Discussion 11

Simulation, Kinematics, Animation

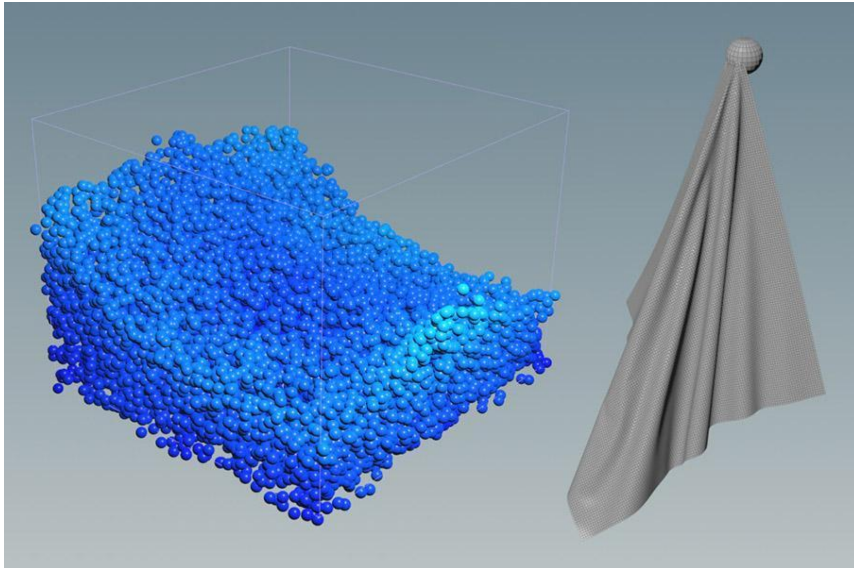
Computer Graphics and Imaging
UC Berkeley CS 184

Week 7 Announcements

- Exam this Thursday (2 hours long)
  - Reminder: Coding question clobber
- HW 3 Finished, HW 2 and 3 Grades out soon
- Final Project Milestone Due Tonight
  - No slip days
- Only 3 attendance credits possible this week
  - Next week's attendance is extra credit!

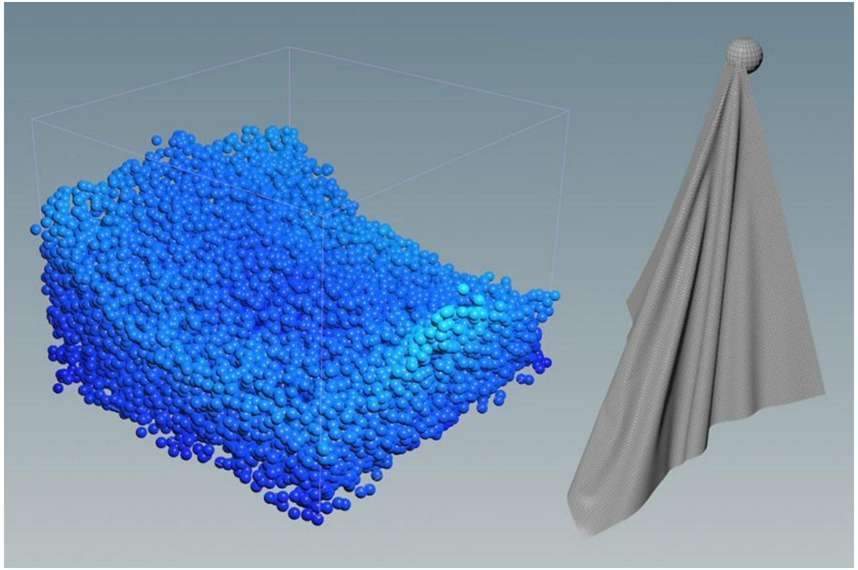
Physical simulation

- We have a physical basis for the motion of single particles: gravity, friction, springs, etc.



Physical simulation

- We have a physical basis for the motion of single particles: gravity, friction, springs, etc.
- How can we simulate this behavior on our laptop within reasonable compute limits?



Euler's method

- Problem: consider a system defined by a differential equation

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, t)$$

- To simulate the system, we want to estimate $\boldsymbol{x}^t$.

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$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \, \dot{\boldsymbol{x}}^t$$
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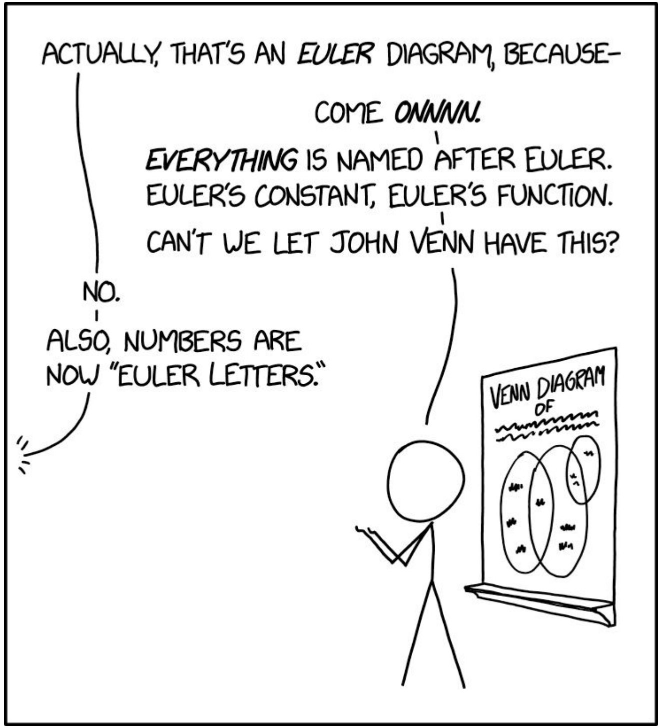


- https://en.wikipedia.org/wiki/List_of_topics_named_after_Leonhard_Euler

Euler's work touched upon so many fields that he is often the earliest written reference on a given matter. In an effort to avoid naming everything after Euler, some discoveries and theorems are attributed to the first person to have proved them *after* Euler.[1][2]

Explicit Euler's Method

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^{t} + \Delta t \dot{\boldsymbol{x}}^{t}$$

