Triangles and Transforms 2

CS 184: Foundations of Computer Graphics

1 Rasterization Potpourri

1. What is the connection between rasterizing a polygon and discretizing a continuous function?

   Solution: A polygon in vector graphics format can be defined by an indicator function on continuous two-dimensional space. (From lecture, **inside**(tri, x, y).) A raster display, often a grid of pixels, is a discrete two-dimensional space.

   To rasterize a polygon is to approximate the indicator function in discrete space, i.e. to discretize the conitnuous function.

2. Why does aliasing occur? Describe or draw an example of aliasing of any kind.

   Solution: Aliasing occurs when high-frequency signals are undersampled (sampled at an insufficient frequency). The sampled function will incorrectly resemble a lower-frequency signal.

   Examples: jaggies, wagon-wheel effect, moiré pattern, etc.

3. What is the connection between applying a box blur to an image and supersampling each pixel within the image?

   Solution: One way to avoid aliasing is to filter out high-frequency signals before sampling.

   A box blur is a low-pass filter with a discrete, uniform kernel. Applied to an image, a box blur filters out high-frequency signals. It's an approximation of a filtered continuous signal.

   Supersampling (then downsampling) approximates sampling a low-pass filtered continuous signal. Per pixel, the average value within the pixel is approximated by taking multiple samples, then taking their average.

   Both the box blur and supersampling approximate the anti-aliasing technique of filtering out high-frequency signals, then sampling.

The cross product between two vectors, **a** and **b**, in three-dimensional space is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

1. Given two vectors, $\mathbf{a} = \begin{pmatrix} 10 \\ 7 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ -11 \\ 0 \end{pmatrix}$, calculate $\mathbf{a} \times \mathbf{b}$. Next, calculate $\mathbf{b} \times \mathbf{a}$.

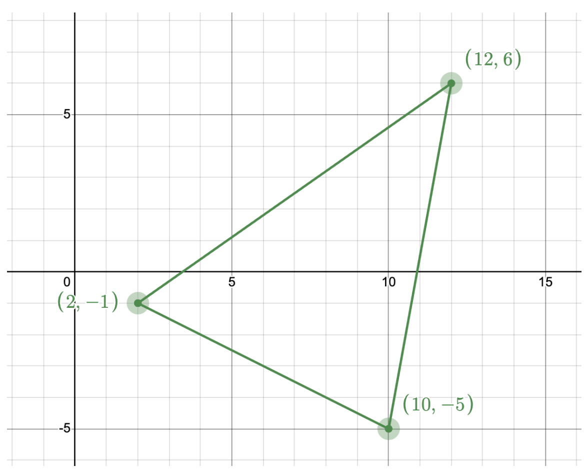
Solution: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 7 & 0 \\ -2 & -11 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -96 \end{pmatrix}$

$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -11 & 0 \\ 10 & 7 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 96 \end{pmatrix}$

Notice that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. So, the magnitude of both cross-product vectors is the same. However, $\mathbf{a} \times \mathbf{b}$ is on the negative z-axis (going into the page), whereas $\mathbf{b} \times \mathbf{a}$ is on the positive z-axis (coming out of the page).

2. Draw the triangle given by points $(2, -1)$, $(12, 6)$, $(10, -5)$. What is the *winding order* of the triangle? In other words, are the points given in clockwise or counter-clockwise order?

Solution: The points are in clockwise order.



3. What are the three vectors defined by the edges of this triangle? Assume the triangle is lying in the xy-plane. Select two of the three vectors. Calculate their cross product.

Solution: $\mathbf{a} = \begin{pmatrix} 10 \\ 7 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ -11 \\ 0 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -8 \\ 4 \\ 0 \end{pmatrix}$.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -11 & 0 \\ -8 & 4 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -96 \end{pmatrix} = -\mathbf{c} \times \mathbf{b}$$

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 4 & 0 \\ 10 & 7 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -96 \end{pmatrix} = -\mathbf{a} \times \mathbf{c}$$

4. In general, given triangle vertices $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, how can winding order be determined?

