Ray Tracing and Radiometry 6

CS 184: Foundations of Computer Graphics

1 Ray-Triangle Intersection

Given a mesh representation of an object, we would like to render it onto a display. To do so, we need to know which parts of the object are visible, where to put shadows, how to apply the scene's lighting, and more. The simplest idea to handle these problems is to take a ray and intersect it with each triangle in the mesh.

Recall that a ray is defined by its origin $\mathbf{O}$ and a direction vector $\mathbf{D}$ and varies with "time" t for $0 \le t < \infty$.

$$\mathbf{r}(t) = \mathbf{O} + t\mathbf{D}. \tag{1}$$

Recall that a point within a triangle $\mathbf{P_0P_1P_2}$ can be represented as

$$\mathbf{P} = \alpha\mathbf{P}_0 + \beta\mathbf{P}_1 + \gamma\mathbf{P}_2, \tag{2}$$

where $\alpha + \beta + \gamma = 1$. Defining $b_1 = \beta$ and $b_2 = \gamma$, we obtain $\alpha = 1 - b_1 - b_2$. Thus, we can rewrite the point $\mathbf{P}$ in barycentric coordinates as:

$$\mathbf{P} = (1 - b_1 - b_2)\mathbf{P}_0 + b_1\mathbf{P}_1 + b_2\mathbf{P}_2. \tag{3}$$

1. Let's solve for the intersection of a ray and a triangle. Specifically, if we arrange the unknowns t, b_1 and b_2 into a column vector $\mathbf{x} = [t, b_1, b_2]^T$, can you get a matrix $\mathbf{M}$ and a column vector $\mathbf{b}$ so that $\mathbf{Mx} = \mathbf{b}$?

Solution:

Since the intersection is both along the ray and on the triangle, we have

$$\begin{aligned} \mathbf{P} &= (1 - b_1 - b_2)\mathbf{P}_0 + b_1\mathbf{P}_1 + b_2\mathbf{P}_2 \\ &= \mathbf{P}_0 - b_1\mathbf{P}_0 - b_2\mathbf{P}_0 + b_1\mathbf{P}_1 + b_2\mathbf{P}_2 && \text{(Expanding terms)} \\ \mathbf{O} + t\mathbf{D} &= \mathbf{P}_0 + b_1(\mathbf{P}_1 - \mathbf{P}_0) + b_2(\mathbf{P}_2 - \mathbf{P}_0) && \text{(Plugging in } \mathbf{P} = \mathbf{O} + t\mathbf{D}) \end{aligned} \tag{4}$$

Thus,

$$\mathbf{O} - \mathbf{P_0} = -t\mathbf{D} + b_1(\mathbf{P_1} - \mathbf{P_0}) + b_2(\mathbf{P_2} - \mathbf{P_0}). \tag{5}$$

Writing it in matrix form, we have

$$\begin{bmatrix} -\mathbf{D} & \mathbf{P_1} - \mathbf{P_0} & \mathbf{P_2} - \mathbf{P_0} \end{bmatrix} \begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \mathbf{O} - \mathbf{P_0} \tag{6}$$

So, we set $\mathbf{M} = [-\mathbf{D}, \mathbf{P_1} - \mathbf{P_0}, \mathbf{P_2} - \mathbf{P_0}]$, and $\mathbf{b} = \mathbf{O} - \mathbf{P_0}$.

2. Now let's derive the **Möller-Trumbore algorithm**!

$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \frac{1}{\mathbf{S_1} \cdot \mathbf{E_1}} \begin{bmatrix} \mathbf{S_2} \cdot \mathbf{E_2} \\ \mathbf{S_1} \cdot \mathbf{S} \\ \mathbf{S_2} \cdot \mathbf{D} \end{bmatrix} \tag{7}$$

where $\mathbf{E_1} = \mathbf{P_1} - \mathbf{P_0}$, $\mathbf{E_2} = \mathbf{P_2} - \mathbf{P_0}$, $\mathbf{S} = \mathbf{O} - \mathbf{P_0}$, $\mathbf{S_1} = \mathbf{D} \times \mathbf{E_2}$, $\mathbf{S_2} = \mathbf{S} \times \mathbf{E_1}$.

