

Lecture 4:

Transforms

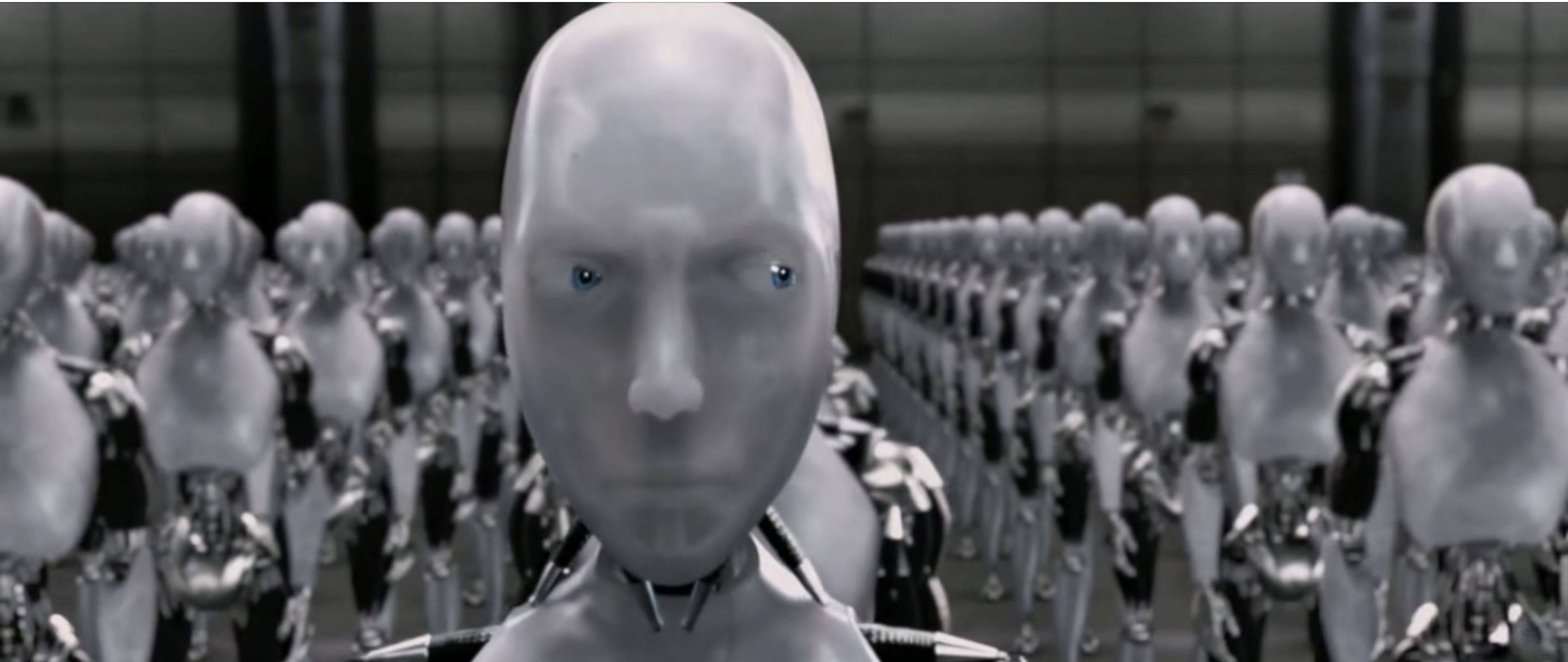
Computer Graphics and Imaging
UC Berkeley CS184/284A

Why Study Transforms?