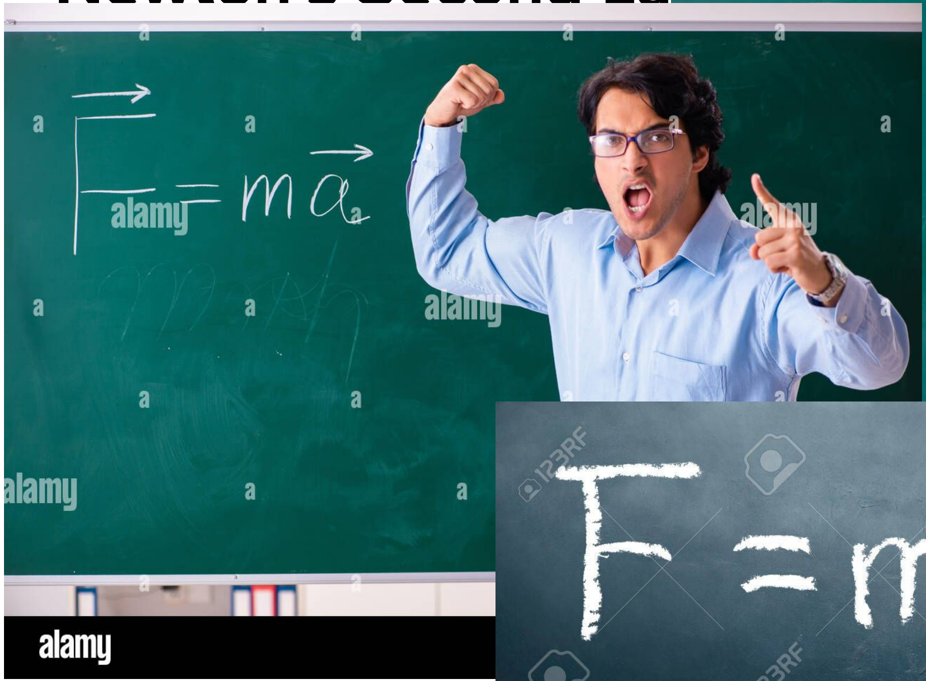
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^{t} + \Delta t \ddot{\boldsymbol{x}}^{t}$$

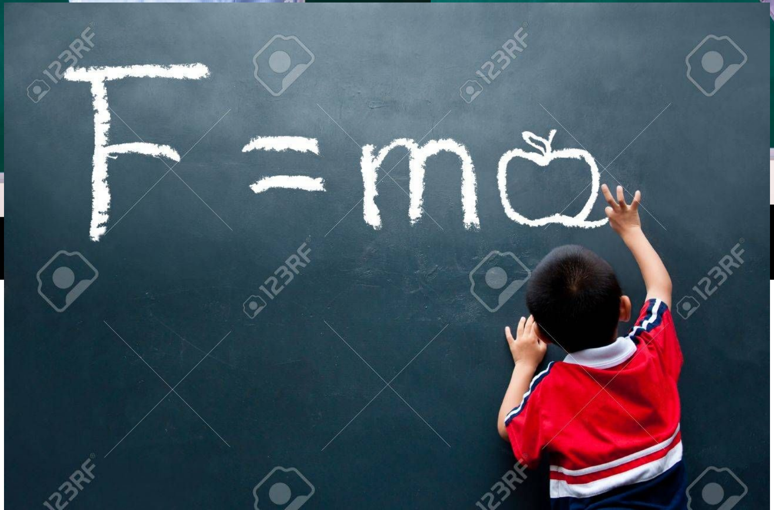
position

velocity

acceleration

Newton's Second La





Question 1

	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1$
$\theta^{\Delta t}$	0.1 rad	0.1 rad	0.1 rad
$\dot{\theta}^{\Delta t}$	−0.1 rad/s	−0.5 rad/s	−1 rad/s
$\theta^{2\Delta t}$	0.09 rad	−0.15 rad	−0.9 rad
$\dot{\theta}^{2\Delta t}$	−0.2 rad/s	−1.0 rad/s	−2.0 rad/s
$\theta^{3\Delta t}$	0.07 rad	−0.65 rad	−2.9 rad
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Not physically possible!
(pendulum cannot swing past $\theta^0 = 0.1$ rad)

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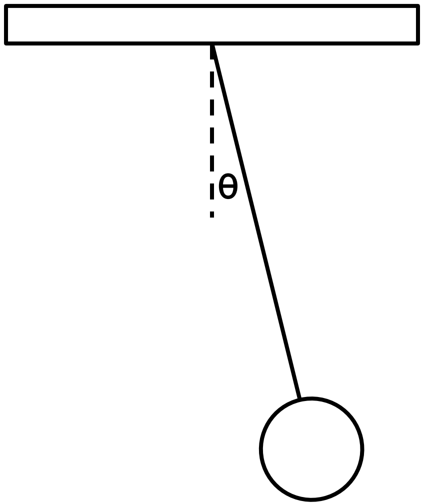
Not physically possible! (pendulum cannot swing past $\theta^0 = 0.1$ rad)

- **Problem with Euler's method:** Errors accumulate! Need very small step size (more steps) to achieve low approximation error.
- See parts 4 and 5 for more detail — turns out Euler's method is <u>never</u> numerically stable for this setting.

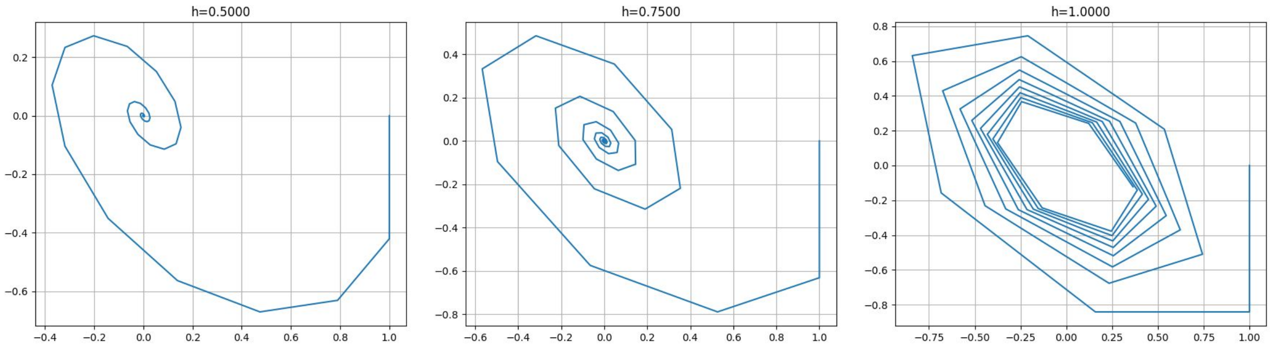

θ

Modifications to Euler’s Method

Modifications to Euler’s Method

Explicit Euler’s method can be unstable, since errors accumulate. Some modifications to explicit Euler’s method include:

