SIMULATION, KINEMATICS, ANIMATION 12

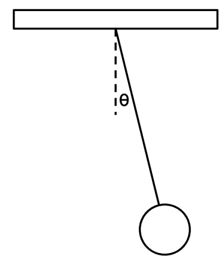
CS 184: FOUNDATIONS OF COMPUTER GRAPHICS

1 Sriram's Pendulum

Sriram is simulating a pendulum. Under the small-angle approximation, the angle between the vertical and the pendulum string, θ, is governed by the following differential equation.

$$\ddot{\theta} = -\frac{g}{L}\theta$$

g is acceleration due to gravity. L is the length of the pendulum string.



1. Sriram plans to use the explicit Euler's method to approximate θ at future points in time. In terms of g, L, θ^t, and $\dot{\theta}^t$, derive the update equations for $\theta^{t+\Delta t}$ and $\dot{\theta}^{t+\Delta t}$.

2. Sriram approximates $g \approx 10\,\text{m/s}^2$. The pendulum string has length $L = 1\,\text{m}$. The system's initial conditions are $\theta^0 = 0.1$ rad and $\dot{\theta}^0 = 0$ rad/s. Fill in the table with the results of running three Euler updates for the given step sizes.

	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1$
$\theta^{\Delta t}$			
$\dot{\theta}^{\Delta t}$			
$\theta^{2\Delta t}$			
$\dot{\theta}^{2\Delta t}$			
$\theta^{3\Delta t}$			
$\dot{\theta}^{3\Delta t}$			

3. How does increasing the step size affect the accuracy of Sriram's approximation of θ?

4. Fill in the 2×2 matrix A below to express the explicit Euler's update — for a general step size Δt — in the form of a linear dynamical system. That is, express the update as:

$$\begin{bmatrix} \theta^{t+\Delta t} \\ \dot{\theta}^{t+\Delta t} \end{bmatrix} = A \begin{bmatrix} \theta^{t} \\ \dot{\theta}^{t} \end{bmatrix}$$

Fill in the matrix A below:

$$\begin{bmatrix} \theta^{t+\Delta t} \\ \\ \dot{\theta}^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} \underline{\hspace{3cm}} & \underline{\hspace{3cm}} \\ \\ \underline{\hspace{3cm}} & \underline{\hspace{3cm}} \end{bmatrix} \begin{bmatrix} \theta^{t} \\ \\ \dot{\theta}^{t} \end{bmatrix}$$

5. A linear dynamical system is guaranteed to be internally stable if the eigenvalues of matrix A satisfy $|\lambda_i| < 1$ for all i. Calculate the two eigenvalues of A. Is the condition for internal stability satisfiable?

2 Multitudes of Euler's Methods

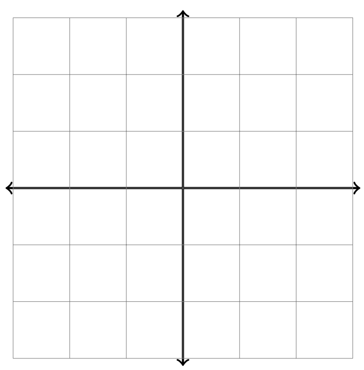
We have a particle with mass 1 kg. It starts at position $\boldsymbol{x}^0 = (0\text{ m}, 1\text{ m})$ with an initial velocity $\dot{\boldsymbol{x}}^0 = (-1\text{ m/s}, 0\text{ m/s})$ and no initial acceleration. The particle is at one end of a spring, whose other end is fixed at $(0\text{ m}, 0\text{ m})$. Its spring constant is $k = 1$ N/m and rest length is 1 m.

Recall the explicit Euler's method, which uses the following update rules

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

$\boldsymbol{x}^t, \dot{\boldsymbol{x}}^t, \ddot{\boldsymbol{x}}^t$ respectively denote the position, velocity, and acceleration at time t.

1. Calculate the particle's position at $t = 3$ using the explicit Euler's method with timestep $\Delta t = 1$. You might find it helpful to plot the particle on the provided grid.



2. For implicit Euler's method, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^{t+\Delta t}$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^{t+\Delta t}$$

   Write the update step for calculating the particle's position at $t = 1$ using implicit Euler's method with timestep $\Delta t = 1$. Why might it be difficult to solve for $\boldsymbol{x}^1$?

3. For modifed Euler's method, the update rules are

$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \frac{\Delta t}{2}(\dot{\boldsymbol{x}}^t + \dot{\boldsymbol{x}}^{t+\Delta t})$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \dot{\boldsymbol{x}}^t + \Delta t \ddot{\boldsymbol{x}}^t$$

Calculate the particle's position at $t = 2$ using modified Euler's method with timestep $\Delta t = 1$.

4. For Verlet integration, the update rules are

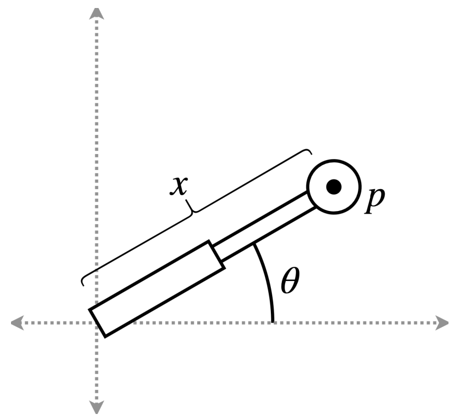
$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \dot{\boldsymbol{x}}^t + \frac{1}{2}(\Delta t)^2 \ddot{\boldsymbol{x}}^t$$
$$\dot{\boldsymbol{x}}^{t+\Delta t} = \frac{\boldsymbol{x}^{t+\Delta t} - \boldsymbol{x}^t}{\Delta t}$$

Use the particle's position at $t = 1$ as calculated with modified Euler's method. Calculate the particle's position at $t = 2$ using Verlet integration with timestep $\Delta t = 1$.

5. What are some pros and cons of using the explicit Euler's method?

3 Animation

Consider the piston as shown in the diagram below. For each x (piston displacement) and θ (piston rotation angle), the end of the piston is at some point p in the 2D plane. θ is constrained to $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$.



1. x and θ are controllable parameters. Express $p = (a, b)$ as a function of x and θ.

2. Given $p = (a, b)$, express x and θ in terms of a and b.

3. What is the difference between forward and inverse kinematics?

4. Are inverse kinematics solutions of this system unique?

5. Why might linear interpolation between rotations (e.g., angles or quaternions) result in unnatural motion?