Coordinating A Robot Army



Modeling A Robot Army



iRobot movie

Transforms can describe position of object instances

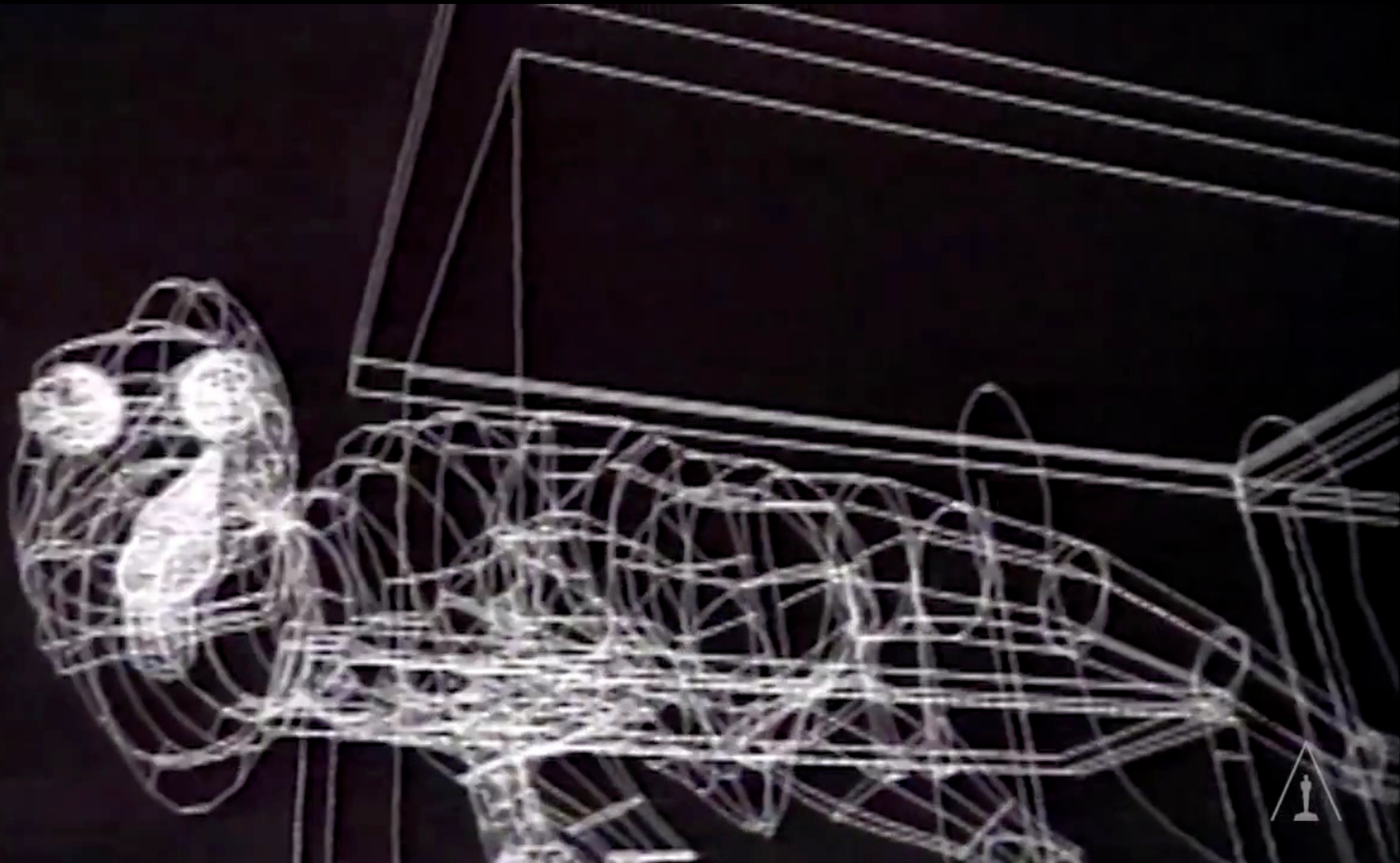
