Monte Carlo and Global Illumination 7

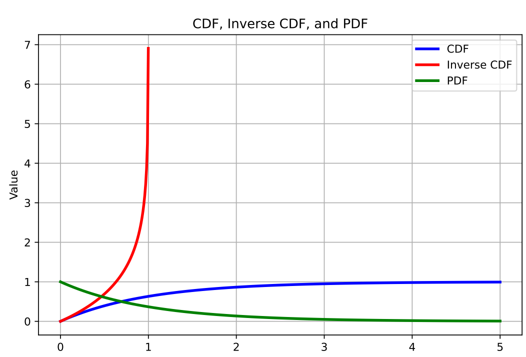
CS 184: Foundations of Computer Graphics

1 Inversion Method

Given a uniform random variable U in the interval $[0, 1]$, we can generate a random variable from any other one dimensional distribution using its cumulative distribution function: $X = F^{-1}(U)$. This is how we choose sample points when running a ray tracing algorithm.

1. What function of U will return a sample from the exponential distribution (with parameter λ)? This distribution has density $p_\lambda(x) = \lambda e^{-\lambda x}$, and is defined for $x \geq 0$.

2. For $\lambda = 1$, the plot below shows the CDF, Inverse CDF, and PDF. If we sample $U = 0.75$, determine the corresponding sampled value using the inverse method. Mark the corresponding point on the CDF curve in the graph.



3. What does the x-axis represent for the blue CDF curve?

2 Unbiased Estimators

1. Let $f : [-2, 2] \times [-2, 2] \rightarrow \mathbb{R}$ be a function. You have a machine that allows you to sample $2n$ values independently and uniformly from the interval $[-2, 2]$. Construct an unbiased Monte Carlo estimator for

$$F = \int_{-2}^{2} \int_{-2}^{2} f(x, y) dx dy.$$

3 Boxing

Preston owns an unpolished wooden cube that scatters light diffusely. In other words, Preston's cube is a *Lambertian* surface with a bidirectional reflectance distribution function (BRDF) of $f_r(\mathrm{p}, \omega_i, \omega_o) = \frac{\rho}{\pi}$.

Recall that reflected radiance, $L_r(\mathrm{p}, \omega_o)$, is equal to $L_e(\mathrm{p}, \omega_o) + \int_{H^2} f_r(\mathrm{p}, \omega_i, \omega_o) L_i(\mathrm{p}, \omega_i) \cos\theta_i \mathrm{d}\omega_i$.

1. What is the emitted radiance, L_e, of the cube?

2. L_i is the incoming radiance. Assume L_i is uniform over the hemisphere, H^2, that surrounds point p, located on the top face of the cube. Solve for $L_r(\mathrm{p}, \omega_o)$.

   (Hint: re-parameterize your integral to be in terms of θ and ϕ, instead of ω.)

3. How does the reflected radiance depend on ω_o? How does the reflected radiance depend on ρ?

4. Preston adjusts his light source so that incoming radiance is no longer uniform. Now, he wants to use a Monte Carlo estimator to approximate $L_r(\mathrm{p}, \omega_i)$. He samples n directions over the hemisphere from $p(\omega)$, a distribution that is proportional to the BRDF.

   Construct the Monte Carlo estimator.

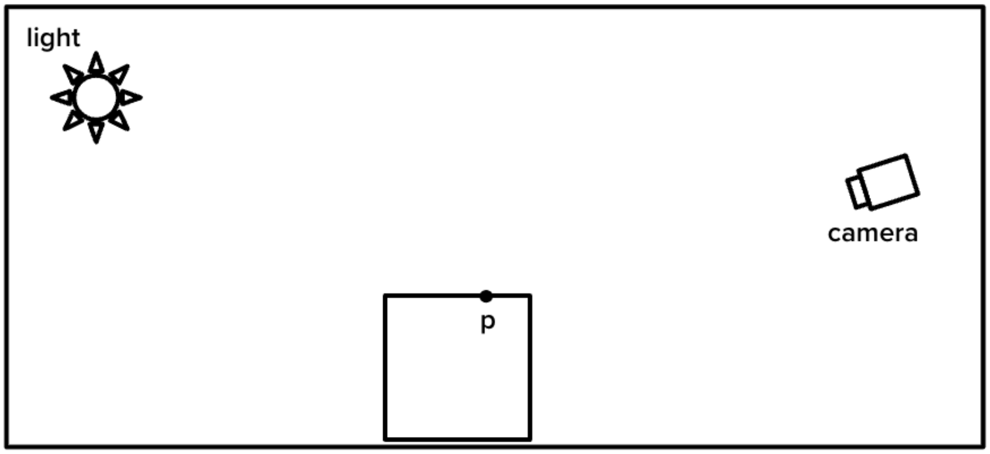
5. In practice, cosine-weighted hemisphere sampling results in better convergence. When $p(\omega) = \frac{\cos\theta}{\pi}$, what is the Monte Carlo estimator for n samples?

