SIMULATION, KINEMATICS, ANIMATION 12

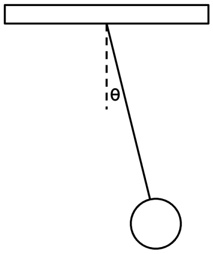
CS 184: FOUNDATIONS OF COMPUTER GRAPHICS

1 Sriram's Pendulum

Sriram is simulating a pendulum. Under the small-angle approximation, the angle between the vertical and the pendulum string, θ, is governed by the following differential equation.

$$\ddot{\theta} = -\frac{g}{L}\theta$$

g is acceleration due to gravity. L is the length of the pendulum string.



1. Sriram plans to use the explicit Euler's method to approximate θ at future points in time. In terms of g, L, θ^t, and $\dot{\theta}^t$, derive the update equations for $\theta^{t+\Delta t}$ and $\dot{\theta}^{t+\Delta t}$.

Solution: Explicit Euler's update equations:

$$\begin{aligned}\theta^{t+\Delta t} &= \theta^t + \Delta t\dot{\theta}^t \\ \dot{\theta}^{t+\Delta t} &= \dot{\theta}^t + \Delta t\ddot{\theta}^t\end{aligned}$$

Sriram's update equations:

$$\begin{aligned}\theta^{t+\Delta t} &= \theta^t + \Delta t\dot{\theta}^t \\ \dot{\theta}^{t+\Delta t} &= \dot{\theta}^t - \Delta t\left(\frac{g}{L}\right)\theta^t\end{aligned}$$

2. Sriram approximates $g \approx 10\,\mathrm{m/s^2}$. The pendulum string has length $L = 1\,\mathrm{m}$. The system's initial conditions are $\theta^0 = 0.1$ rad and $\dot{\theta}^0 = 0$ rad/s. Fill in the table with the results of running three Euler updates for the given step sizes.

Solution:

Using our equations (with known values plugged in),

$$\begin{aligned}\theta^{t+\Delta t} &= \theta^t + \Delta t\dot{\theta}^t \\ \dot{\theta}^{t+\Delta t} &= \dot{\theta}^t - 10\Delta t\theta^t,\end{aligned}$$

	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1$
$\theta^{\Delta t}$			
$\dot{\theta}^{\Delta t}$			
$\theta^{2\Delta t}$			
$\dot{\theta}^{2\Delta t}$			
$\theta^{3\Delta t}$			
$\dot{\theta}^{3\Delta t}$			

the table should look like the following:

	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1$
$\theta^{\Delta t}$	0.1 rad	0.1 rad	0.1 rad
$\dot{\theta}^{\Delta t}$	−0.1 rad/s	−0.5 rad/s	−1 rad/s
$\theta^{2\Delta t}$	0.09 rad	−0.15 rad	−0.9 rad
$\dot{\theta}^{2\Delta t}$	−0.2 rad/s	−1.0 rad/s	−2.0 rad/s
$\theta^{3\Delta t}$	0.07 rad	−0.65 rad	−2.9 rad
$\dot{\theta}^{3\Delta t}$	−0.29 rad/s	−0.25 rad/s	7.0 rad/s

3. How does increasing the step size affect the accuracy of Sriram's approximation of θ?

Solution: As the step size increases, errors accumulate, and the approximation becomes less accurate. Notice that when $\Delta t = 1$, the pendulum swings past -0.1 rad, which is not even physically possible.

4. Fill in the 2×2 matrix A below to express the explicit Euler's update — for a general step size Δt — in the form of a linear dynamical system. That is, express the update as:

$$\begin{bmatrix} \theta^{t+\Delta t} \\ \dot{\theta}^{t+\Delta t} \end{bmatrix} = A \begin{bmatrix} \theta^{t} \\ \dot{\theta}^{t} \end{bmatrix}$$

Fill in the matrix A below:

$$\begin{bmatrix} \theta^{t+\Delta t} \\ \\ \dot{\theta}^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \\ \\ \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \end{bmatrix} \begin{bmatrix} \theta^{t} \\ \\ \dot{\theta}^{t} \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & \Delta t \\ \\ -\frac{g}{L}\Delta t & 1 \end{bmatrix}$$

5. A linear dynamical system is guaranteed to be internally stable if the eigenvalues of matrix A satisfy $|\lambda_i| < 1$ for all i. Calculate the two eigenvalues of A. Is the condition for internal stability satisfiable?

