5. (Total : 15 points) Monte Carlo Integration

(5a) Monte Carlo Integration with a Non-Uniform Distribution

Consider the following Monte Carlo integration problem, where we wish to integrate a function $F(x)$ over the interval $[-\pi/2, \pi/2]$, which is known to have a humped shape near the origin. The goal is to compute:

$$I = \int_{-\pi/2}^{\pi/2} F(x)\, dx$$

We are given a proposal distribution $G(x) = \cos(x)$ and will perform importance sampling proportional to this function to estimate the integral of $F(x)$.

5a.i. [9 points] Sampling Proportional to $G(x)$: First, we need to sample from $G(x)$. Follow the inversion method described in class: normalize $G(x)$ to get the PDF $p_G(x)$, compute its cumulative distribution function (CDF), $P_G(x)$, and invert the CDF to get a function that, given a uniform random variable $u \in [0, 1]$, $P_G^{-1}(u)$ is a random sample in the domain $[-\pi/2, \pi/2]$ that is chosen randomly with a probability that is proportion to $G(x)$. Derive expressions for $p_G(x)$, $P_G(x)$ and $P_G^{-1}(u)$.

Solution:

1. Normalize $G(x) = \cos(x)$ to get the PDF $p_G(x)$. The normalization constant Z is given by the integral of $G(x)$ over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

$$Z = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\, dx = [\sin(x)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - (-1) = 2$$

Thus, the normalized PDF is:

$$p_G(x) = \frac{1}{2}\cos(x)$$

2. Compute the CDF $P_G(x)$ by integrating $p_G(x)$:

$$P_G(x) = \int_{-\frac{\pi}{2}}^{x} \frac{1}{2}\cos(t)\, dt = \frac{1}{2}(\sin(x) + 1) \text{for} \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

3. Invert the CDF to get the inverse function $P_G^{-1}(u)$. Given $u \in [0, 1]$, solve for x:

$$u = P_G(x) = \frac{1}{2}(\sin(x) + 1)$$

Solving for x, we get:

$$x = \sin^{-1}(2u - 1)$$

Thus, the inverse CDF is:

$$P_G^{-1}(u) = \sin^{-1}(2u - 1)$$

5a.ii. [6 points] Monte Carlo Estimator for $I = \int_{-\pi/2}^{\pi/2} F(x)dx$:
Assume that we have a sequence of uniformly random samples X_i drawn from $[-\pi/2, \pi/2]$. As discussed in class, a basic Monte Carlo estimate for I is

$$\hat{I}_{\text{basic}} = \frac{\pi}{N} \sum_{i=1}^{N} F(X_i).$$

Now, use your results from the previous part to instead derive an expression for an importance-sampled Monte Carlo estimator. Use the sequence of randomly samples X_i. Hint: transform these uniformly random samples into random samples drawn from the PDF $p_G(x)$.

Solution:
The Monte Carlo estimator for the integral of $F(x)$ using importance sampling is given by:

$$\hat{I}_{\text{importance}} = \frac{1}{N} \sum_{i=1}^{N} \frac{F(P_G^{-1}(X_i/\pi + 1/2))}{p_G(P_G^{-1}(X_i/\pi + 1/2))} = \frac{1}{N} \sum_{i=1}^{N} \frac{F(\sin^{-1}(2X_i/\pi))}{\frac{1}{2}\cos(\sin^{-1}(2X_i/\pi))}$$

8. (Total : 12 points) Monte Carlo Integration

For each row in the table below, you are given a definite integral to evaluate with Monte Carlo integration, the probability density function (PDF) to draw random samples from and the number of samples to take. Your job is to write down an equation for the corresponding unbiased Monte Carlo estimator in the last column. The first two rows are completed as examples. In this question, H^2 denotes the hemisphere. Simplify your answer as much as possible, and show your work for partial credit.

(8a) (12 points) Monte Carlo Estimators

	Definite Integral	**Number of Samples**	**Random Sampling**	**Estimator**
	$\int_a^b f(x)\,dx$	1	Random X drawn uniformly at random from $[a,b]$	$(b-a)f(X)$
	$\int_a^b f(x)\,dx$	N	Random X_i drawn uniformly at random from $[a,b]$	$\frac{b-a}{N}\sum_{i=1}^{N} f(X_i)$
(a)	$\int_0^\pi f(x)\,dx$	1	Random X drawn from $[0,\pi]$ with probability proportional to $\sin x$.	
(b)	$\int_{H^2} L(\omega)\cos\theta\,d\omega$	N	Random ω_i drawn from H^2 with probability proportional to $\cos(\theta_i)$*	
(c)	$\int_a^b\int_c^d h(x,y)\,dx\,dy$	N	Random (X_i, Y_i) drawn within the domain of integration with probability proportional to function $g(X_i, Y_i)$.	

Solution:

(a) $\frac{2f(x)}{\sin x}$

(b) $\frac{\pi}{N}\sum_{i=1}^{N} L(\omega_i)$

(c) $\frac{\int_a^b\int_c^d g(x,y)\,dx\,dy}{N}\sum_{i=1}^{N}\frac{h(X_i,Y_i)}{g(X_i,Y_i)}$