Hint 1: (Cramer's rule) Linear equations $\mathbf{Mx} = \mathbf{b}$ can be simply solved using determinants of matrices as:

$$\mathbf{x} = \frac{1}{|\mathbf{M}|} \begin{bmatrix} |\mathbf{M_1}| \\ |\mathbf{M_2}| \\ |\mathbf{M_3}| \end{bmatrix}, \tag{8}$$

where $\mathbf{M_i}$ is the matrix $\mathbf{M}$ with its i-th column replaced by $\mathbf{b}$.

Hint 2: Suppose $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are column vectors, the determinant of the 3×3 matrix $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$ satisfy:

$$|\mathbf{A}, \mathbf{B}, \mathbf{C}| = -(\mathbf{A} \times \mathbf{C}) \cdot \mathbf{B} = -(\mathbf{C} \times \mathbf{B}) \cdot \mathbf{A} = -(\mathbf{B} \times \mathbf{A}) \cdot \mathbf{C}. \tag{9}$$

Solution: Applying Cramer's rule, we immediately have

$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \frac{1}{|\mathbf{M}|} \begin{bmatrix} |\mathbf{M_1}| \\ |\mathbf{M_2}| \\ |\mathbf{M_3}| \end{bmatrix} \tag{10}$$

$$= \frac{1}{|\ -\mathbf{D} \quad \mathbf{P_1} - \mathbf{P_0} \quad \mathbf{P_2} - \mathbf{P_0}\ |} \begin{bmatrix} |\ \mathbf{O} - \mathbf{P_0} \quad \mathbf{P_1} - \mathbf{P_0} \quad \mathbf{P_2} - \mathbf{P_0}\ | \\ |\ -\mathbf{D} \quad \mathbf{O} - \mathbf{P_0} \quad \mathbf{P_2} - \mathbf{P_0}\ | \\ |\ -\mathbf{D} \quad \mathbf{P_1} - \mathbf{P_0} \quad \mathbf{O} - \mathbf{P_0}\ | \end{bmatrix} \tag{11}$$

$$= \frac{1}{|\ -\mathbf{D} \quad \mathbf{E_1} \quad \mathbf{E_2}\ |} \begin{bmatrix} |\ \mathbf{S} \quad \mathbf{E_1} \quad \mathbf{E_2}\ | \\ |\ -\mathbf{D} \quad \mathbf{S} \quad \mathbf{E_2}\ | \\ |\ -\mathbf{D} \quad \mathbf{E_1} \quad \mathbf{S}\ | \end{bmatrix} \tag{12}$$

Now let's take a look at these determinants, we have

$$|\ -\mathbf{D} \quad \mathbf{E_1} \quad \mathbf{E_2}\ | = -\left(-\mathbf{D} \times \mathbf{E_2}\right) \cdot \mathbf{E_1} = \mathbf{S_1} \cdot \mathbf{E_1}, \tag{13}$$

$$|\ \mathbf{S} \quad \mathbf{E_1} \quad \mathbf{E_2}\ | = -\left(\mathbf{E_1} \times \mathbf{S}\right) \cdot \mathbf{E_2} = \mathbf{S_2} \cdot \mathbf{E_2}, \tag{14}$$

$$|\ -\mathbf{D} \quad \mathbf{S} \quad \mathbf{E_2}\ | = -\left(-\mathbf{D} \times \mathbf{E_2}\right) \cdot \mathbf{S} = \mathbf{S_1} \cdot \mathbf{S}, \tag{15}$$

$$|\ -\mathbf{D} \quad \mathbf{E_1} \quad \mathbf{S}\ | = -\left(\mathbf{S} \times \mathbf{E_1}\right) \cdot -\mathbf{D} = \mathbf{S_2} \cdot \mathbf{D}. \tag{16}$$

Plugging these back in, we have the Möller-Trumbore algorithm!

$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \frac{1}{\mathbf{S_1} \cdot \mathbf{E_1}} \begin{bmatrix} \mathbf{S_2} \cdot \mathbf{E_2} \\ \mathbf{S_1} \cdot \mathbf{S} \\ \mathbf{S_2} \cdot \mathbf{D} \end{bmatrix} \tag{17}$$

3. Once you've solved for t, b_1 and b_2, what conditions must be satisfied so that you have a valid ray-triangle intersection?

> **Solution:** $t \geq 0$, $0 \leq b_1 \leq 1$, $0 \leq b_2 \leq 1$, $0 \leq 1 - b_1 - b_2 \leq 1$.