Posing a Character's Skeleton



iRobot movie

Transforms can describe relative position of connected body parts

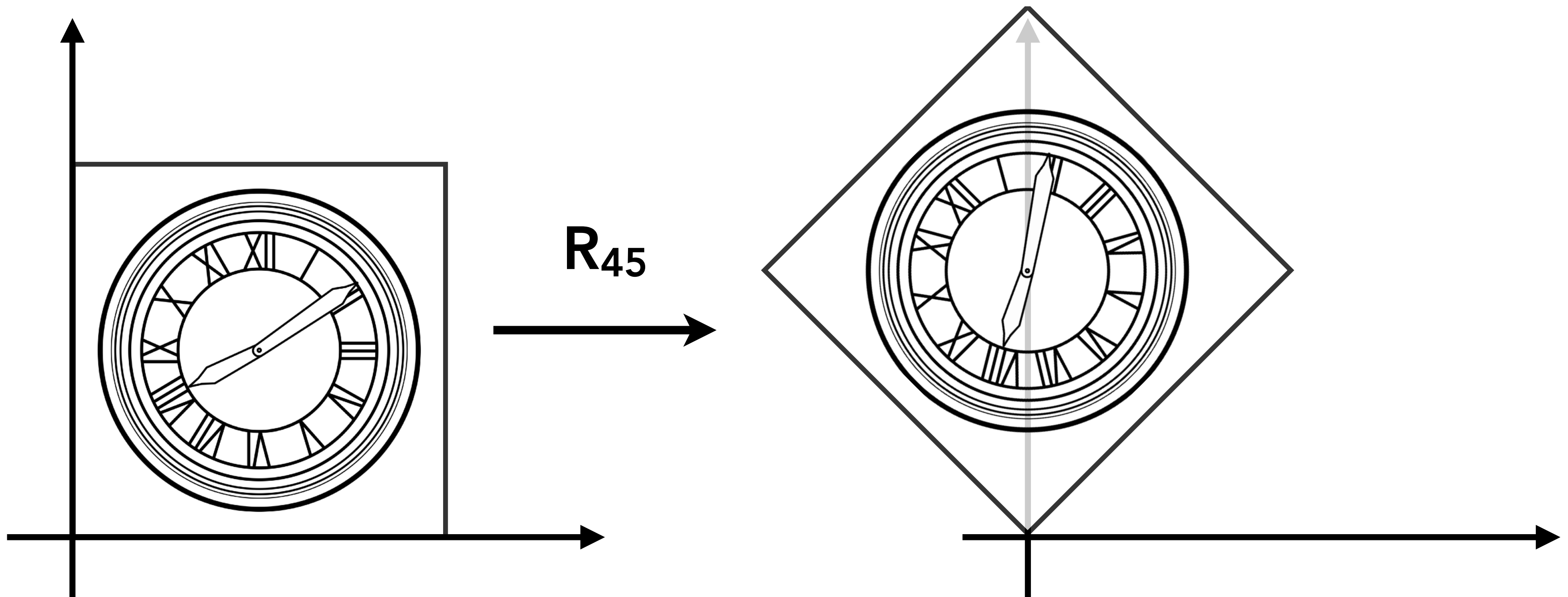