1. Implicit Euler’s method (AKA backward Euler’s method).
2. Modified Euler’s method.
3. Verlet integration.



Modifications to Euler’s Method

1. Implicit Euler’s method (AKA backward Euler’s method).

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^{t+\Delta t}$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^{t+\Delta t}$$

1. Modified Euler’s method.

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \frac{\Delta t}{2}(\dot{\boldsymbol{x}}^t + \dot{\boldsymbol{x}}^{t+\Delta t})$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

1. Verlet integration.

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t + \frac{1}{2}(\Delta t)^2 \ddot{\boldsymbol{x}}^t$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \frac{\boldsymbol{x}^{t+\Delta t} - \boldsymbol{x}^t}{\Delta t}$$

Modifications to Euler’s Method

1. Implicit Euler’s method (AKA backward Euler’s method).
    - More stable than explicit Euler, better for stiff systems
    - Requires solving non-linear equations (and more complex)
    - Uses velocity at next time step
1. Modified Euler’s method.
    - More stable than explicit Euler, better for stiff systems
    - Average velocities at start and endpoint
1. Verlet integration.
    - Doesn’t require velocity (position-based)
    - Not physically based, dissipates energy (error)

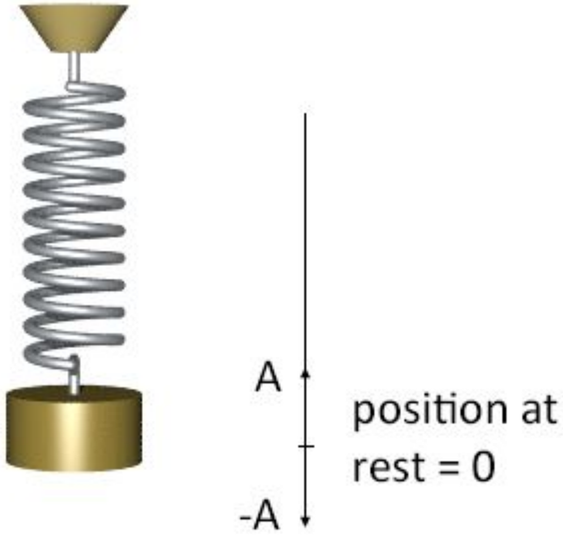
Hooke's Law

- Recall that Hooke's Law is used to calculate <u>the force acting on a particle</u> due to a spring.

$$\boldsymbol{f}_{a \to b} = k_s \frac{\boldsymbol{b} - \boldsymbol{a}}{||\boldsymbol{b} - \boldsymbol{a}||} (||\boldsymbol{b} - \boldsymbol{a}|| - l)$$

Rest length

$$\boldsymbol{f}_{b \to a} = -\boldsymbol{f}_{a \to b}$$



💡 Remember that acceleration is the double derivative! Therefore, acceleration = Force/mass

Question 2

Q2 Multitudes of Euler's Methods

We have a particle with mass 1 kg. It starts at position $\boldsymbol{x}^0 = (0\,\text{m}, 1\,\text{m})$ with an initial velocity $\dot{\boldsymbol{x}}^0 = (-\,\text{m/s}, 0\,\text{m/s})$ and no initial acceleration. The particle is at one end of a spring, whose other end is fixed at $(0\,\text{m}, 0\,\text{m})$. Its spring constant is $k = 1\,\text{N/m}$ and rest length is $1\,\text{m}$.

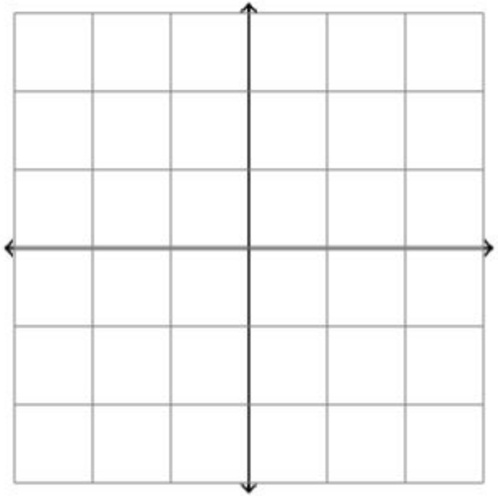
Recall the explicit Euler's method, which uses the following update rules

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

$\boldsymbol{x}^t, \dot{\boldsymbol{x}}^t, \ddot{\boldsymbol{x}}^t$ respectively denote the position, velocity, and acceleration at time t.

Q2.1 Multitudes of Euler's Methods

1. Calculate the particle's position at $t = 3$ using the explicit Euler's method with timestep $\Delta t = 1$. You might find it helpful to plot the particle on the provided grid.



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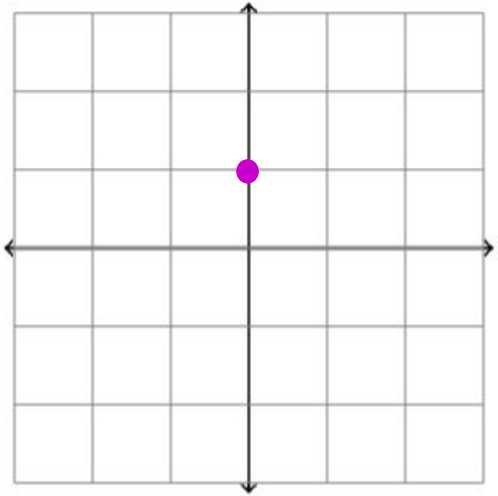

$\boldsymbol{x}^0 = (0, 1)$ $\quad \dot{\boldsymbol{x}}^0 = (-1, 0)$ $\quad \ddot{\boldsymbol{x}} = (0, 0)$