4. What does it mean when $\mathbf{S}_1 \cdot \mathbf{E}_1 = 0$ in the context of the Möller–Trumbore algorithm?

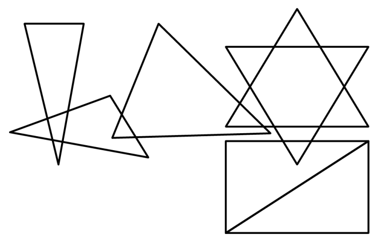
> **Solution:** If $\mathbf{S}_1 \cdot \mathbf{E}_1 = 0$, it means $(\mathbf{D} \times \mathbf{E}_2)$ is perpendicular to $(\mathbf{P}_1 - \mathbf{P}_0)$, which geometrically indicates that the ray $\mathbf{D}$ is parallel to the plane of the triangle (or that the triangle is degenerate). Hence, there is no unique intersection solution in this case.

2 Building Beautifully Balanced Bounding Boxes

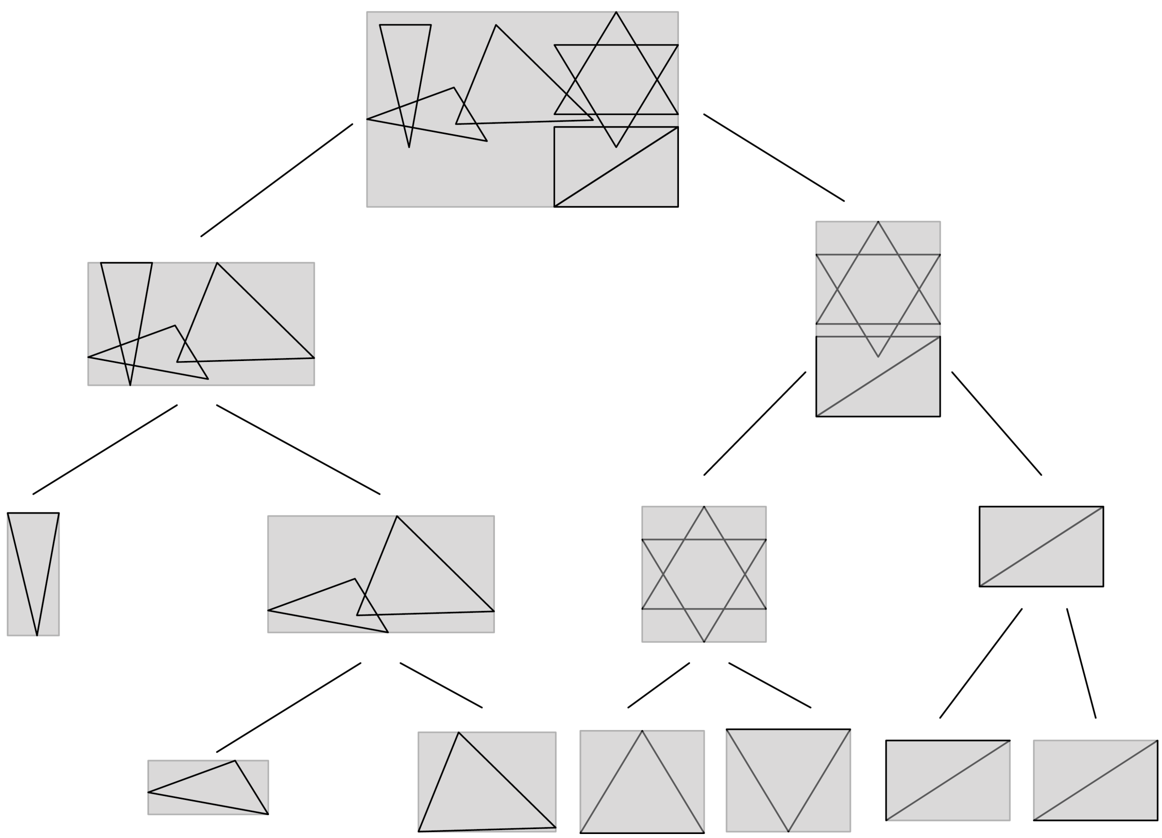
Bounding volumes are used to accelerate ray-triangle intersection tests. If a ray doesn't intersect a bounding volume, we can conclude that it doesn't intersect any triangles contained within.

