, k+2] \to \mathbb{R}^N$$

Assuming integer partitions here, can generalize



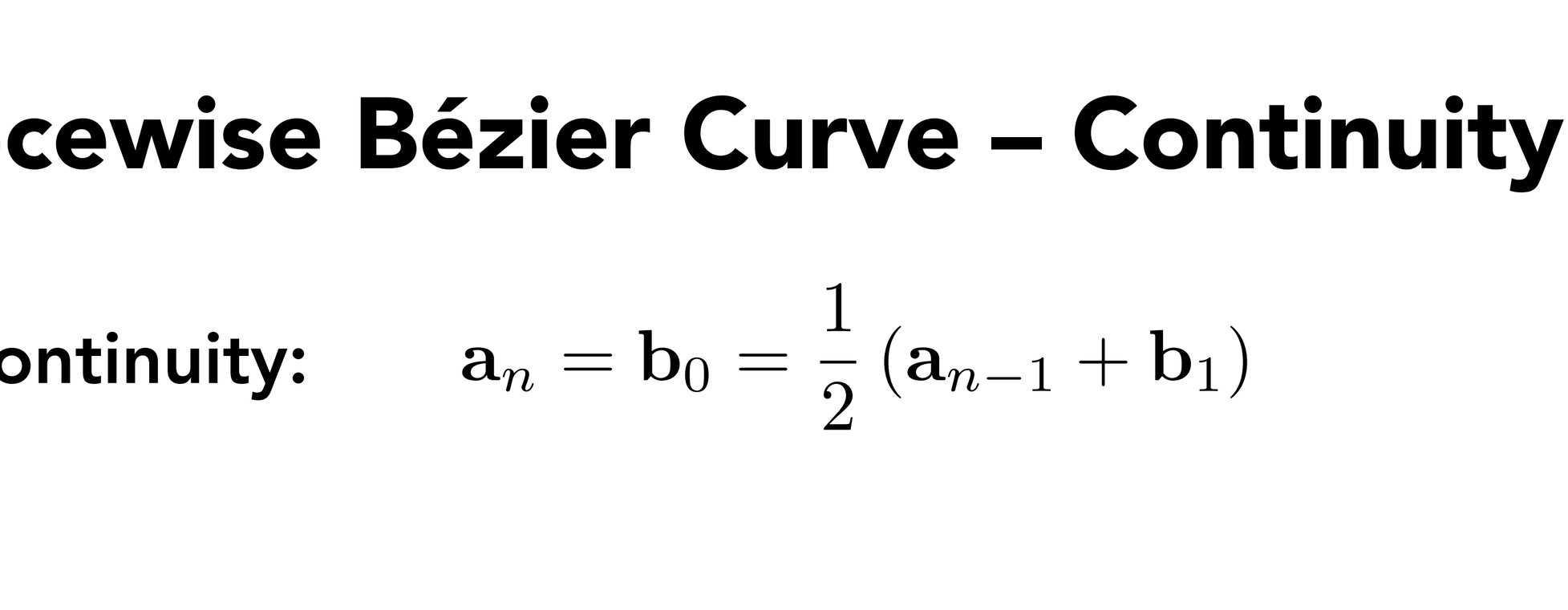
Piecewise Bézier Curve – Continuity

C⁰ continuity: $\mathbf{a}_n = \mathbf{b}_0$



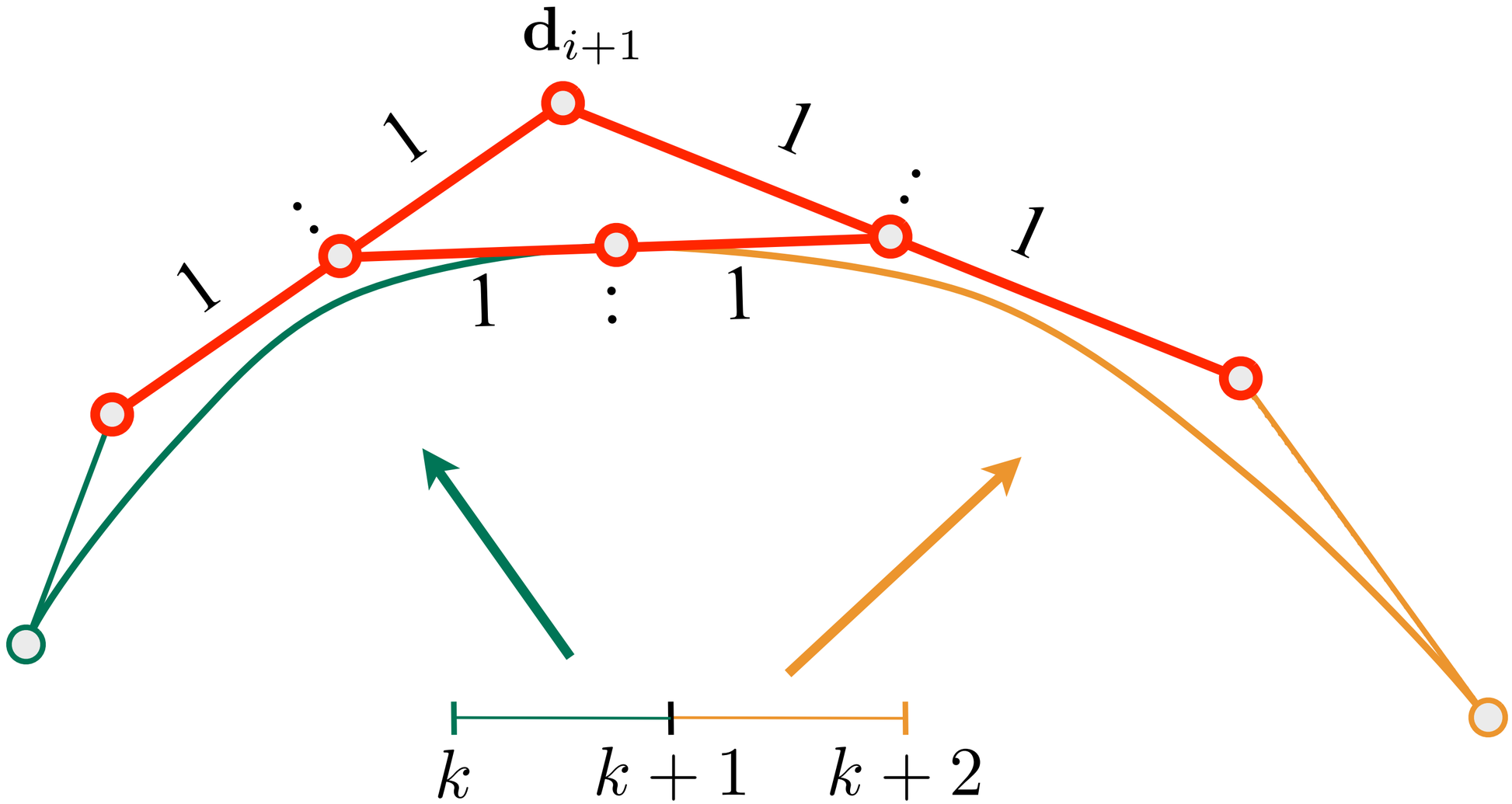
Piecewise Bézier Curve – Continuity

C¹ continuity: $\mathbf{a}_n = \mathbf{b}_0 = \frac{1}{2}(\mathbf{a}_{n-1} + \mathbf{b}_1)$



Piecewise Bézier Curve – Continuity

C^2 continuity: "A-frame" construction



Properties of Bézier Curves

Interpolates endpoints

- For cubic Bézier: $\mathbf{b}(0) = \mathbf{b}_0; \quad \mathbf{b}(1) = \mathbf{b}_3$

Tangent to end segments

- Cubic case: $\mathbf{b}'(0) = 3(\mathbf{b}_1 - \mathbf{b}_0); \quad \mathbf{b}'(1) = 3(\mathbf{b}_3 - \mathbf{b}_2)$