Project Polygons in 3D to 2D Screen



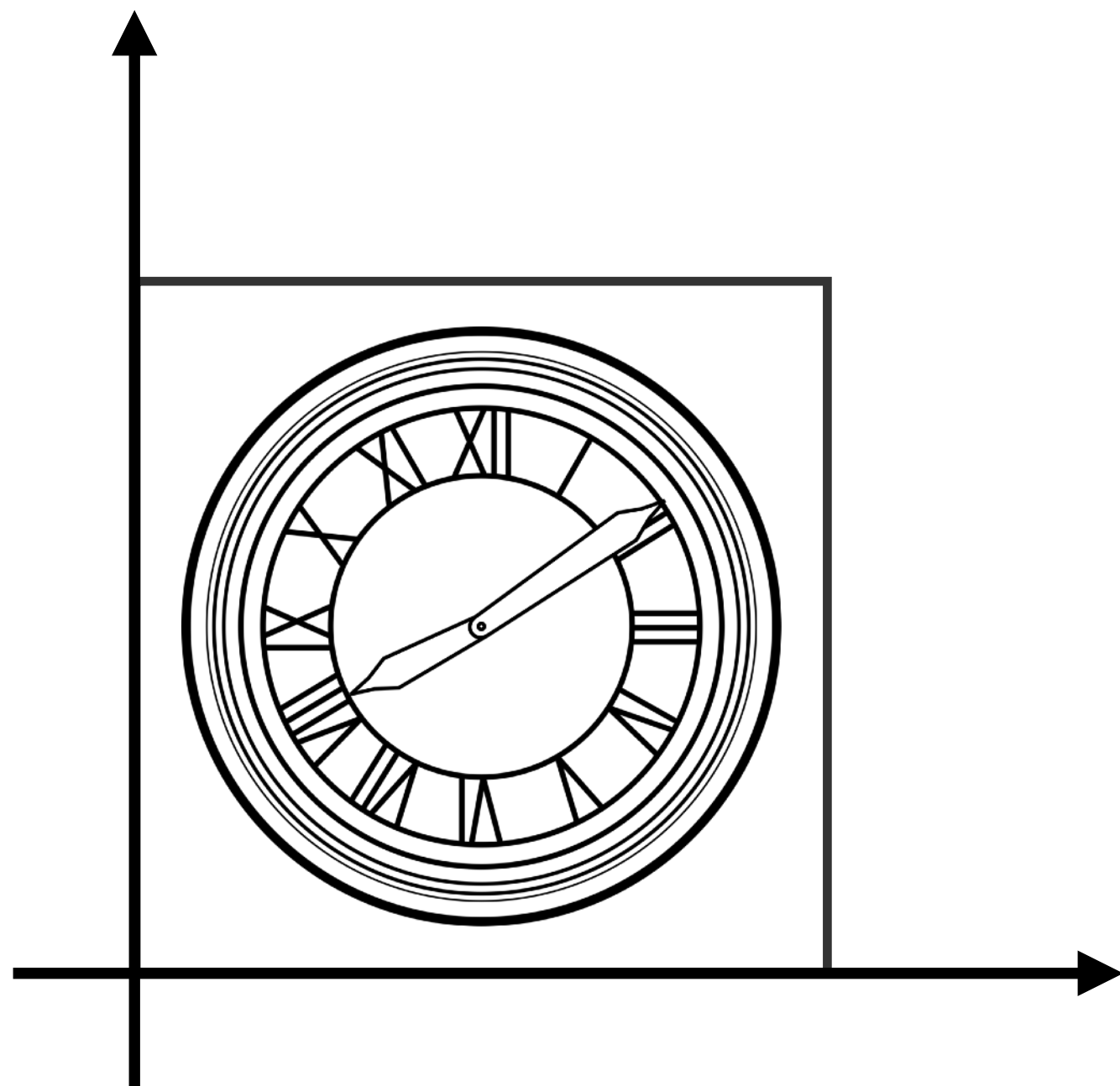
Moments That Changed The Movies: Jurassic Park
<https://www.youtube.com/watch?v=KWsbcbvYqN8>