6. Conceptually, why does cosine-weighted hemisphere sampling outperform uniform sampling over a hemisphere?

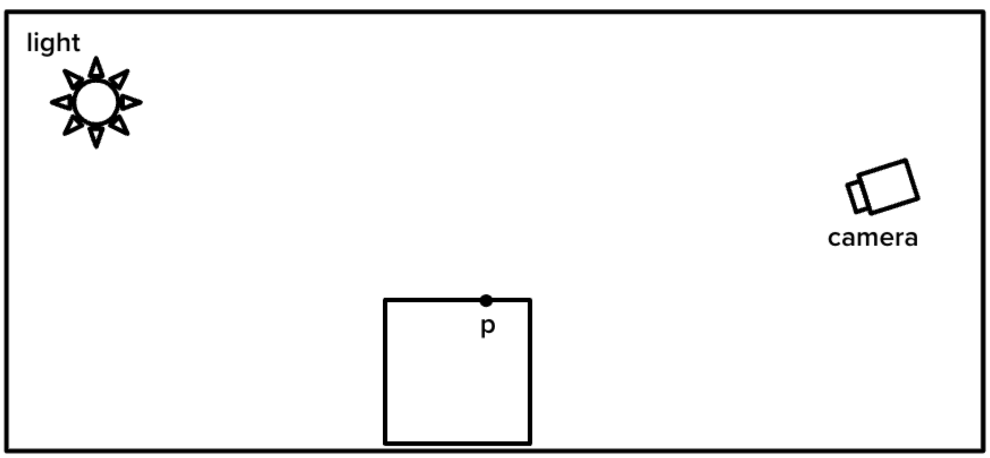
4 Tracing Outside of the Box

The Monte Carlo estimator derived earlier is great for direct illumination from the light source (1-bounce). For *indirect* illumination, it's not enough to sample directions — we need to sample *paths*.

1. Again, Preston's cube has a BRDF of $f_r(\mathrm{p}, \omega_i, \omega_o) = \frac{\rho}{\pi}$. Perform path-tracing by drawing multiple 2-bounce paths for point p. Label ω_o. Light can scatter off walls. Assume the walls are also Lambertian.



2. Draw multiple 3-bounce paths. Label ω_o.



3. Suppose we trace n non-occluded 2-bounce paths from camera, to p, to p_j, to the light source, where $j = 1, 2, \ldots, n$. Express the Monte Carlo estimator for outgoing radiance at p in terms of outgoing radiance at p_j. Assume incoming directions to p, $\omega_{i,j}$, are sampled from $p(\omega)$.

5 Ray or Nay?

With Russian Roulette, we randomly terminate each ray with probability $1 - p_{\text{rr}}$ (equivalently, continue with probability p_{rr}). Additionally, if the original estimator was X, we update it to

$$X_{\text{rr}} = \begin{cases} X/p_{\text{rr}} & \text{with probability } p_{\text{rr}} \\ 0 & \text{else,} \end{cases}$$

so that $\mathbb{E}[X_{\text{rr}}] = \mathbb{E}[X]$.

For ray tracing, X might be the estimated incoming radiance for a given (p, ω_i).

1. Suppose $p_{\text{rr}} = \frac{4}{5}$, and let $N \geq 0$ be a random variable representing the number of bounces.
   (i) What is $\mathbb{P}(N = 0)$?

   (ii) What is $\mathbb{P}(N = 2)$?

   (iii) What is the expected value of N? (Hint: Recall that if $Z \sim \text{Geometric}(p)$, then $\mathbb{P}(Z = z) = (1-p)^{z-1}p$ for $z \geq 1$, and $\mathbb{E}[Z] = \frac{1}{p}$.)

2. If $p_{\text{rr}} = \frac{1}{5}$, then is the expected value of N?

3. In general, how does increasing p_{rr} affect the expected number of bounces $\mathbb{E}[N]$?