Affine transformation property

- Transform curve by transforming control points

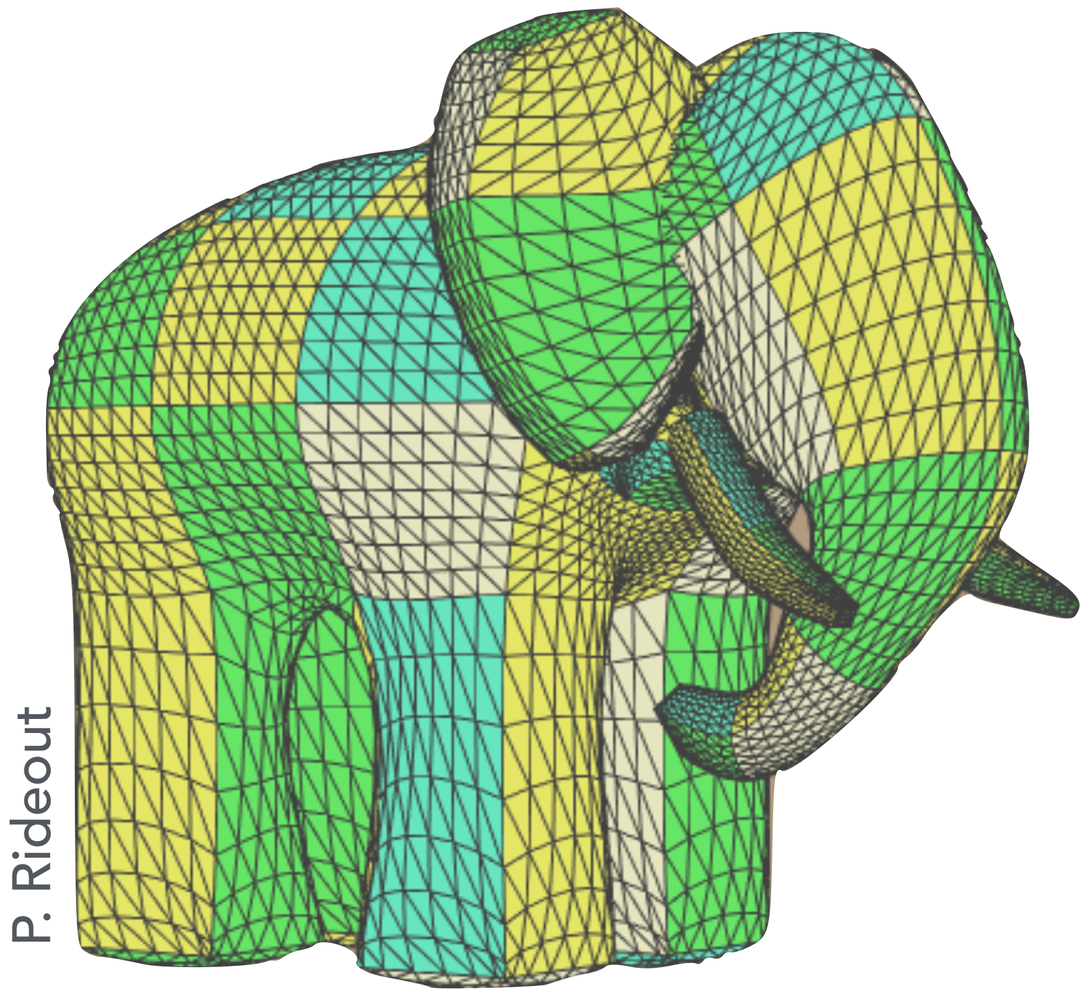
Convex hull property

- Curve is within convex hull of control points

Bézier Surfaces

Bézier Surfaces

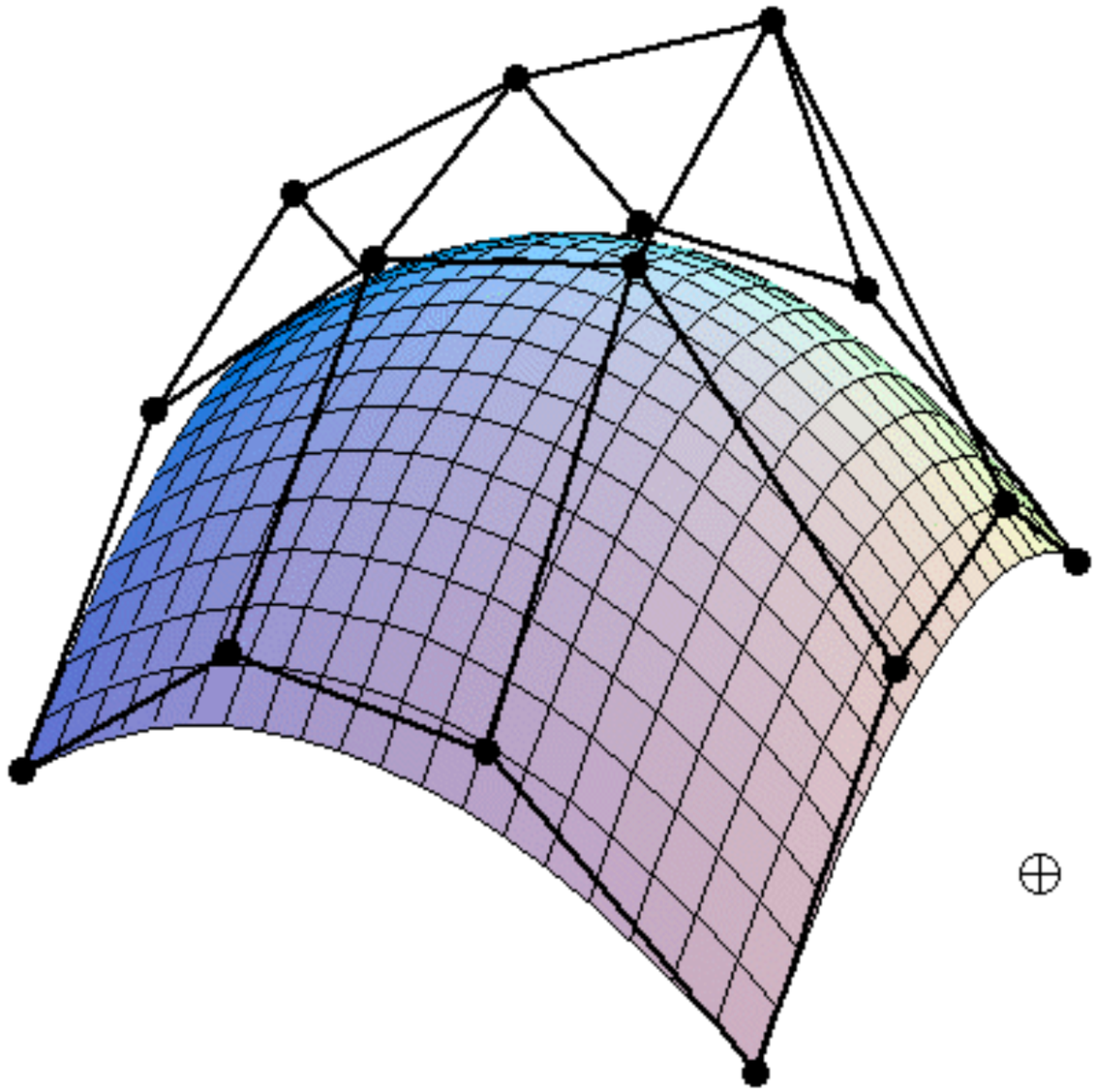
Extend Bézier curves to surfaces



Ed Catmull's "Gumbo" model

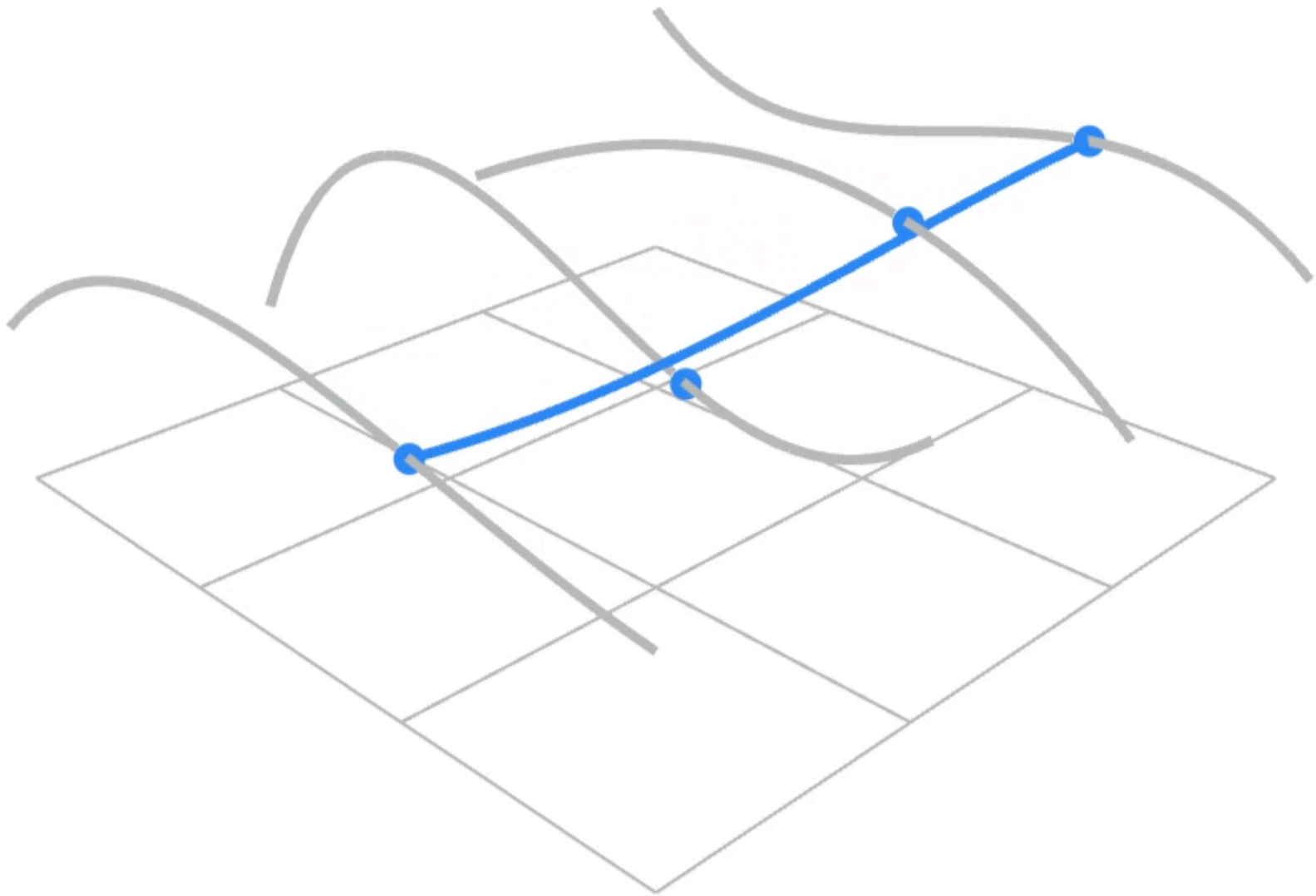
Utah Teapot

Bicubic Bézier Surface Patch



Bezier surface and 4 x 4 array of control points

Visualizing Bicubic Bézier Surface Patch



Animation: Steven Wittens, Making Things with Maths, http://acko.net

Visualizing Bicubic Bézier Surface Patch

4x4 control points

- Each 4x1 control points in u define a Bezier curve
  - (4 Bezier curves in u)
- Corresponding points on these 4 Bezier curves define 4 control points for a “moving curve” in v
  - This “moving” curve sweeps out the 2D surface