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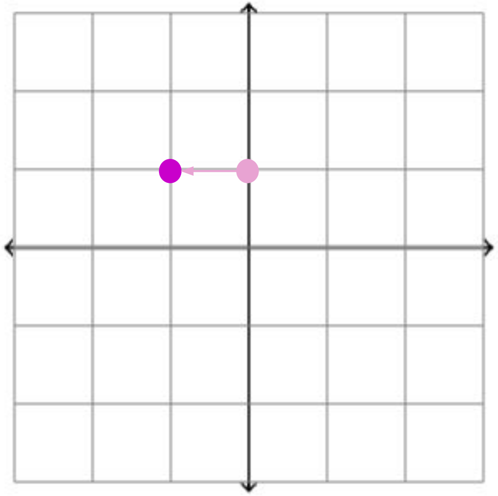


$$\boldsymbol{x}^0 = (0, 1) \qquad \dot{\boldsymbol{x}}^0 = (-1, 0) \qquad \ddot{\boldsymbol{x}} = (0, 0)$$

$$x^1 = x^0 + \Delta t\dot{x}^0 = (0, 1) + (-1, 0) = (-1, 1)$$

$$\dot{x}^1 = x^0 +^{\cdot} \Delta t\ddot{x}^0 = (-1, 0) + (0, 0) = (-1, 0)$$

$$\ddot{x}^1 = ?$$

Hint! 💡 F = ma so a = Force/mass

Modeling the Problem with a Spring System

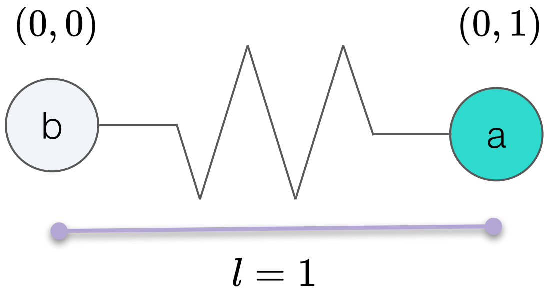
$$\boldsymbol{x}^0 = (0,1) \qquad \dot{\boldsymbol{x}}^0 = (-1,0) \qquad \ddot{\boldsymbol{x}} = (0,0)$$

$$x^1 = x^0 + \Delta t \dot{x}^0 = (-1,1)$$

$$\dot{x}^1 = \dot{x}^0 + \Delta t \ddot{x}^0 = (-1,0)$$

$$\ddot{x}^1 = \frac{k_s}{m}\frac{b-a}{\|b-a\|}(\|b-a\| - l)$$

$$= \frac{1}{1}\frac{(1,-1)}{\sqrt{2}}(\sqrt{2}-1) = (1-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1)$$



$$F = m * a$$

$$a = \frac{F}{m}$$

$$a = \frac{k_s}{m}\frac{b-a}{\|b-a\|}(\|b-a\| - l)$$

At t=1, we want to solve for position and velocity using Euler's, and acceleration via the spring equation

Q2.1 Multitudes of Euler's Methods

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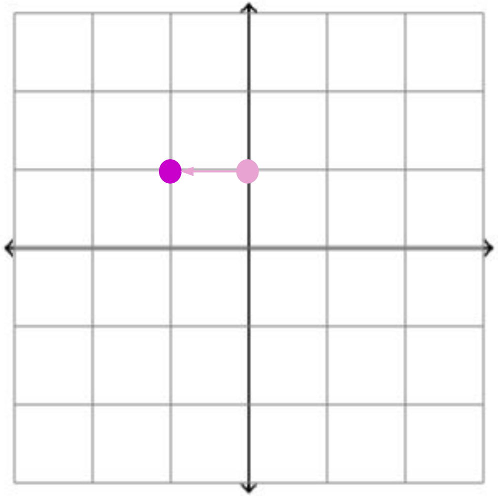

$$\boldsymbol{x}^0 = (0, 1) \qquad \dot{\boldsymbol{x}}^0 = (-1, 0) \qquad \ddot{\boldsymbol{x}} = (0, 0)$$

$$\boldsymbol{x}^1 = (-1, 1) \qquad \dot{\boldsymbol{x}}^1 = (-1, 0)$$

$$\ddot{\boldsymbol{x}}^1 = \frac{F_s}{m} \cdot \frac{(0,0) - (-1,1)}{||(0,0) - (-1,1)||} = \frac{1 \cdot (\sqrt{2} - 1)}{1} \cdot \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \left(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right)$$

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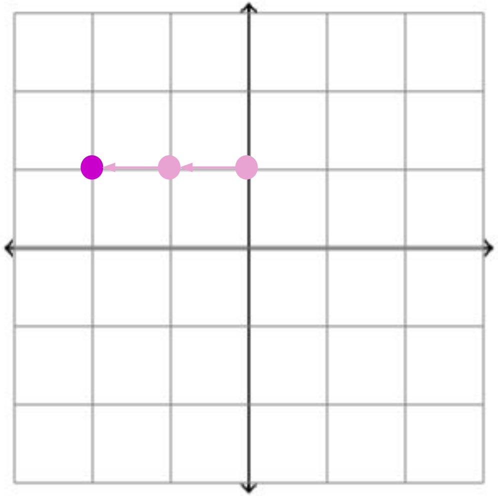

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$$\boldsymbol{x}^1 = (-1,1) \qquad \dot{\boldsymbol{x}}^1 = (-1,0)$$

$$\ddot{\boldsymbol{x}}^1 = \frac{F_s}{m} \cdot \frac{(0,0)-(-1,1)}{||(0,0)-(-1,1)||} = \frac{1 \cdot (\sqrt{2}-1)}{1} \cdot \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \left(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right)$$

$$\boldsymbol{x}^2 = (-2,1) \qquad \dot{\boldsymbol{x}}^2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right) \qquad \ddot{\boldsymbol{x}}^2 = \text{unneeded}$$

Why unneeded? Since we only care about x^3 (which doesn't depend on $\ddot{x}^2$).

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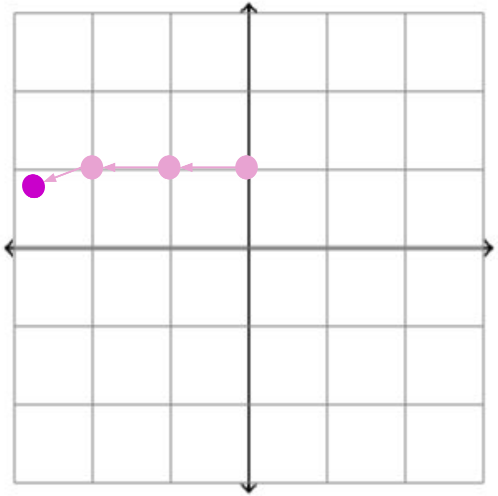

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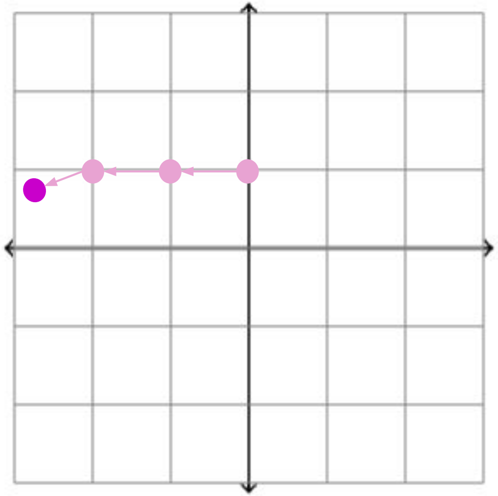
Q2.1 Multitudes of Euler's Methods

1. Calculate the particle's position at $t = 3$ using the explicit Euler's method with timestep $\Delta t = 1$. You might find it helpful to plot the particle on the provided grid.