Solution: For

$$A = \begin{bmatrix} 1 & \Delta t \\ -\dfrac{g}{L}\Delta t & 1 \end{bmatrix},$$

the eigenvalues satisfy

$$\det\bigl(A - \lambda I\bigr) = \begin{vmatrix} 1-\lambda & \Delta t \\ -\dfrac{g}{L}\Delta t & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + \frac{g}{L}\Delta t^2 = 0.$$

$$\therefore \quad \boxed{\lambda_{1,2} = 1 \pm i\sqrt{\frac{g}{L}}\,\Delta t}.$$

Their magnitudes are

$$|\lambda_{1,2}| = \sqrt{1 + \frac{g}{L}\Delta t^2} > 1 \quad \text{for every } \Delta t > 0.$$

Hence the internal–stability condition $|\lambda_i| < 1$ cannot be met for any positive time–step; explicit Euler is never internally stable for this oscillatory system (except in the limit $\Delta t \to 0$).

2 Multitudes of Euler's Methods

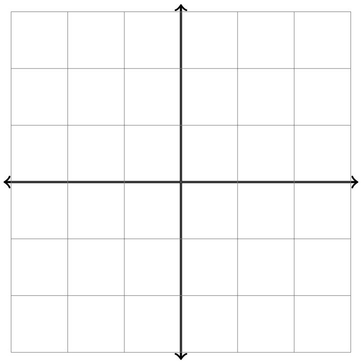
We have a particle with mass $1\,\text{kg}$. It starts at position $\boldsymbol{x}^0 = (0\,\text{m}, 1\,\text{m})$ with an initial velocity $\dot{\boldsymbol{x}}^0 = (-1\,\text{m/s}, 0\,\text{m/s})$ and no initial acceleration. The particle is at one end of a spring, whose other end is fixed at $(0\,\text{m}, 0\,\text{m})$. Its spring constant is $k = 1\,\text{N/m}$ and rest length is $1\,\text{m}$.

Recall the explicit Euler's method, which uses the following update rules

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

$\boldsymbol{x}^t, \dot{\boldsymbol{x}}^t, \ddot{\boldsymbol{x}}^t$ respectively denote the position, velocity, and acceleration at time t.

1. Calculate the particle's position at $t = 3$ using the explicit Euler's method with timestep $\Delta t = 1$. You might find it helpful to plot the particle on the provided grid.



Solution:

$$\boldsymbol{x}^0 = (0,1) \qquad \dot{\boldsymbol{x}}^0 = (-1,0) \qquad \ddot{\boldsymbol{x}} = (0,0)$$

$$\boldsymbol{x}^1 = (-1,1) \qquad \dot{\boldsymbol{x}}^1 = (-1,0)$$

$$\ddot{\boldsymbol{x}}^1 = \frac{F_s}{m} \cdot \frac{(0,0)-(-1,1)}{||(0,0)-(-1,1)||} = \frac{1 \cdot (\sqrt{2}-1)}{1} \cdot \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \left(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right)$$

$$\boldsymbol{x}^2 = (-2,1) \qquad \dot{\boldsymbol{x}}^2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right) \qquad \ddot{\boldsymbol{x}}^2 = \text{unneeded}$$

$$\boldsymbol{x}^3 = \left(-2 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \qquad \dot{\boldsymbol{x}}^3 = \text{unneeded} \qquad \ddot{\boldsymbol{x}}^3 = \text{unneeded}$$

2. For implicit Euler's method, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^{t+\Delta t}$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^{t} + \Delta t \ddot{\boldsymbol{x}}^{t+\Delta t}$$

Write the update step for calculating the particle's position at $t = 1$ using implicit Euler's method with timestep $\Delta t = 1$. Why might it be difficult to solve for $\boldsymbol{x}^1$?

Solution:

$$\ddot{x}^1 = (||(0,0) - x^1|| - 1)\frac{(0,0) - x^1}{||(0,0) - x^1||} = (1 - ||x^1||)\frac{x^1}{||x^1||} = \frac{x^1}{||x^1||} - x^1$$

$$\dot{x}^1 = \dot{x}^0 + \ddot{x}^1 = \dot{x}^0 + \frac{x^1}{||x^1||} - x^1$$

$$x^1 = x^0 + \dot{x}^1 = x^0 + \dot{x}^0 + \frac{x^1}{||x^1||} - x^1 = (-1,1) + \frac{x^1}{||x^1||} - x^1$$

To find x^1, need to solve a non-linear equation. That's pretty difficult.