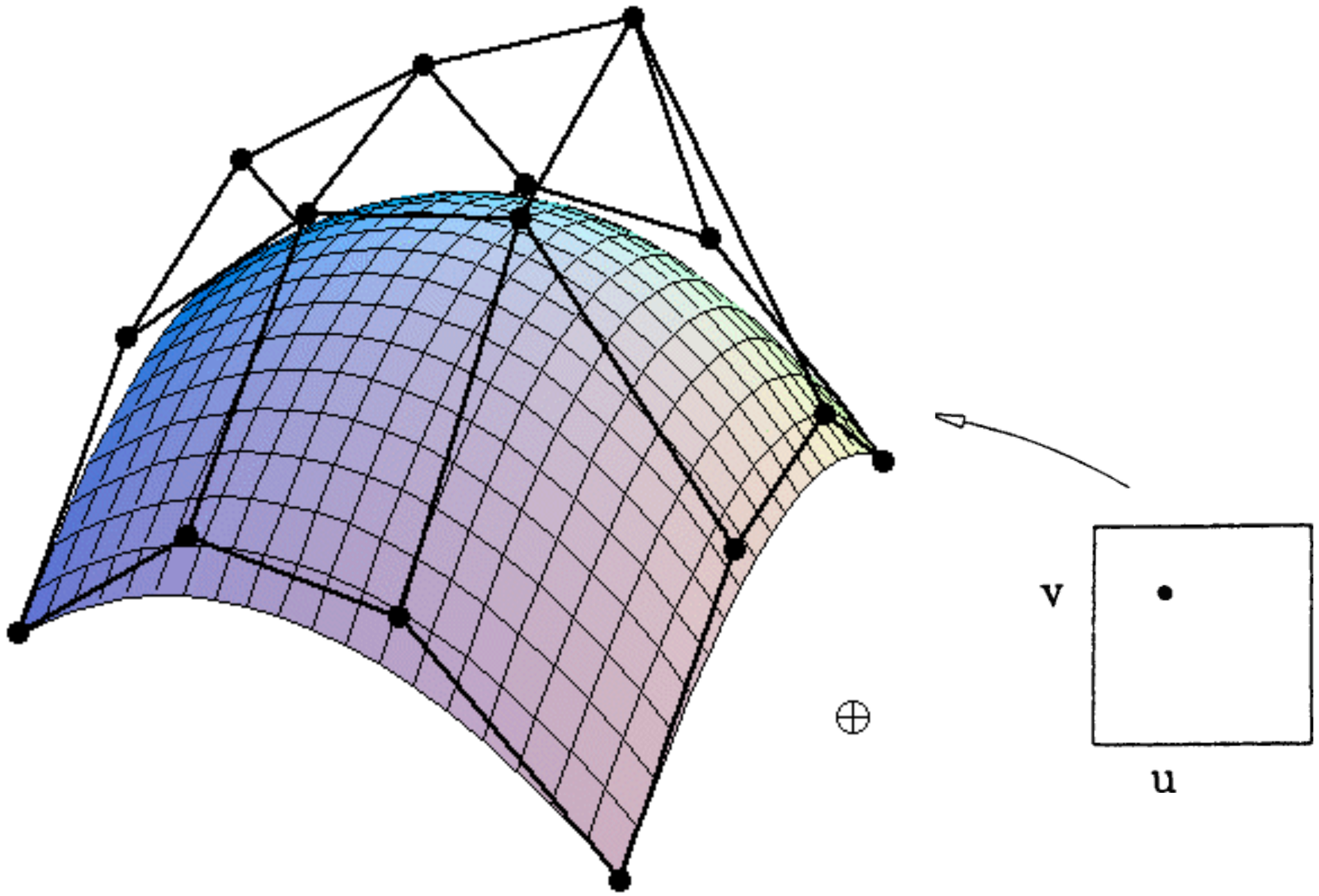
Evaluating Bézier Surfaces

Evaluating Surface Position For Parameters (u,v)

