Photometry & Monte Carlo 8

CS 184: Foundations of Computer Graphics

1 Lumens and Joules and Nits — Oh My!

1. Fill in the table below. In the right-most column, R denotes the distance to the light source.

Symbol/Name	Radiometry Unit/Name	Photometry Unit/Name	Effect of Increased R
Q: Energy	Radiant Energy Joules (W·s)	Luminous Energy Lumen·sec	$\uparrow$ (=) $\downarrow$
Φ: Flux (Power)			$\uparrow$ = $\downarrow$
I: Angular Flux Density			$\uparrow$ = $\downarrow$
E: Spatial Flux Density			$\uparrow$ = $\downarrow$
L: Spatio-Angular Flux Density			$\uparrow$ = $\downarrow$

Solution:

Symbol/Name	Radiometry Unit/Name	Photometry Unit/Name	Effect of Increased R
Q: Energy	Radiant Energy Joules (W·s)	Luminous Energy Lumen·sec	=
Φ: Flux (Power)	Radiant Power W	Luminous Power Candela·sr	=
I: Angular Flux Density	Radiant Intensity W/sr	Luminous Intensity Candela = Lumen/sr	=
E: Spatial Flux Density	Irradiance (in), Radiosity (out) W/m^2	Illuminance (in), Luminosity (out) Lux = $Lumen/m^2$	$\downarrow$
L: Spatio-Angular Flux Density	Radiance $W/m^2/sr$	Luminance Nit = $Candela/m^2$	=

As justification for the rightmost column:

- Q and Φ are properties of the light source itself, and do not depend on R.
- I is dependent on the light source and solid angle, but not the distance.
- $E = \frac{\Phi}{A}\cos\theta$. Since Φ and θ do not change with R.
- L is invariant along a ray in a vacuum, i.e. there is no dependence on R.

2. For these questions, feel free to use infinitesimal quantities, e.g. dA, to represent a small area around a point, and θ to represent the angle between the light direction and surface normal.

   (a) What is Φ in terms of Q and time t?

   (b) What is $I(p,\omega)$ at a point p in terms of flux and solid angle?

   (c) What is irradiance $E(p)$ at a point p in terms of flux and area?

   (d) What is surface radiance $L(p,\omega)$ at a point p in a direction ω in terms of flux, area, and solid angle?

   (e) How can surface radiance $L(p,\omega)$ be expressed in terms of irradiance $E(p)$?

   (f) How can surface radiance $L(p,\omega)$ be expressed in terms of intensity $I(p,\omega)$?

Solution:

(a) $\Phi = \dfrac{dQ}{dt}$

(b) $I(p,\omega) = \dfrac{d\Phi}{d\omega}$

(c) $E(p) = \dfrac{d\Phi(p)}{dA}$

(d) $L(p,\omega) = \dfrac{d^2\Phi(p,\omega)}{d\omega\, dA\cos\theta}$

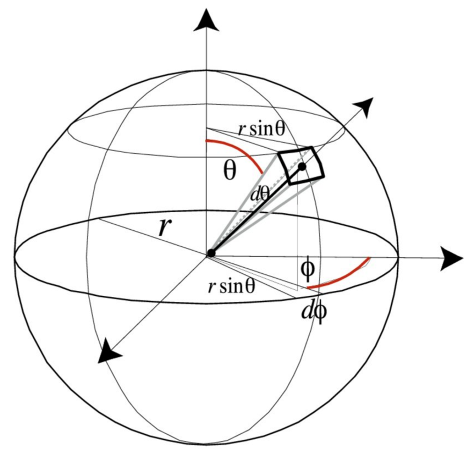
(e) $L(p,\omega) = \dfrac{dE(p)}{d\omega\cos\theta}$

(f) $L(p,\omega) = \dfrac{dI(p,\omega)}{dA\cos\theta}$

2 Shedding Some Light

1. Suppose we use (θ, ϕ)-parameterization of directions. Recall that the solid angle represents the ratio of the subtended area on a sphere to the radius squared, $\Omega = \frac{A}{r^2}$. Estimate the solid angle subtended by a patch that covers $\theta \in [\pi/6 - \pi/12, \pi/6 + \pi/12]$ and $\phi \in [\pi/5 - \pi/24, \pi/5 + \pi/24]$?

   (Hint: you may assume that the patch is small enough. Recall or derive the differential solid angle $d\omega$, then use the values given.)



> **Solution:** Under (θ, ϕ)-parameterization, we know that the differential solid angle is $\mathrm{d}\omega = \sin\theta\, \mathrm{d}\theta \mathrm{d}\phi$. When a patch is small enough, we can use this to approximate its solid angle as
>
> $$\Delta\omega = \sin\theta\, \Delta\theta\Delta\phi, \tag{1}$$
>
> where ϕ is the azimuth angle, and θ the elevation angle, at the center of the patch.
>
> In our specific case, the solid angle subtended by the patch is now
>
> $$\Delta\omega \approx \sin\frac{\pi}{6} \cdot \left[\frac{\pi}{12} - \left(-\frac{\pi}{12}\right)\right] \cdot \left[\frac{\pi}{24} - \left(-\frac{\pi}{24}\right)\right] \tag{2}$$
>
> $$= \frac{\pi^2}{144}. \tag{3}$$

2. A point light at position $(6, 0, 8)$ (in meters) with radiant flux (power) of 100 watts directs all its light uniformly into the hemisphere directly below it. Some of this light falls on a flat, tilted surface passing through the origin, with surface normal vector $(1, 1, 1)$.

   What is the irradiance at the origin? Show your work and use the correct units.