Solution:

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Q2.2 Multitudes of Euler's Methods

2. For implicit Euler's method, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^{t+\Delta t}$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^{t+\Delta t}$$

   Write the update step for calculating the particle's position at $t = 1$ using implicit Euler's method with timestep $\Delta t = 1$. Why might it be difficult to solve for $\vec{x}^1$?

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Write the update step for calculating the particle's position at $t = 1$ using implicit Euler's method with timestep $\Delta t = 1$. Why might it be difficult to solve for $\vec{x}^1$?

Solution:

$$\ddot{x}^1 = (||(0,0) - x^1|| - 1)\frac{(0,0) - x^1}{||(0,0) - x^1||} = (1 - ||x^1||)\frac{x^1}{||x^1||} = \frac{x^1}{||x^1||} - x^1$$

$$\dot{x}^1 = \dot{x}^0 + \ddot{x}^1 = \dot{x}^0 + \frac{x^1}{||x^1||} - x^1$$

$$x^1 = x^0 + \dot{x}^1 = x^0 + \dot{x}^0 + \frac{x^1}{||x^1||} - x^1 = (-1, 1) + \frac{x^1}{||x^1||} - x^1$$

To find x^1, need to solve a non-linear equation. That's pretty difficult.

Q2.3 Multitudes of Euler's Methods

3. For modifed Euler's method, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \frac{\Delta t}{2}(\dot{\boldsymbol{x}}^t + \dot{\boldsymbol{x}}^{t+\Delta t})$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

Calculate the particle's position at $t = 2$ using modified Euler's method with timestep $\Delta t = 1$.

Q2.3 Multitudes of Euler's Methods

3. For modifed Euler's method, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \frac{\Delta t}{2}(\dot{\boldsymbol{x}}^t + \dot{\boldsymbol{x}}^{t+\Delta t})$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

Calculate the particle's position at $t = 2$ using modified Euler's method with timestep $\Delta t = 1$.

Solution:

$$\dot{\boldsymbol{x}}^1 = (-1, 0) + (0, 0) = (-1, 0)$$

$$\boldsymbol{x}^1 = (0, 1) + \frac{1}{2}((-1, 0) + (-1, 0)) = (-1, 1)$$

$$\dot{\boldsymbol{x}}^2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right)$$

from the work done in part 1. Finally, solve for

$$\boldsymbol{x}^2 = (-1, 1) + \frac{1}{2}\left((-1, 0) + \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right)\right) = \left(-\frac{3}{2} - \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} + \frac{1}{2}\right)$$

Q2.4 Multitudes of Euler’s Methods

4. For Verlet integration, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t + \frac{1}{2}(\Delta t)^2 \ddot{\boldsymbol{x}}^t$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \frac{\boldsymbol{x}^{t+\Delta t} - \boldsymbol{x}^t}{\Delta t}$$

Use the particle’s position at $t = 1$ as calculated with modified Euler’s method. Calculate the particle’s position at $t = 2$ using Verlet integration with timestep $\Delta t = 1$.

Q2.4 Multitudes of Euler's Methods

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$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t + \frac{1}{2}(\Delta t)^2 \ddot{\boldsymbol{x}}^t$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \frac{\boldsymbol{x}^{t+\Delta t} - \boldsymbol{x}^t}{\Delta t}$$

Use the particle's position at $t = 1$ as calculated with modified Euler's method. Calculate the particle's position at $t = 2$ using Verlet integration with timestep $\Delta t = 1$.

Solution:

$$\boldsymbol{x}^1 = (-1, 1)$$

$$\dot{\boldsymbol{x}}^1 = (-1, 1) - (0, 1) = (-1, 0)$$

$$\boldsymbol{x}^2 = (-1, 1) + (-1, 0) + \frac{1}{2}\ddot{\boldsymbol{x}}^1 = (-2, 1) + \left(\frac{1}{2} - \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} - \frac{1}{2}\right) = \left(-\frac{3}{2} - \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} + \frac{1}{2}\right)$$

Q2.5 Multitudes of Euler's Methods

5. What are some pros and cons of using the explicit Euler's method?

Q2.5 Multitudes of Euler's Methods

5. What are some pros and cons of using the explicit Euler's method?

Solution: Pros – simple and easy to compute, we don't always care about precision in graphics (i.e. close enough is good enough).

Cons – inaccurate and unstable, which frequently leads to divergent results, especially as time progresses and errors accumulate.

Q3:

Recall the forward, or explicit Euler method, which uses the following update rules:

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

where $\boldsymbol{x}^t, \dot{\boldsymbol{x}}^t, \ddot{\boldsymbol{x}}^t$ respectively denote the position, velocity, and acceleration at time *t*.

(a) Give some pros and cons of using the explicit Euler method.

(b) Say we have a particle with mass 1 starting at position $\boldsymbol{x}^0 = (0, 1)$ with an initial velocity $\dot{\boldsymbol{x}}^0 = (-1, 0)$ and no initial acceleration. The particle is at one end of a spring, whose other end is the origin (0, 0), and whose spring constant is $k = 1$ and rest length is 1. Calculate particle's position at $t = 3$ using the explicit Euler method with timestep $\Delta t = 1$

Q3:

Recall the forward, or explicit Euler method, which uses the following update rules:

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

where $\boldsymbol{x}^t, \dot{\boldsymbol{x}}^t, \ddot{\boldsymbol{x}}^t$ respectively denote the position, velocity, and acceleration at time *t*.

(a) Give some pros and cons of using the explicit Euler method.

Pros - simple and easy to compute, we don't always care about precision in graphics (i.e. close enough is good enough).