For bi-cubic Bezier surface patch,

Input: 4x4 control points

Output is 2D surface parameterized by (u,v) in $[0,1]^2$

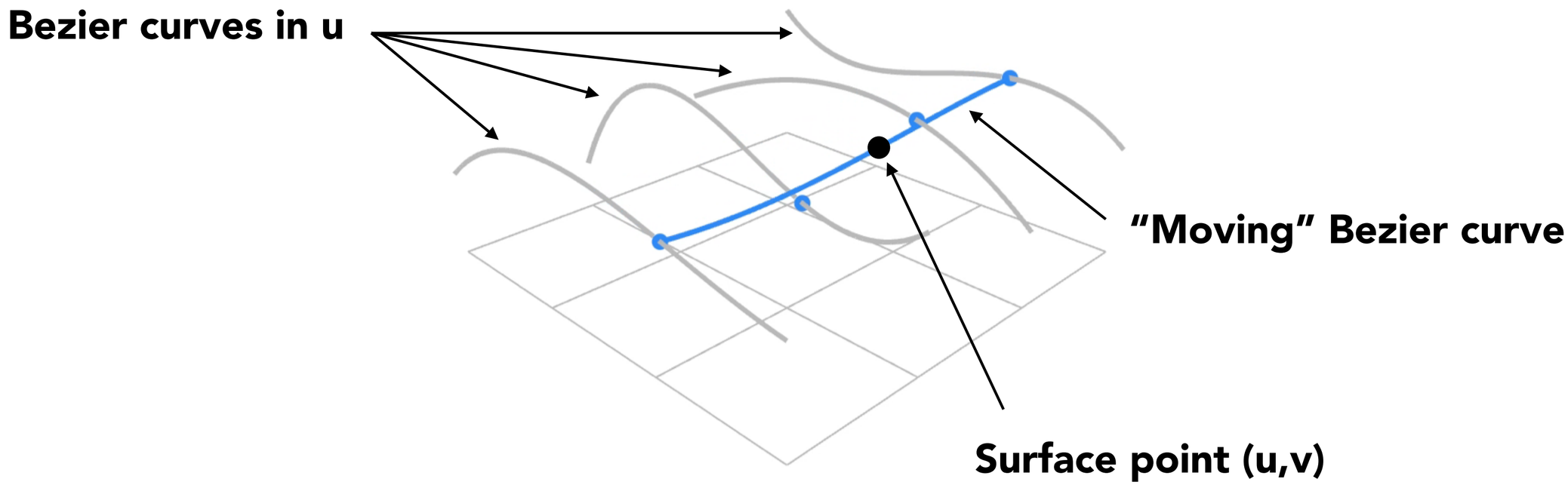


Method 1: Separable 1D de Casteljau Algorithm

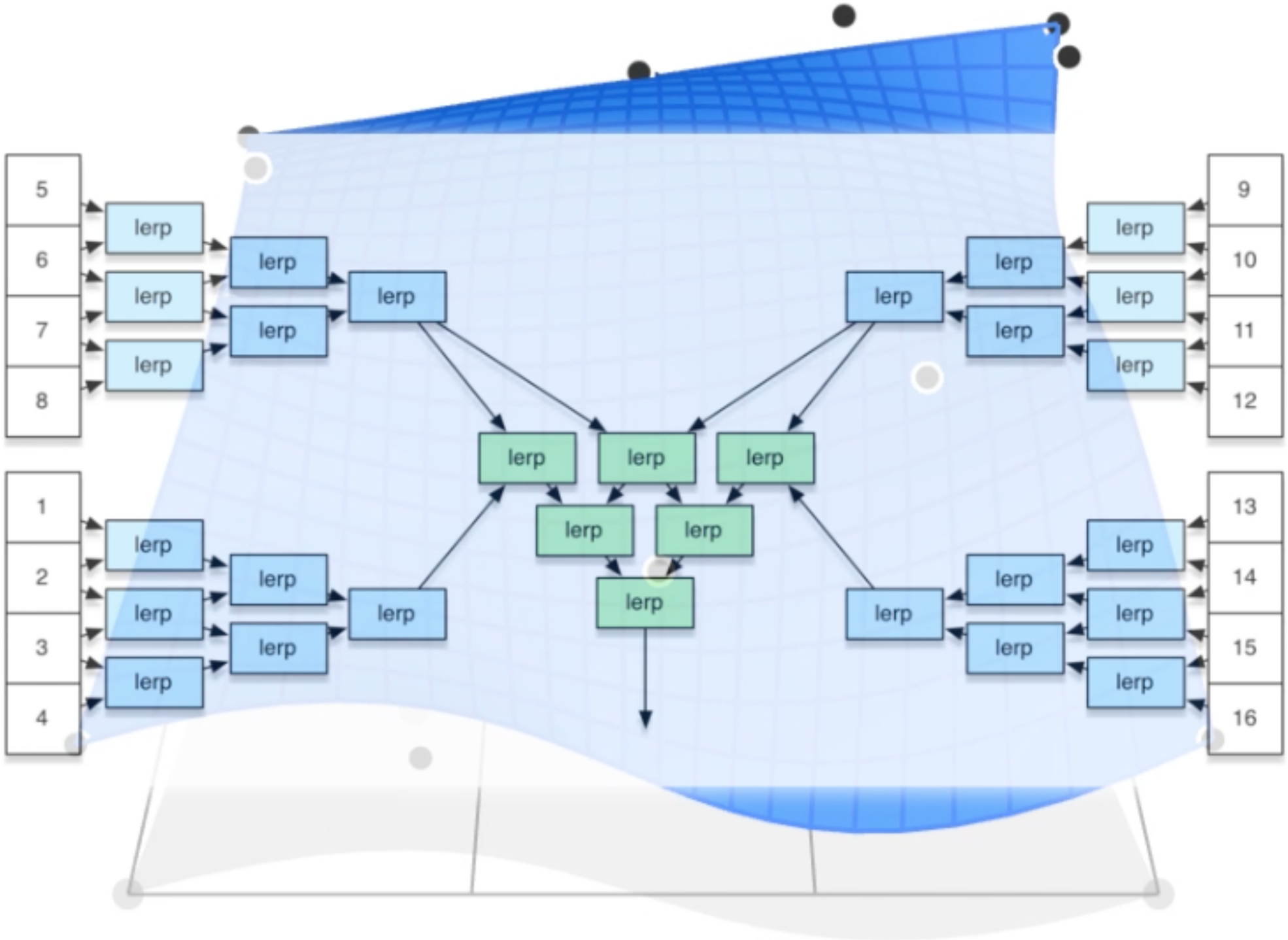
Goal: Evaluate surface position corresponding to (u,v)

(u,v)-separable application of de Casteljau algorithm

- Use de Casteljau to evaluate point u on each of the 4 Bezier curves in u. This gives 4 control points for the “moving” Bezier curve
- Use 1D de Casteljau to evaluate point v on the “moving” curve



Method 1: Separable 1D de Casteljau Algorithm



Method 2: Algebraic Evaluation

Let the moving curve be a degree m Bézier curve

$$\mathbf{b}^m(u) = \sum_{i=0}^{m} \mathbf{b}_i B_i^m(u)$$

(remember, Bernstein polynomials)

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Let each control point $\mathbf{b}_i$ be moving along a Bézier curve of degree n