> **Solution:** First, we have
>
> $$r = \sqrt{6^2 + 8^2} = 10.$$
>
> The hemisphere below the light has solid angle equal to 2π. The cosine angle between the surface

normal and the light position is given by the dot product of their normalized vectors:

$$\cos(\theta) = \frac{1}{10}(6, 0, 8) \cdot \frac{1}{\sqrt{3}}(1, 1, 1) = \frac{14}{10\sqrt{3}}.$$

Finally,

$$E = \frac{\Phi}{2\pi r^2}\cos(\theta) = \frac{100}{200\pi}\frac{14}{10\sqrt{3}}\ \mathrm{W/m^2} = \frac{7}{10\sqrt{3}\pi}\ \mathrm{W/m^2}.$$

3 Inversion Method

Given a uniform random variable U in the interval $[0,1]$, we can generate a random variable from any other one dimensional distribution using its cumulative distribution function: $X = F^{-1}(U)$. This is how we choose sample points when running a ray tracing algorithm.

1. What function of U will return a sample from the exponential distribution (with parameter λ)? This distribution has density $p_\lambda(x) = \lambda e^{-\lambda x}$, and is defined for $x \geq 0$.

Solution: First, we need to calculate the CDF:

$$F_\lambda(x) = \int_0^x p_\lambda(t)dt = \int_0^x \lambda e^{-\lambda t}dt = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}$$

Set the CDF equal to U:

$$U = 1 - e^{-\lambda x}$$

Solving for x:

$$e^{-\lambda x} = 1 - U$$

$$-\lambda x = \ln(1-U)$$

$$x = -\frac{\ln(1-U)}{\lambda}$$

Thus, the inverse function is given by:

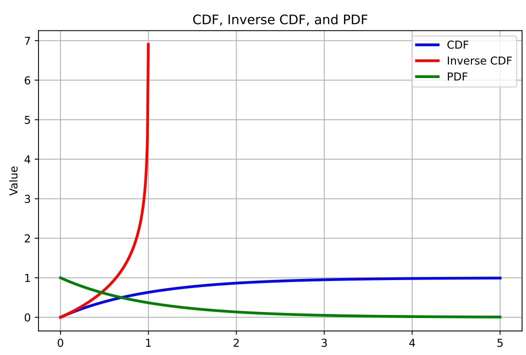
$$F_\lambda^{-1}(x) = -\frac{\ln(1-x)}{\lambda}$$

and we can return:

$$-\frac{\ln(1-U)}{\lambda}$$

to sample from the exponential distribution.

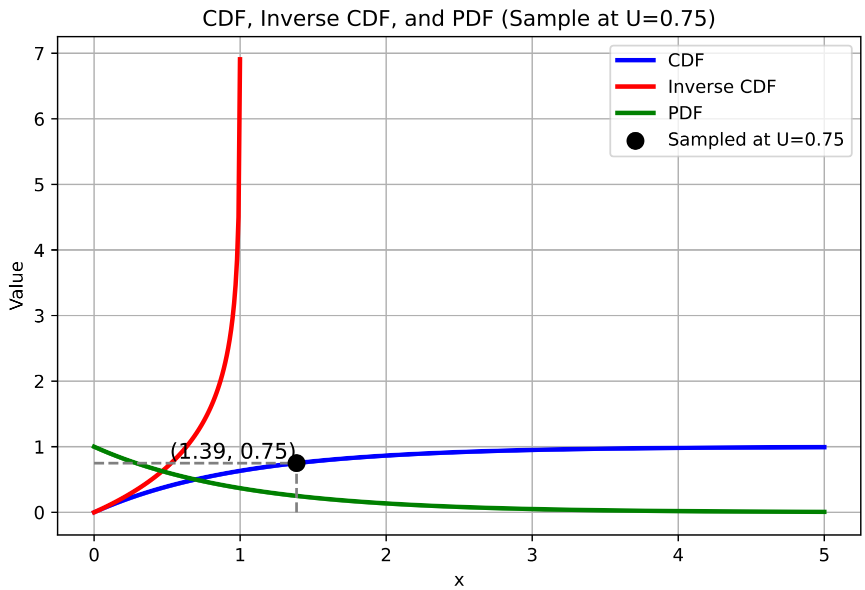
2. For $\lambda = 1$, the plot below shows the CDF, Inverse CDF, and PDF. If we sample $U = 0.75$, determine the corresponding sampled value using the inverse method. Mark the corresponding point on the CDF curve in the graph.



Solution: Using the inverse CDF function:

$$x = -\ln(1 - 0.75) = -\ln(0.25) \approx 1.386.$$

Thus, the sampled value is approximately $x \approx 1.386$.



3. What does the x-axis represent for the blue CDF curve?

Solution: The x-axis represents the possible values that can be sampled from the exponential distribution. The CDF gives the probability that a randomly drawn value from this distribution is less than or equal to a given x. Therefore, the x-axis represents the range of values that an exponential random variable can take.

4 Unbiased Estimators

1. Let $f : [-2,2] \times [-2,2] \to \mathbb{R}$ be a function. You have a machine that allows you to sample $2n$ values independently and uniformly from the interval $[-2,2]$. Construct an unbiased Monte Carlo estimator for

$$F = \int_{-2}^{2} \int_{-2}^{2} f(x,y) dx dy.$$

Solution: Draw n random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. Then we take

$$\begin{aligned}\langle F_n \rangle &:= \frac{1}{n} \sum_{i=1}^{n} \frac{f(X_i, Y_i)}{p(X_i, Y_i)} \text{, where } p(X_i, Y_i) = p(X_i)p(Y_i) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16} \\ &= \frac{1}{n} \sum_{i=1}^{n} \frac{f(X_i, Y_i)}{\frac{1}{16}} \\ &= \frac{16}{n} \sum_{i=1}^{n} f(X_i, Y_i).\end{aligned}$$