6. (Total : 16 points) Rendering

(6a) (10 points) GLaDOS writes a ray tracing program in which rays bounce randomly inside a diffuse sphere of unit radius with a hole of surface area A. Let X be the number of bounces before the average ray exits. Please fill in the following blanks. Show your work. For parts ii and iii, you can refer to your answer in part i as the probability p.

6a.i. The probability that $X = 0$ is:

6a.i. ____________

6a.ii. The probability that $X = 2$ is:

6a.ii. ____________

6a.iii. The expected value of X is:

6a.iii. ____________

UPLOAD PHOTO **On a sheet of your own paper, write your answer. Take a photo of your page and upload to GRADESCOPE EXAM 1 PART B.**

Solution:
i. $p = \frac{A}{4\pi}$ ii. $p(1-p)^2$ iii. $\frac{1-p}{p}$
Let $\bar{p} = 1 - p$
$E[X] = (\bar{p} + 2\bar{p}^2 + 3\bar{p}^3 + \ldots)p$
$\bar{p}E[X] + (\bar{p}^2 + \bar{p}^3 + \ldots)p = (2\bar{p}^2 + 3\bar{p}^3 + \ldots)p$
$\bar{p}E[X] + \bar{p}^2 = E[x] - \bar{p}p$
$E[X] = \frac{1-p}{p}$

(6b) (6 points) In path tracing, we discussed two types of importance sampling: sampling over the surface of light sources, or sampling the BRDF in the directions where it is large.

GLaDOS is considering a point on the surface of a material at which there is a very shiny BRDF and light source that subtends a large solid angle. To obtain a more accurate image, would it be better for GLaDOS to importance sample the lights or the BRDF, and why?

Solution: Better to sample the BRDF. In importance sampling we want the pdf to sample proportional to the integrand function. In this case, we have an integral over directions where the integrand is the product of the intensity of the lights and the magnitude of the BRDF. We are told the lights have large support in this domain, and the BRDF being shiny means it has narrow support. The integrand, being the product of these two functions, has a support that is closer in size to the BRDF's smaller support.

(5b) (4 points) Photometric Units

Which of these units is NOT equivalent to Lux (circle all that apply)?

(i) Watt / meter^2

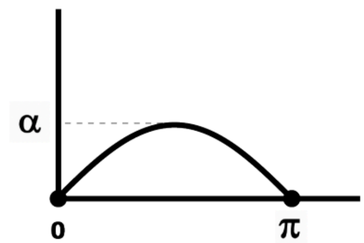
(ii) Lumen / meter^2

(iii) Candela / meter^2

(iv) Nit $\cdot$ steradians

Solution: Watt / meter^2 and Candela / meter^2

(5c) (6 points) Random Sampling



Lucas is writing a program to do Monte Carlo integral estimation of a function $f(x)$ for x between 0 and π, and wants to use importance sampling. His psychic powers give him a hunch that $f(x)$ rises and falls over the interval, so he decides to try importance sampling with a probability density function proportional to $\sin(x)$ as shown in the function graphed above.

Given v, a uniform random value between 0 and 1, derive an expression for a random value x (as a function of v) that satisfies Lucas' desired probability density function.

Solution:

$$\begin{aligned} v &\sim U(0, \pi) \\ x &\sim f(x) \end{aligned}$$

$$\begin{aligned} f(x) &= \alpha \sin(x) \\ \int_0^\pi f(x)dx &= \int_0^\pi \alpha \sin(x)dx \\ &= -\alpha \cos(x)|_0^\pi \\ &= -\alpha(\cos(\pi) - \cos(0)) \\ &= -\alpha(-1 - 1) \\ &= 2\alpha = 1 \\ \implies \alpha &= \frac{1}{2} \\ \implies f(x) &= \frac{1}{2} \sin(x) \end{aligned}$$

$$\begin{aligned} F(x) &= \int_0^x \frac{1}{2} \sin(x)dx \\ &= -\frac{1}{2} \cos(x)|_0^x \\ &= -\frac{1}{2}(\cos(x) - \cos(0)) \\ &= -\frac{1}{2}(\cos(x) - 1) \\ &= \frac{1}{2}(1 - \cos(x)) = v \\ \implies x &= \arccos(1 - 2v) \end{aligned}$$

(5d) (4 points) Solid Angle

As a vanguard for the attack against Earth, the alien overlord Giygas sends a satellite to our solar system. The satellite uses a flat, disk-shaped solar sail that is 10 meters in diameter.

When Ness, an inhabitant of earth, saw the satellite yesterday, the satellite's sail was facing directly towards earth, and was 400 million kilometers away. Today, Ness observes that the satellite is 1600 million kilometers away. Ness' science-y friend Jeff estimates that today, the sail subtends a solid angle of only $\frac{1}{32}$ compared to yesterday. What angle away from the earth is the satellite's sail pointing today? Briefly explain your answer.

UPLOAD PHOTO **On a sheet of your own paper, write your answer. Take a photo of your page and upload to GRADESCOPE EXAM 1 PART B.**

Solution: The solid angle formula is $\frac{A}{r^2}$. The radius of the satellite is 4 times as large as before, meaning the denominator will be 16 times as large. Therefore, in order for the solid angle today to be 1/32 of the angle yesterday, the area must be half as large. By Lambert's law, the visible area is proportional to the cosine of the angle so $\cos(\theta) = 0.5$, so $\theta = 60°$