$$\mathbf{b}_i = \mathbf{b}_i(v) = \sum_{j=0}^{n} \mathbf{b}_{i,j} B_j^n(v)$$

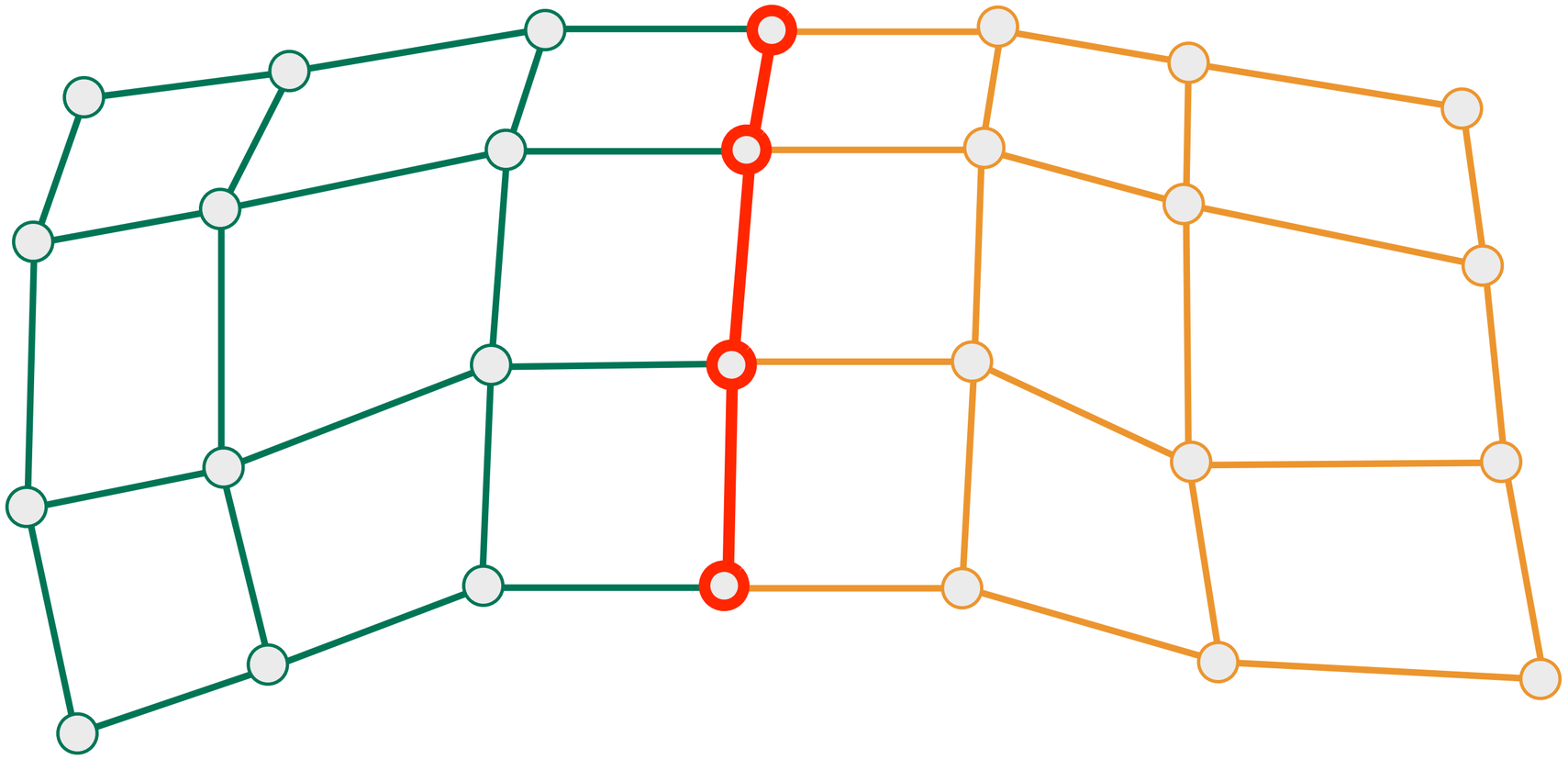
Tensor product Bézier patch

$$\mathbf{b}^{m,n}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{b}_{i,j} B_i^m(u) B_j^n(v)$$