A bounding volume hierarchy is a tree of bounding volumes. The bounding volume at a node encloses the bounding volumes of its children. The ray tracing algorithm traverses this hierarchy to determine if the ray intersects an object.

1. Given a set of planar triangles, build a BVH following these rules:
    - Always pick the longest axis to divide.
    - Use center of mass of triangles to decide their relative positions.
    - Keep the BVH balanced. Try to ensure the same number of triangles for children nodes.



Solution:



2. A box has corners $(-2, -2, -2)$ and $(2, 2, 2)$. A ray has origin $(-3, 4, 5)$ and direction $(1, -1, -2)$. Compute the value(s) of t at which the ray insersects a yz-slab of the box.

Solution: Intersecting the yz-slabs, we have

$$t_{x,1} = (-2 - (-3))/1 = 1, \tag{18}$$
$$t_{x,2} = (2 - (-3))/1 = 5. \tag{19}$$

3. Compute the value(s) of t at which the ray insersects a xz-slab of the box.

Solution: Intersecting the xz-slabs, we have

$$t_{y,1} = (-2 - 4)/(-1) = 6, \tag{20}$$
$$t_{y,2} = (2 - 4)/(-1) = 2. \tag{21}$$

4. Compute the value(s) of t at which the ray insersects a xy-slab of the box.

Solution: Intersecting the xy-slabs, we have

$$t_{z,1} = (-2 - 5)/(-2) = 3.5, \tag{22}$$
$$t_{z,2} = (2 - 5/(-2) = 1.5. \tag{23}$$

5. Compute the entry and exit *points* of the ray for the given box.

Solution:

So we have

$$t_{x,\min} = 1,\ t_{x,max} = 5, \tag{24}$$
$$t_{y,\min} = 2,\ t_{y,max} = 6, \tag{25}$$
$$t_{z,\min} = 1.5,\ t_{z,max} = 3.5. \tag{26}$$

Then

$$t_{\min} = \max\{t_{x,\min}, t_{y,\min}, t_{z,\min}\} = 2, \tag{27}$$
$$t_{\max} = \min\{t_{x,\max}, t_{y,\max}, t_{z,\max}\} = 3.5. \tag{28}$$

Since $t_{\min} <= t_{\max}$ and $t_{\min} > 0$ and $t_{\max} > 0$, we have two intersections. The entry and exit points are at

$$(-3, 4, 5) + t_{\min}(1, -1, -2) = (-1, 2, 1) \tag{29}$$

and

$$(-3, 4, 5) + t_{\max}(1, -1, -2) = (0.5, 0.5, -2). \tag{30}$$

3 Lumens and Joules and Nits — Oh My!

1. Fill in the table below. In the right-most column, R denotes the distance to the light source.

Symbol/Name	Radiometry Unit/Name	Photometry Unit/Name	Effect of Increased R
Q: Energy	Radiant Energy Joules (W·s)	Luminous Energy Lumen·sec	↑ (=) ↓
Φ: Flux (Power)			↑ = ↓
I: Angular Flux Density			↑ = ↓
E: Spatial Flux Density			↑ = ↓
L: Spatio-Angular Flux Density			↑ = ↓

Solution:

Symbol/Name	Radiometry Unit/Name	Photometry Unit/Name	Effect of Increased R
Q: Energy	Radiant Energy Joules (W·s)	Luminous Energy Lumen·sec	=
Φ: Flux (Power)	Radiant Power W	Luminous Power Candela·sr	=
I: Angular Flux Density	Radiant Intensity W/sr	Luminous Intensity Candela = Lumen/sr	=
E: Spatial Flux Density	Irradiance (in), Radiosity (out) W/m^2	Illuminance (in), Luminosity (out) Lux = $Lumen/m^2$	↓
L: Spatio-Angular Flux Density	Radiance $W/m^2/sr$	Luminance Nit = $Candela/m^2$	=

As justification for the rightmost column:

- Q and Φ are properties of the light source itself, and do not depend on R.
- I is dependent on the light source and solid angle, but not the distance.
- $E = \frac{\Phi}{A}\cos\theta$. Since Φ and θ do not change with R.
- L is invariant along a ray in a vacuum, i.e. there is no dependence on R.

2. For these questions, feel free to use infinitesimal quantities, e.g. dA, to represent a small area around a

point, and θ to represent the angle between the light direction and surface normal.