Cons - inaccurate and unstable, which frequently leads to divergent results, especially as time progresses and errors accumulate.

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(b) Say we have a particle with mass 1 starting at position $\boldsymbol{x}^0 = (0, 1)$ with an initial velocity $\dot{\boldsymbol{x}}^0 = (-1, 0)$ and no initial acceleration. The particle is at one end of a spring, whose other end is the origin (0, 0), and whose spring constant is $k = 1$ and rest length is 1. Calculate particle's position at $t = 3$ using the explicit Euler method with timestep $\Delta t = 1$

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Rest length

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$$\ddot{\boldsymbol{x}}^1 = \frac{F_s}{m} * \frac{(0,0)-(-1,1)}{||(0,0)-(-1,1)||} = \frac{1 * (\sqrt{2}-1)}{1} * (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = (1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1)$$

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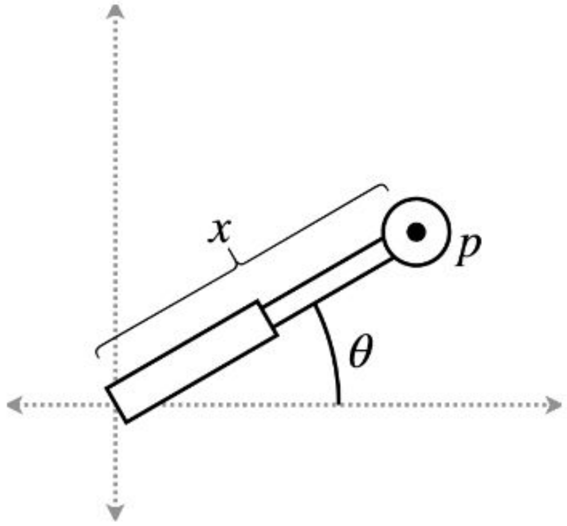
$$\boldsymbol{x}^3 = (-2-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \qquad \dot{\boldsymbol{x}}^3 = \text{unneeded} \qquad \ddot{\boldsymbol{x}}^3 = \text{unneeded}$$

Animation & Inverse Kinematics

Question 3

1 Animation

Consider the piston as shown in the diagram below. For each x (piston displacement) and θ (piston rotation angle), the end of the piston is at some point p in the 2D plane. θ is constrained to $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$.



1. x and θ are controllable parameters. Express $p = (a, b)$ as a function of x and θ.

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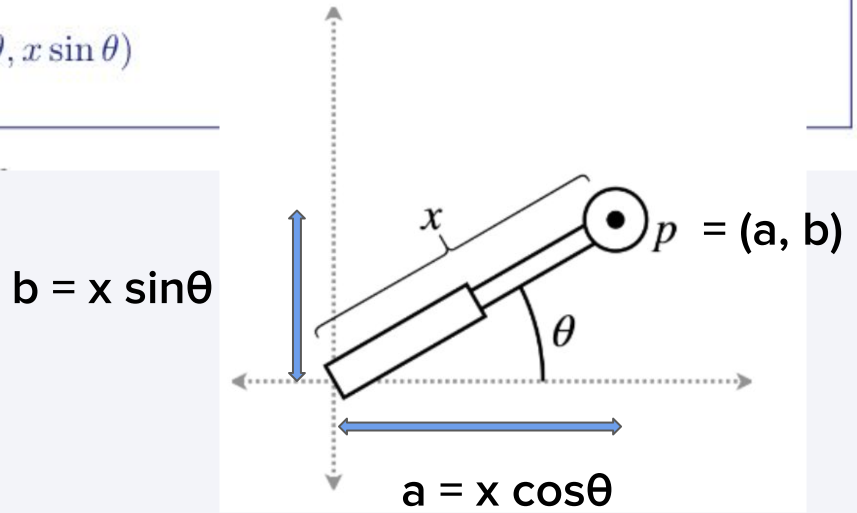
Solution:

$$(x \cos\theta, x \sin\theta)$$



2. Given $p = (a, b)$, express x and θ in terms of a and b.

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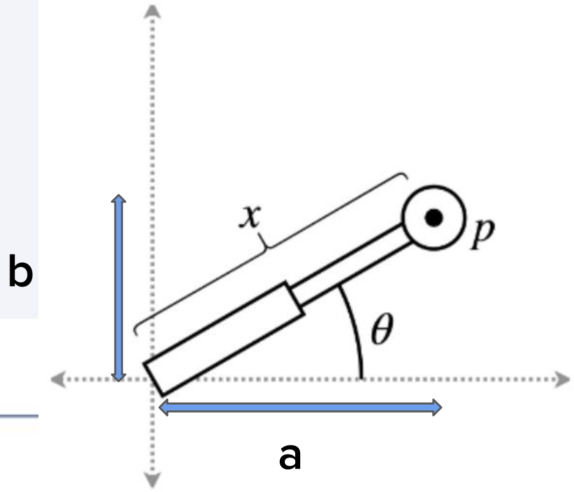

Solution:

Because $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$,

$$\theta = \arctan(b/a), \quad x = \sqrt{a^2 + b^2}.$$

3. What is the difference between forward and inverse kinematics?

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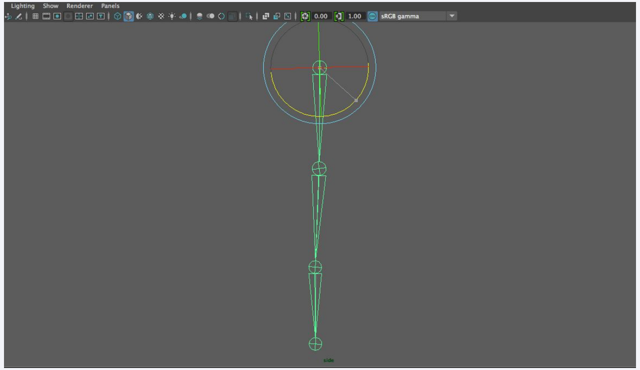
Solution: Forward - we provide angles (e.g. for joints), computer determines final position (e.g. of end of limbs).