Bézier Surface Continuity

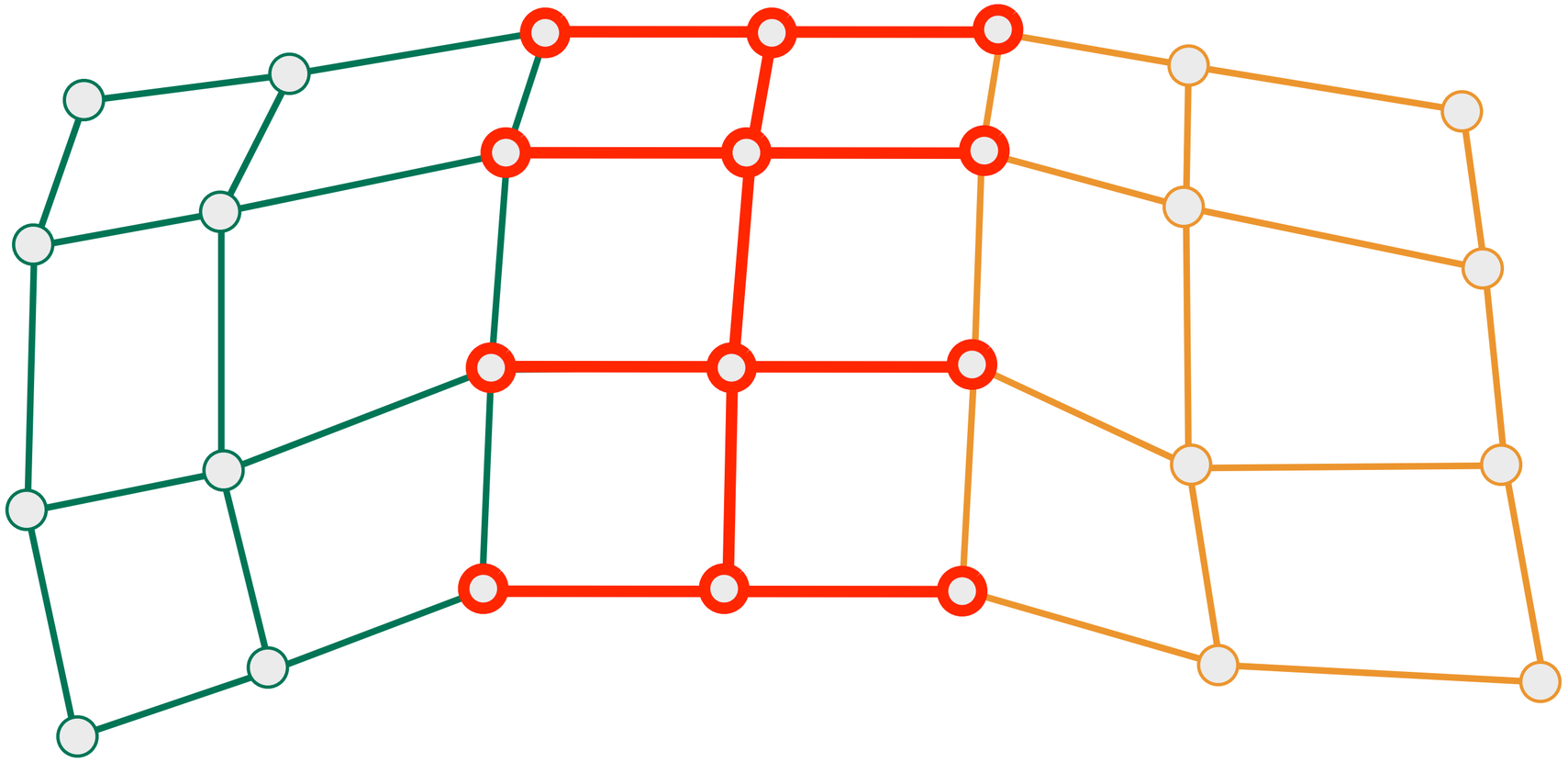
Piecewise Bézier Surfaces

C^0 continuity: Boundary curves



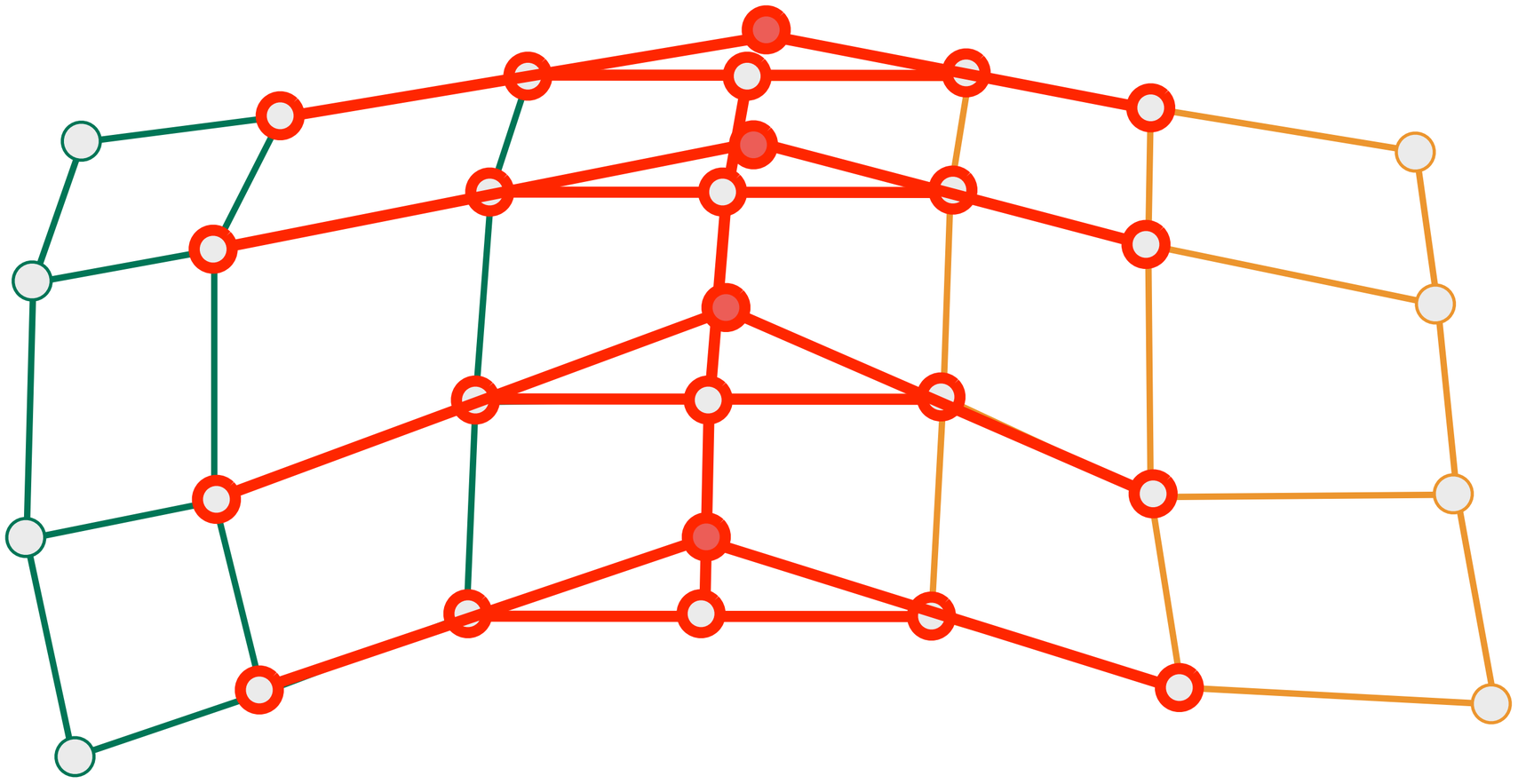
Piecewise Bézier Surfaces

C^1 continuity: Collinearity



Piecewise Bézier Surfaces

C^2 continuity: A-frames



Things to Remember

Splines

- Cubic Hermite and Catmull-Rom interpolation
- Matrix representation of cubic polynomials

Bézier curves

- Easy-to-control spline
- Recursive linear interpolation – de Casteljau algorithm
- Properties of Bézier curves
- Piecewise Bézier curve – continuity types and how to achieve

Bézier surfaces

- Bicubic Bézier patches – tensor product surface
- 2D de Casteljau algorithm

Acknowledgments

Thanks to Pat Hanrahan, Mark Pauly and Steve Marschner for presentation resources.