

1 A Polynomial Interpolation

Our goal is to fit a polynomial to given points and derivatives of a curve. We can solve this problem by formulating it as a system of linear equations in the coefficients of the polynomial.

1. List all degree 2 polynomials satisfying: $f(0) = 1, f(1) = 2, f(2) = 5$.

2. How many degree 3 polynomials satisfy the given constraints?

3. Suppose we have a list of constraints:

$$f(0) = p_0, f'(0) = d_0, f(1) = p_1, f'(1) = d_1, \dots, f(k) = p_k, f'(k) = d_k.$$

Since we have $2(k + 1)$ constraints (one function and one derivative condition per point), the unique interpolating polynomial must have degree $2k + 1$.

For a function f , what are the tradeoffs when either

- solving for a single degree $2k + 1$ polynomial, versus
- taking the point and derivative constraints at i and $i - 1$ for $i = 1, \dots, k$ and using them to fit k cubic Hermite splines?

4. Consider a cubic polynomial $f(t) = at^3 + bt^2 + ct + d$ that satisfies the following conditions:

$$\begin{aligned}f(0) &= f_0, \\f(1) &= f_1, \\f''(0) &= f_0'', \\f''(1) &= f_1''.\end{aligned}$$

Write the matrix that, when inverted and applied to the vector $(f_0, f_1, f_0'', f_1'')^T$, allows you to recover the coefficients a, b, c , and d of the polynomial.

5. Given the numerical inverse of the matrix:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1/6 & 1/6 \\ 0 & 0 & 1/2 & 0 \\ -1 & 1 & -1/3 & -1/6 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_0'' \\ f_1'' \end{pmatrix}$$

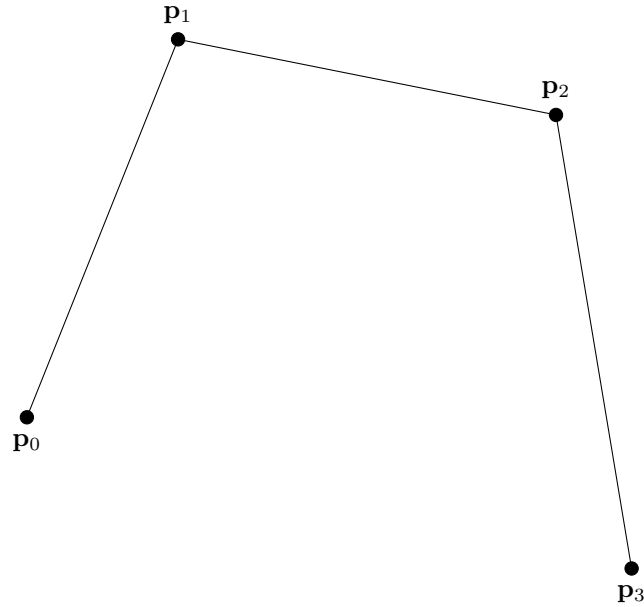
what are the **basis polynomials** that allow expressing $f(t)$ in terms of f_0, f_1, f_0'', f_1'' ?

2 de Casteljau's algorithm

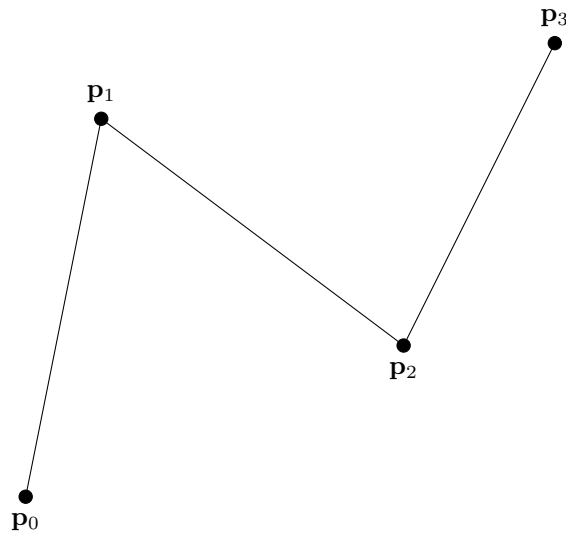
Given $k + 1$ points $\mathbf{p}_0, \dots, \mathbf{p}_k$, create a new set of k points $\mathbf{p}'_0, \dots, \mathbf{p}'_{k-1}$ by computing $\mathbf{p}'_i = \text{lerp}(\mathbf{p}_i, \mathbf{p}_{i+1}, t)$, where $\text{lerp}(\mathbf{p}_i, \mathbf{p}_{i+1}, t) = (1 - t)\mathbf{p}_i + t\mathbf{p}_{i+1}$.

Perform k times to yield a single point, $f(t)$.

1. Use de Casteljau's algorithm to construct $f(1/2)$ with the given control points.



2. Use de Casteljau's algorithm to construct $f(1/3)$ with the given control points.



3. Show that the point with parameter t on the Bézier curve with control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is given by $s^3\mathbf{p}_0 + 3s^2t\mathbf{p}_1 + 3st^2\mathbf{p}_2 + t^3\mathbf{p}_3$, where $s = 1 - t$. (Hint: apply de Casteljau's algorithm algebraically to the control points. With this setup, linear interpolation between two points \mathbf{q}_0 and \mathbf{q}_1 looks like $s\mathbf{q}_0 + t\mathbf{q}_1$.)

4. What is this matrix product? (Hint: *don't* expand it. Instead, think about what each matrix in the product does. How are they related to de Casteljau's algorithm?)

$$(s \ t) \begin{pmatrix} s & t & 0 \\ 0 & s & t \end{pmatrix} \begin{pmatrix} s & t & 0 & 0 \\ 0 & s & t & 0 \\ 0 & 0 & s & t \end{pmatrix}$$

5. For a Bézier curve defined by 3 control points, what is the degree of the polynomial that results from de Casteljau's algorithm? What about for k points?

¹Here, we consider the control points \mathbf{p}_j to be variables and the resulting function to be a polynomial in t . If you can fix \mathbf{p}_j 's, it is possible to obtain a polynomial with lower degree. For example, if every $\mathbf{p}_j = \mathbf{0}$, then the resulting polynomial is $\mathbf{0}$.