3. For modifed Euler's method, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^{t} + \frac{\Delta t}{2}(\dot{\boldsymbol{x}}^{t} + \dot{\boldsymbol{x}}^{t+\Delta t})$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^{t} + \Delta t \ddot{\boldsymbol{x}}^{t}$$

Calculate the particle's position at $t = 2$ using modified Euler's method with timestep $\Delta t = 1$.

Solution:

$$\dot{\boldsymbol{x}}^1 = (-1,0) + (0,0) = (-1,0)$$

$$\boldsymbol{x}^1 = (0,1) + \frac{1}{2}((-1,0) + (-1,0)) = (-1,1)$$

$$\dot{\boldsymbol{x}}^2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right)$$

from the work done in part 1. Finally, solve for

$$\boldsymbol{x}^2 = (-1,1) + \frac{1}{2}\left((-1,0) + \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 1\right)\right) = \left(-\frac{3}{2} - \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} + \frac{1}{2}\right)$$

4. For Verlet integration, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^{t} + \Delta t \dot{\boldsymbol{x}}^{t} + \frac{1}{2}(\Delta t)^2 \ddot{\boldsymbol{x}}^{t}$$

$$\dot{\boldsymbol{x}}^{t+\Delta t} = \frac{\boldsymbol{x}^{t+\Delta t} - \boldsymbol{x}^{t}}{\Delta t}$$

Use the particle's position at $t = 1$ as calculated with modified Euler's method. Calculate the particle's position at $t = 2$ using Verlet integration with timestep $\Delta t = 1$.

Solution:

$$\boldsymbol{x}^1 = (-1, 1)$$

$$\dot{\boldsymbol{x}}^1 = (-1, 1) - (0, 1) = (-1, 0)$$

$$\boldsymbol{x}^2 = (-1, 1) + (-1, 0) + \frac{1}{2}\ddot{\boldsymbol{x}}^1 = (-2, 1) + \left(\frac{1}{2} - \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} - \frac{1}{2}\right) = \left(-\frac{3}{2} - \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} + \frac{1}{2}\right)$$

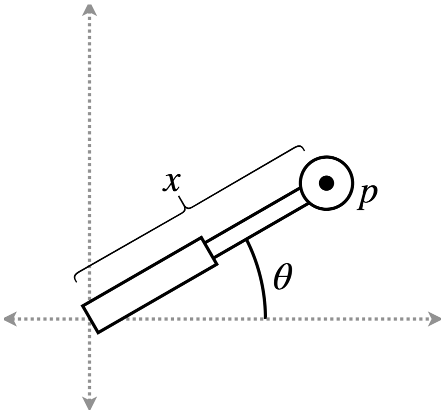
5. What are some pros and cons of using the explicit Euler's method?

Solution: Pros – simple and easy to compute, we don't always care about precision in graphics (i.e. close enough is good enough).

Cons – inaccurate and unstable, which frequently leads to divergent results, especially as time progresses and errors accumulate.

3 Animation

Consider the piston as shown in the diagram below. For each x (piston displacement) and θ (piston rotation angle), the end of the piston is at some point p in the 2D plane. θ is constrained to $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$.



1. x and θ are controllable parameters. Express $p = (a, b)$ as a function of x and θ.

Solution:

$$(x \cos\theta, x \sin\theta)$$

2. Given $p = (a, b)$, express x and θ in terms of a and b.

Solution:

Because $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\theta = \arctan(b/a), \quad x = \sqrt{a^2 + b^2}.$$

3. What is the difference between forward and inverse kinematics?

Solution: Forward – we provide angles (e.g. for joints), computer determines final position (e.g. of end of limbs).

Inverse – we provide ending position, need to compute the joint angles to reach the position.

Difficulties with inverse kinematics – sometimes has multiple possible solutions (sometimes connected to each other, sometimes separate), sometimes has no solutions, want to make sure solution found is realistic, etc.

4. Are inverse kinematics solutions of this system unique?

Solution: Yes, but it depends how we constrain θ. Here, because $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, a point p is uniquely determined by the polar coordinates corresponding to the point. The point $p \neq (0, 0)$ determines x uniquely as $x = \sqrt{a^2 + b^2}$, but θ is only determined up to an additive multiple of 2π. If we restrict

θ to a principal domain like $(-\pi, \pi]$, then it becomes unique for any non-zero p.

5. Why might linear interpolation between rotations (e.g., angles or quaternions) result in unnatural motion?

Solution: Linear interpolation treats angles as if they lie on a straight line, but rotations lie on a circle. For example, consider motion along a line at constant velocity. As $t \to \infty$, θ should eventually converge.