(a) What is Φ in terms of Q and time t?

(b) What is $I(p, \omega)$ at a point p in terms of flux and solid angle?

(c) What is irradiance $E(p)$ at a point p in terms of flux and area?

(d) What is surface radiance $L(p, \omega)$ at a point p in a direction ω in terms of flux, area, and solid angle?

(e) How can surface radiance $L(p, \omega)$ be expressed in terms of irradiance $E(p)$?

(f) How can surface radiance $L(p, \omega)$ be expressed in terms of intensity $I(p, \omega)$?

Solution:

(a) $\Phi = \dfrac{dQ}{dt}$

(b) $I(p, \omega) = \dfrac{d\Phi}{d\omega}$

(c) $E(p) = \dfrac{d\Phi(p)}{dA}$

(d) $L(p, \omega) = \dfrac{d^2\Phi(p, \omega)}{d\omega\, dA \cos\theta}$

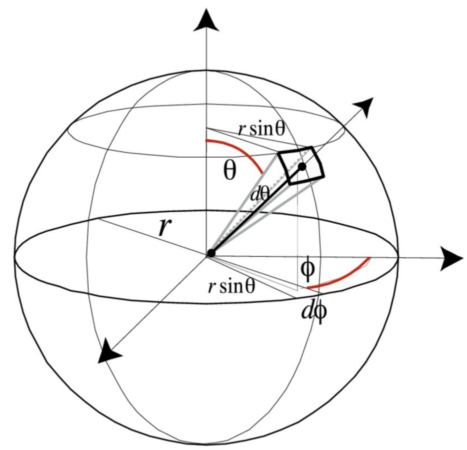
(e) $L(p, \omega) = \dfrac{dE(p)}{d\omega \cos\theta}$

(f) $L(p, \omega) = \dfrac{dI(p, \omega)}{dA \cos\theta}$

4 Shedding Some Light

1. Suppose we use (θ, ϕ)-parameterization of directions. Recall that the solid angle represents the ratio of the subtended area on a sphere to the radius squared, $\Omega = \frac{A}{r^2}$. Estimate the solid angle subtended by a patch that covers $\theta \in [\pi/6 - \pi/12, \pi/6 + \pi/12]$ and $\phi \in [\pi/5 - \pi/24, \pi/5 + \pi/24]$?

   (Hint: you may assume that the patch is small enough. Recall or derive the differential solid angle $d\omega$, then use the values given.)



Solution: Under (θ, ϕ)-parameterization, we know that the differential solid angle is $\mathrm{d}\omega = \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi$. When a patch is small enough, we can use this to approximate its solid angle as

$$\Delta\omega = \sin\theta \, \Delta\theta\Delta\phi, \tag{31}$$

where ϕ is the azimuth angle, and θ the elevation angle, at the center of the patch.

In our specific case, the solid angle subtended by the patch is now

$$\Delta\omega \approx \sin\frac{\pi}{6} \cdot \left[\frac{\pi}{12} - \left(-\frac{\pi}{12}\right)\right] \cdot \left[\frac{\pi}{24} - \left(-\frac{\pi}{24}\right)\right] \tag{32}$$

$$= \frac{\pi^2}{144}. \tag{33}$$

2. A point light at position $(6, 0, 8)$ (in meters) with radiant flux (power) of 100 watts directs all its light uniformly into the hemisphere directly below it. Some of this light falls on a flat, tilted surface passing through the origin, with surface normal vector $(1, 1, 1)$.

   What is the irradiance at the origin? Show your work and use the correct units.

Solution: First, we have

$$r = \sqrt{6^2 + 8^2} = 10.$$

The hemisphere below the light has solid angle equal to 2π. The cosine angle between the surface normal and the light position is given by the dot product of their normalized vectors:

$$\cos(\theta) = \frac{1}{10}(6, 0, 8) \cdot \frac{1}{\sqrt{3}}(1, 1, 1) = \frac{14}{10\sqrt{3}}.$$

Finally,

$$E = \frac{\Phi}{2\pi r^2}\cos(\theta) = \frac{100}{200\pi}\frac{14}{10\sqrt{3}}\ \mathrm{W/m^2} = \frac{7}{10\sqrt{3}\pi}\ \mathrm{W/m^2}.$$