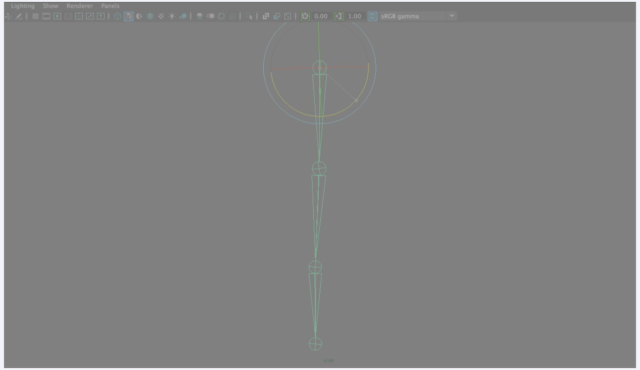


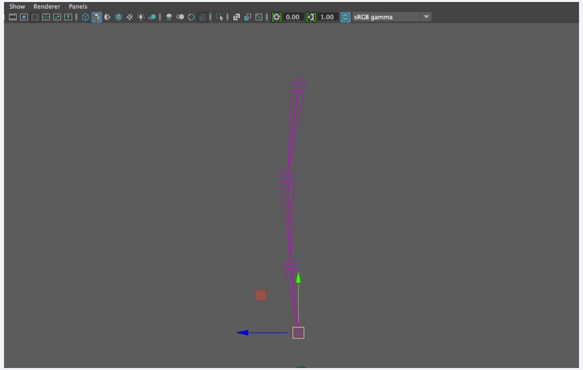
3. What is the difference between forward and inverse kinematics?

Solution: Forward - we provide angles (e.g. for joints), computer determines final position (e.g. of end of limbs).

Inverse - we provide ending position, need to compute the joint angles to reach the position.

Difficulties with inverse kinematics - sometimes has multiple possible solutions (sometimes connected to each other, sometimes separate), sometimes has no solutions, want to make sure solution found is realistic, etc.





4. Are inverse kinematics solutions of this system unique?

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Solution: Yes, but it depends how we constrain θ. Here, because $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, a point p is uniquely determined by the polar coordinates corresponding to the point. The point $p \neq (0, 0)$ determines x uniquely as $x = \sqrt{a^2 + b^2}$, but θ is only determined up to an additive multiple of 2π. If we restrict θ to a principal domain like $(-\pi, \pi]$, then it becomes unique for any non-zero p.

5. What are keyframes and what do we do with them?

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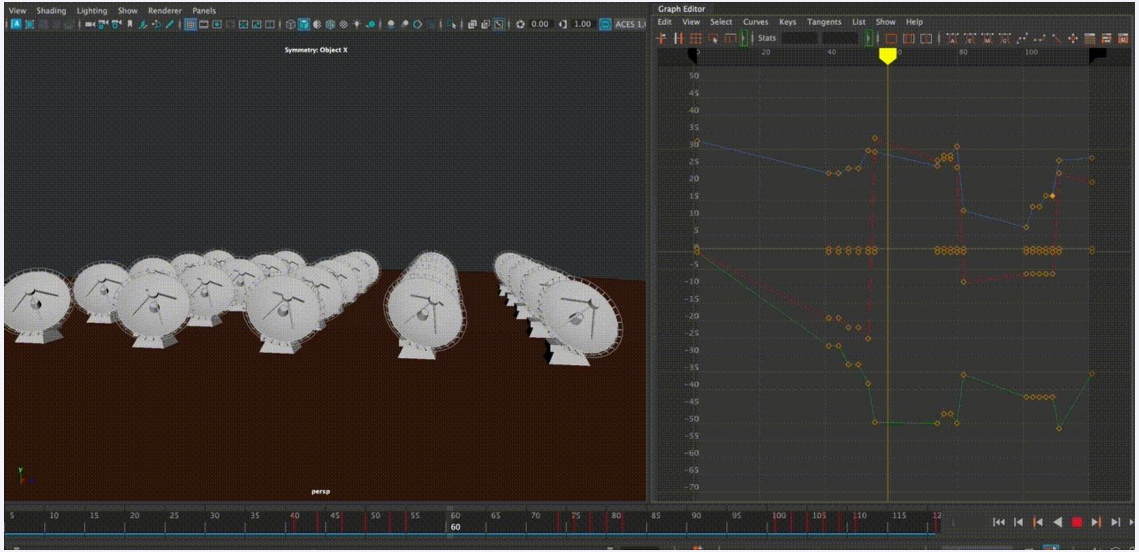
Solution: Keyframes, as their name suggests, are the important moments in some transition or motion, usually the starting and ending points. We usually interpolate between them (though not linearly, because usually too complex) in order to create the frames between the keyframes to form fluid animation - the process of creating the intermediate frames is known as tweening.



6. Why might linear interpolation between rotations (e.g., angles or quaternions) result in unnatural motion?

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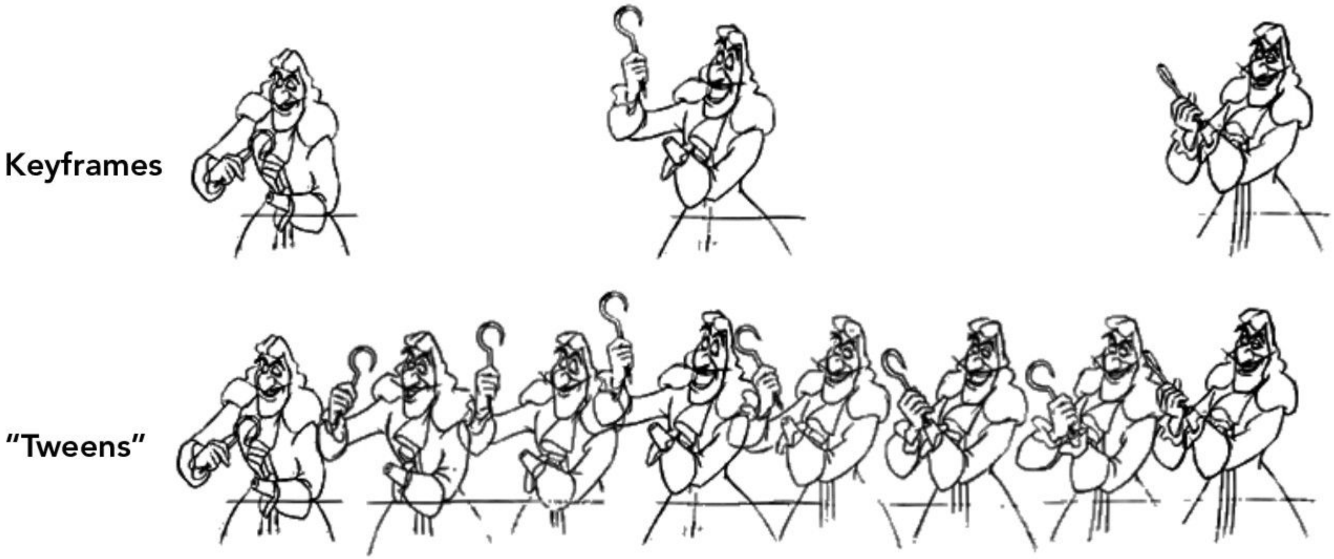
Solution: Linear interpolation treats angles as if they lie on a straight line, but rotations lie on a circle. This can cause unnatural motion, such as taking the long path around or changing speed unevenly. In 3D, similar issues arise with quaternions—spherical linear interpolation (slerp) is used to fix this by following the shortest arc at constant speed.