Solution:

One method is to calculate the cross product of two consecutive edges. For example, $(\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_1)$. This is equivalent to calculating $((x_1-x_0)(y_2-y_1)-(x_2-x_1)(y_1-y_0))\mathbf{k}$. If the component on the z-axis is negative, winding order is clockwise. If the component on the z-axis is positive, winding order is counterclockwise.

There are other approaches that can be taken. For example, $(\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0) = (\mathbf{p}_1 - \mathbf{p}_0) \times -(\mathbf{p}_0 - \mathbf{p}_2) = (\mathbf{p}_0 - \mathbf{p}_2) \times -(\mathbf{p}_1 - \mathbf{p}_0)$. This is equivalent to the first approach — that is, calculating the cross-product in the order that the vertices are presented in.

3 Basic Transforms

1. Isometric transformations preserve the distances between every pair of points on an object. Which transformations are isometric? Which transformations aren't?

Solution: Rotations, translations, and reflections (and their combinations) all fulfill this property. They are what we call isometries or rigid transformations (as if we were manipulating an object that cannot be bent or broken).

Scaling and shearing are examples of transformations that are not isometric.

2. Using homogeneous coordinates, define a point, $\mathbf{p} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$, a vector, $\mathbf{v} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$, and a translation transformation, $\mathbf{T} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$. What is the result of applying $\mathbf{T}$ to $\mathbf{p}$?

Solution:

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 + t_x \\ 5 + t_y \\ 1 \end{pmatrix}$$

Point $\mathbf{p}$ is shifted t_x to the right and t_y up.

3. What is the result of applying $\mathbf{T}$ to $\mathbf{v}$? Justify this result.

Solution:

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$$

A vector represents a direction, not a point in space, so it should be unaffected by translation transformations. (Translating a *direction* should not change anything about the *direction*.)

4. Write the transformation matrix for a 2D object that is reflected across the y-axis, translated up by 1 unit and right by 3 units, then rotated around the origin by $90°$ counterclockwise.

Solution:

An important thing to remember here is that the transformation matrices are multiplied right to left. Keeping the order correct is important!

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

4 Rebecca Takes a Look

In world space coordinates, a teapot sits at $(0, 0, 0)$. Rebecca holds her camera at position $(1, 0, 2)$ — the eye point, **e**, of the camera.

1. Rebecca points her camera at the teapot, intending to take a picture. What is the view direction, **v**, of her camera?

Solution: $\begin{pmatrix}0\\0\\0\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix}$. To normalize, $\frac{1}{\sqrt{1+0+4}}\begin{pmatrix}-1\\0\\-2\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}-1\\0\\-2\end{pmatrix}$.

2. In world space coordinates, in what direction does the positive z-axis of Rebecca's "standard" camera space point?

Solution: $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}$

3. The up vector, **u**, points in the direction of the positive y-axis of "standard" camera space. In world space coordinates, $\mathbf{u} = \begin{pmatrix}0 & 1 & 0\end{pmatrix}^\mathsf{T}$. Calculate the right vector, **r**, which points in the direction of the positive x-axis of "standard" camera space.

Solution: Perform the cross-product $\mathbf{u} \times -\mathbf{v}$. This is essentially $\mathbf{j} \times \mathbf{k}$ to find $\mathbf{i}$.

$$\frac{1}{\sqrt{5}}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\0 & 1 & 0\\1 & 0 & 2\end{vmatrix} = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{k}).$$

$$\mathbf{r} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\-1\end{pmatrix}$$

4. Write the matrix that transforms coordinates from "standard" camera space to world space. The inverse of this matrix is the *Look-At* matrix. What coordinate system transformation does the *Look-At* matrix perform?

Solution:

$$\begin{pmatrix}2/\sqrt{5} & 0 & 1/\sqrt{5} & 1\\0 & 1 & 0 & 0\\-1/\sqrt{5} & 0 & 2/\sqrt{5} & 2\\0 & 0 & 0 & 1\end{pmatrix}$$

Inverse performs world space to camera space transformation.