Q1:

What are keyframes and what do we do with them?



Q1:

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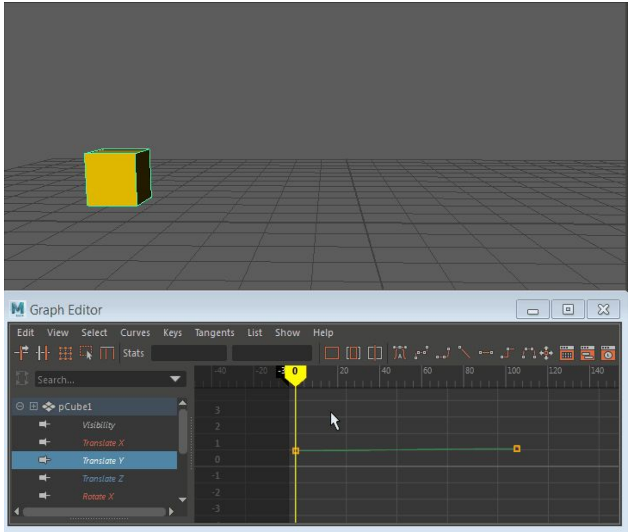
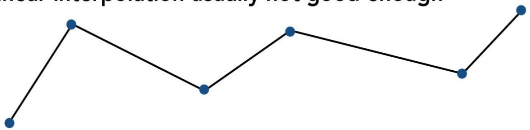
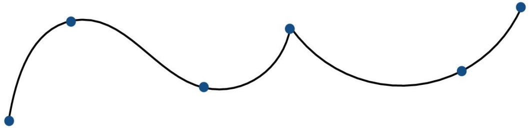
Important moments in some transition/motion, usually starting and ending points. We **interpolate** between them (though not linearly, usually too complex) to create frames between the keyframes to form fluid video - known as **tweening**.



Linear interpolation usually not good enough

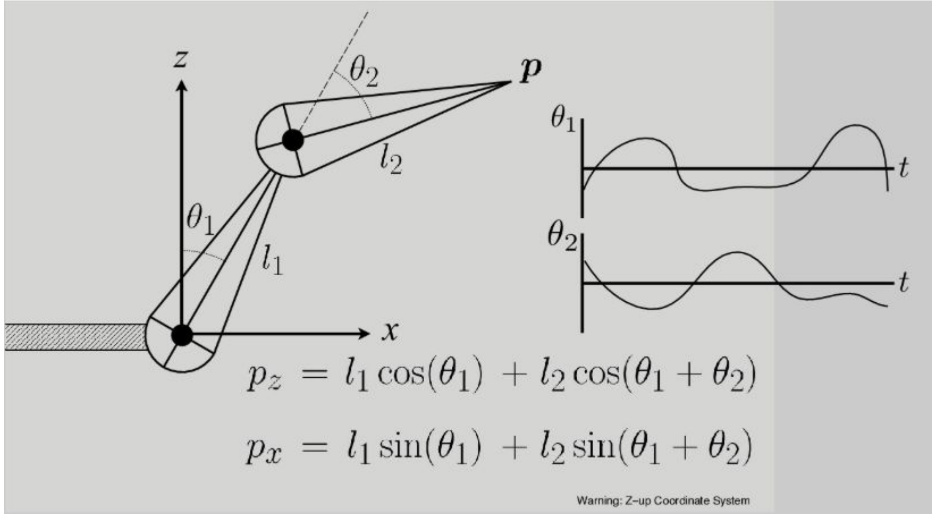


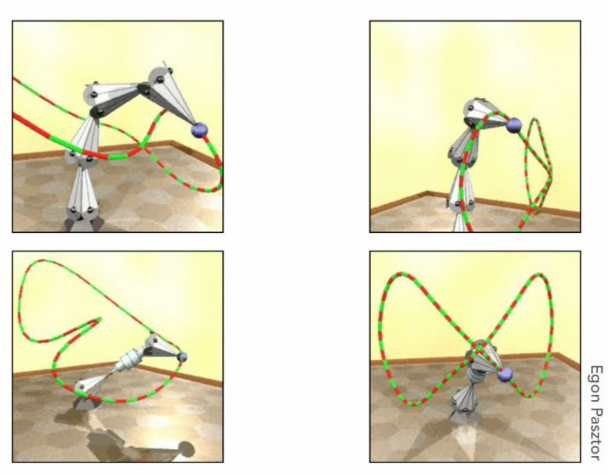
Recall splines for smooth / controllable interpolation



Q2.1:

What is the difference between forward and inverse kinematics?





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https://github.com/cal-cs184/rope_simulation

This is a smaller, toy version of Project 4 to get intuition about simulation

README.md

Rope Simulation

In this discussion, you will be creating a simplified version of the upcoming cloth simulation project.

Part 1: Linking the Rope Constraints (10 mins)

In `rope.cpp`, implement the Rope constructor. It should create a new `Rope` object starting at `start` and ending at `end`, containing `num_nodes` nodes. That is, something along the following diagram:

Spring with non-zero rest length

$$\boldsymbol{f}_{a \to b} = k_S \frac{\boldsymbol{b} - \boldsymbol{a}}{||\boldsymbol{b} - \boldsymbol{a}||} (||\boldsymbol{b} - \boldsymbol{a}|| - l)$$

Rest length

Euler's Method (a.k.a. Forward Euler, Explicit)

- Simple iterative method
- Commonly used
- Very inaccurate
- Most often goes unstable

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \, \dot{\boldsymbol{x}}^t$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \, \ddot{\boldsymbol{x}}^t$$

Let’s Take Attendance.

- Be sure to select Week 12 and input your TA’s secret word 😉
- Any feedback? Let us know!