Basic Transforms

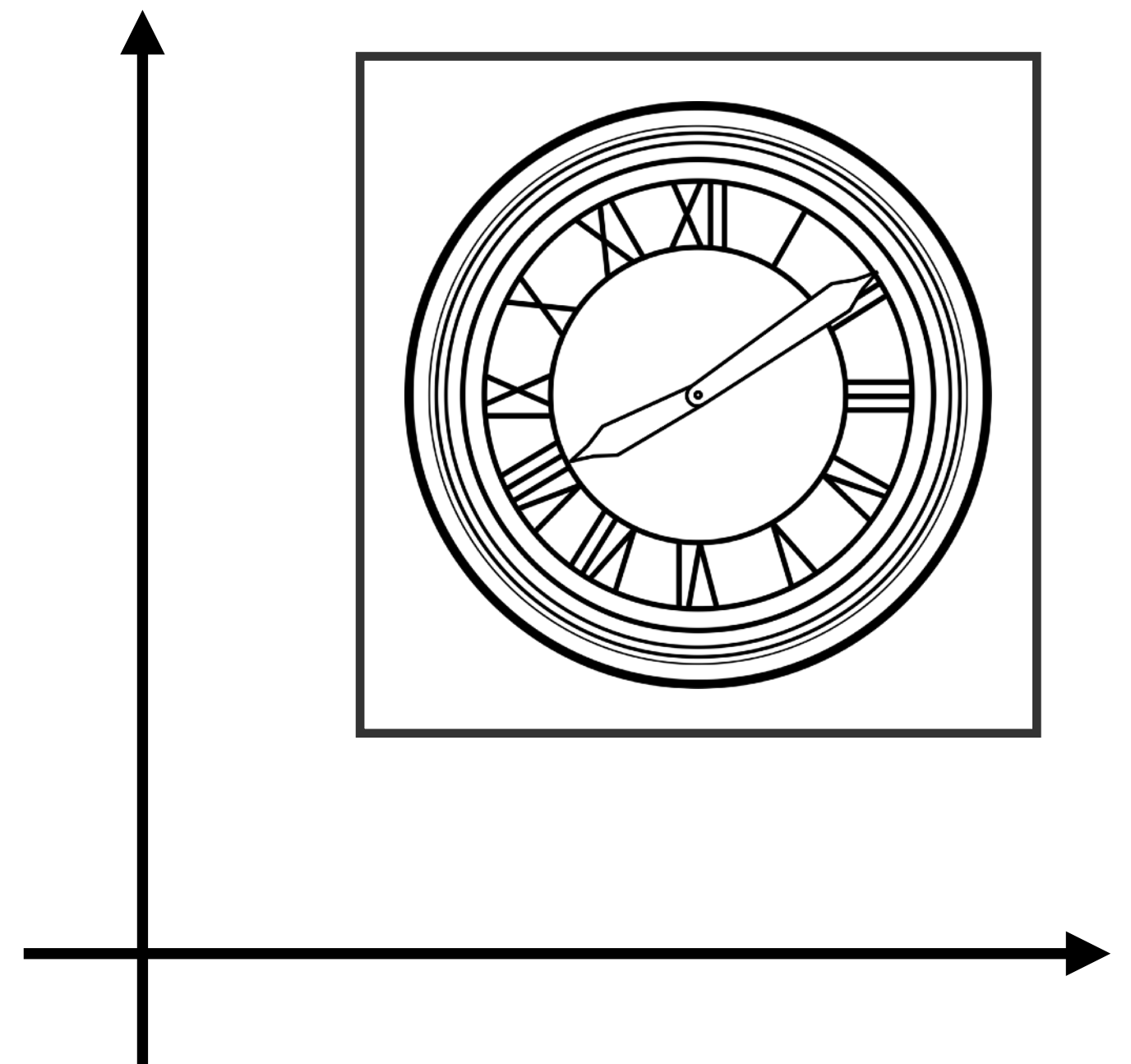
Rotate



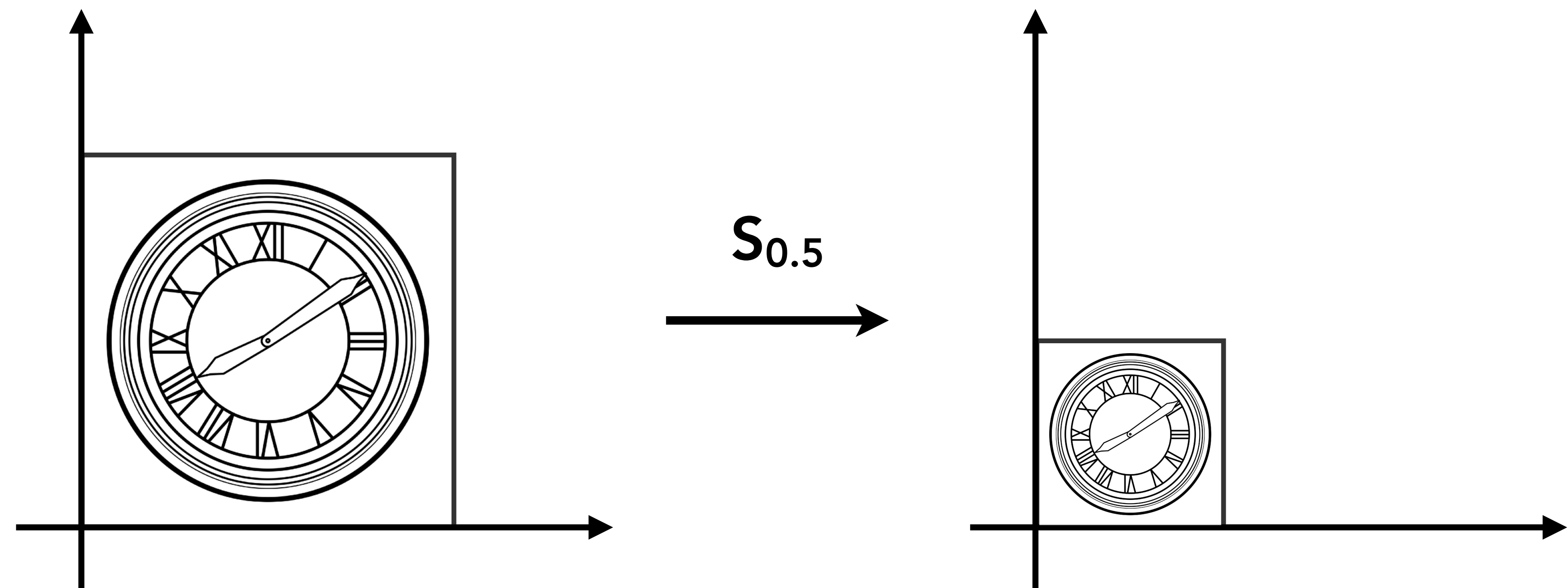
Translate



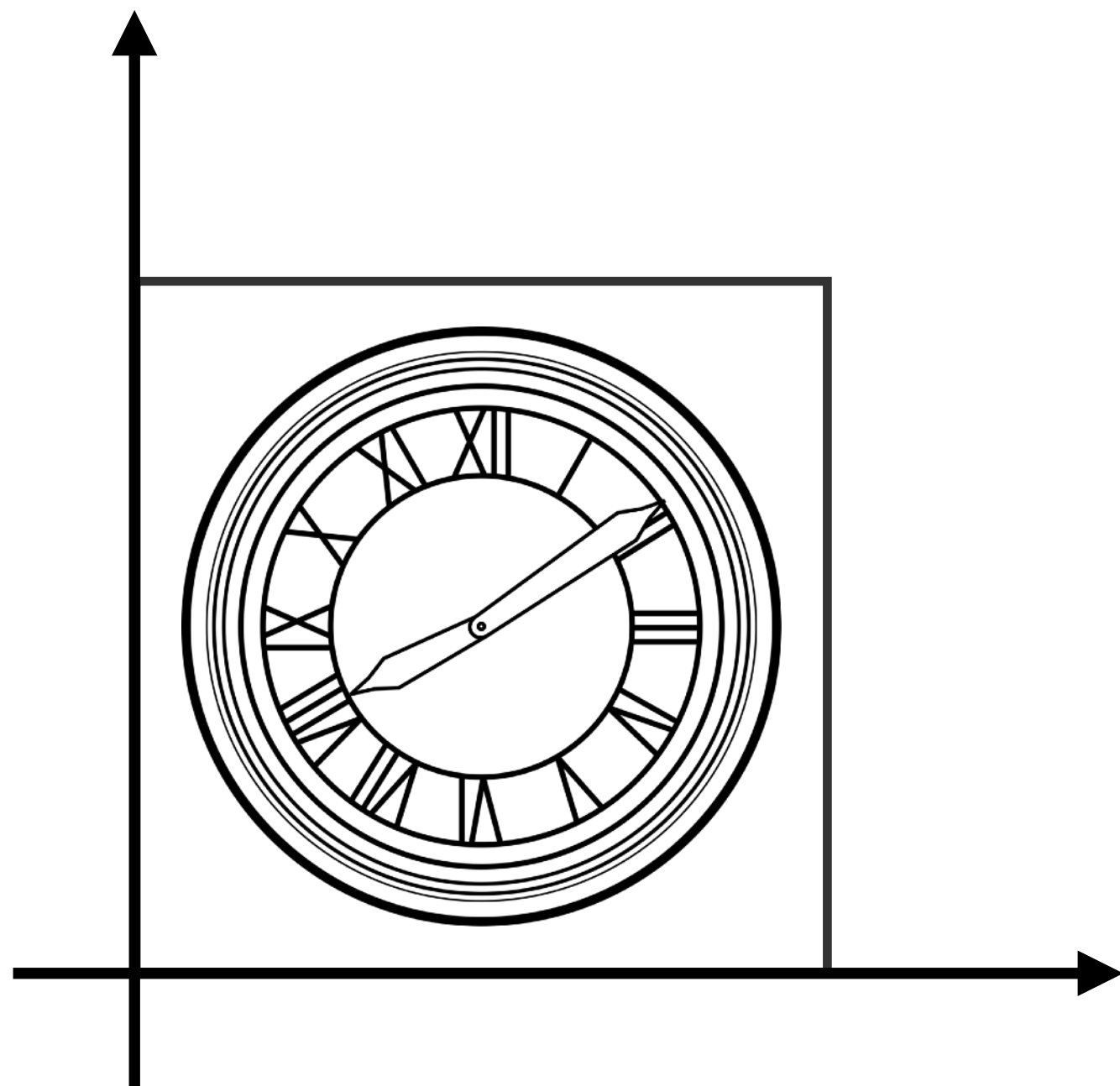
$T_{1,1}$



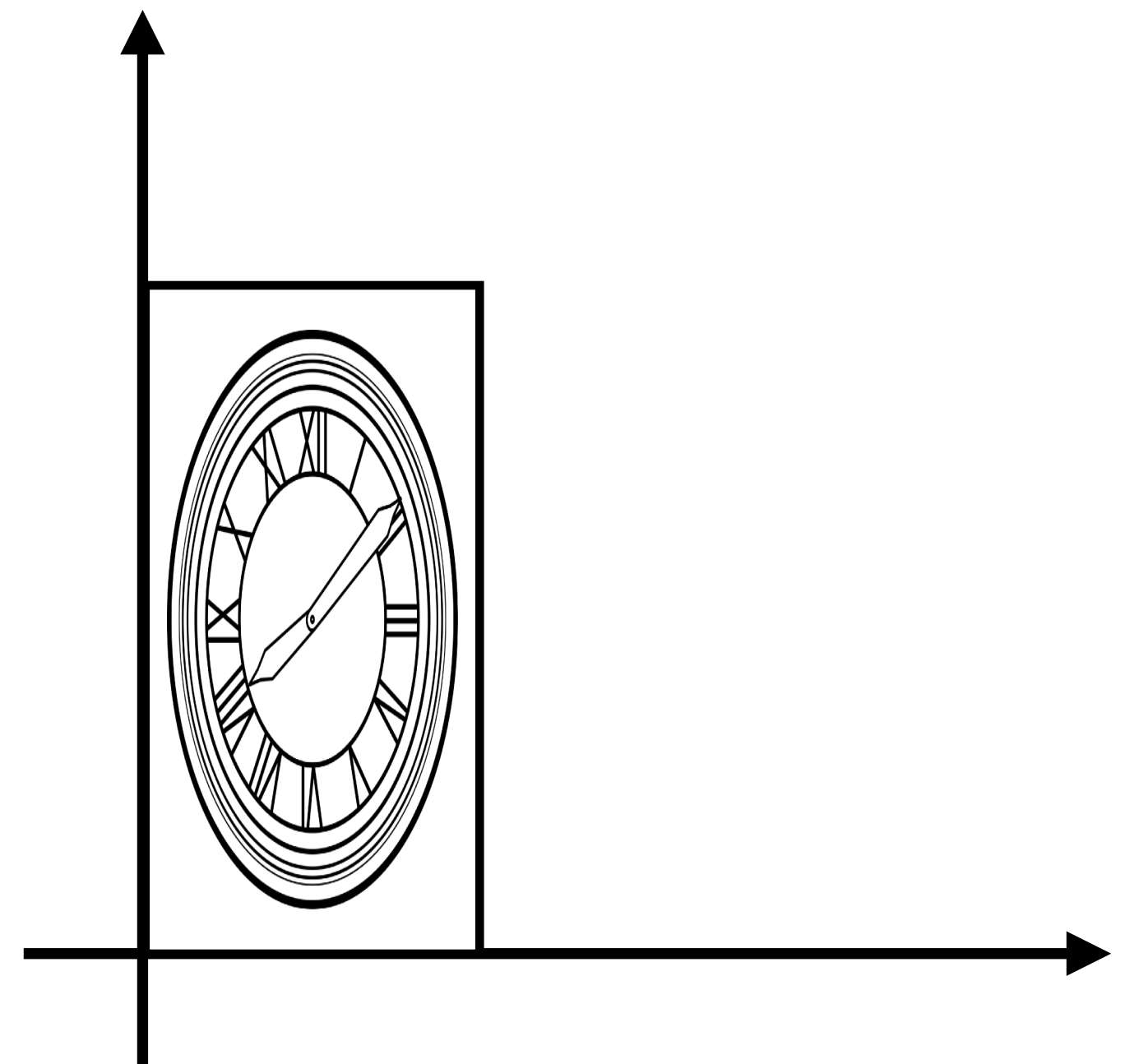
Scale



Scale (Non-Uniform)



$S_{0.5,1.0}$



Why Study Transforms?

Modeling

- Define shapes in convenient coordinates
- Enable multiple copies of the same object
- Efficiently represent hierarchical scenes

Viewing

- World coordinates to camera coordinates
- Parallel / perspective projections from 3D to 2D

Lecture Outline

How to think about and use transformations

- Types: rotate, translate, scale, ...
- Coordinate frames
- Composing multiple transformations
- Hierarchical transforms
- Perspective projection

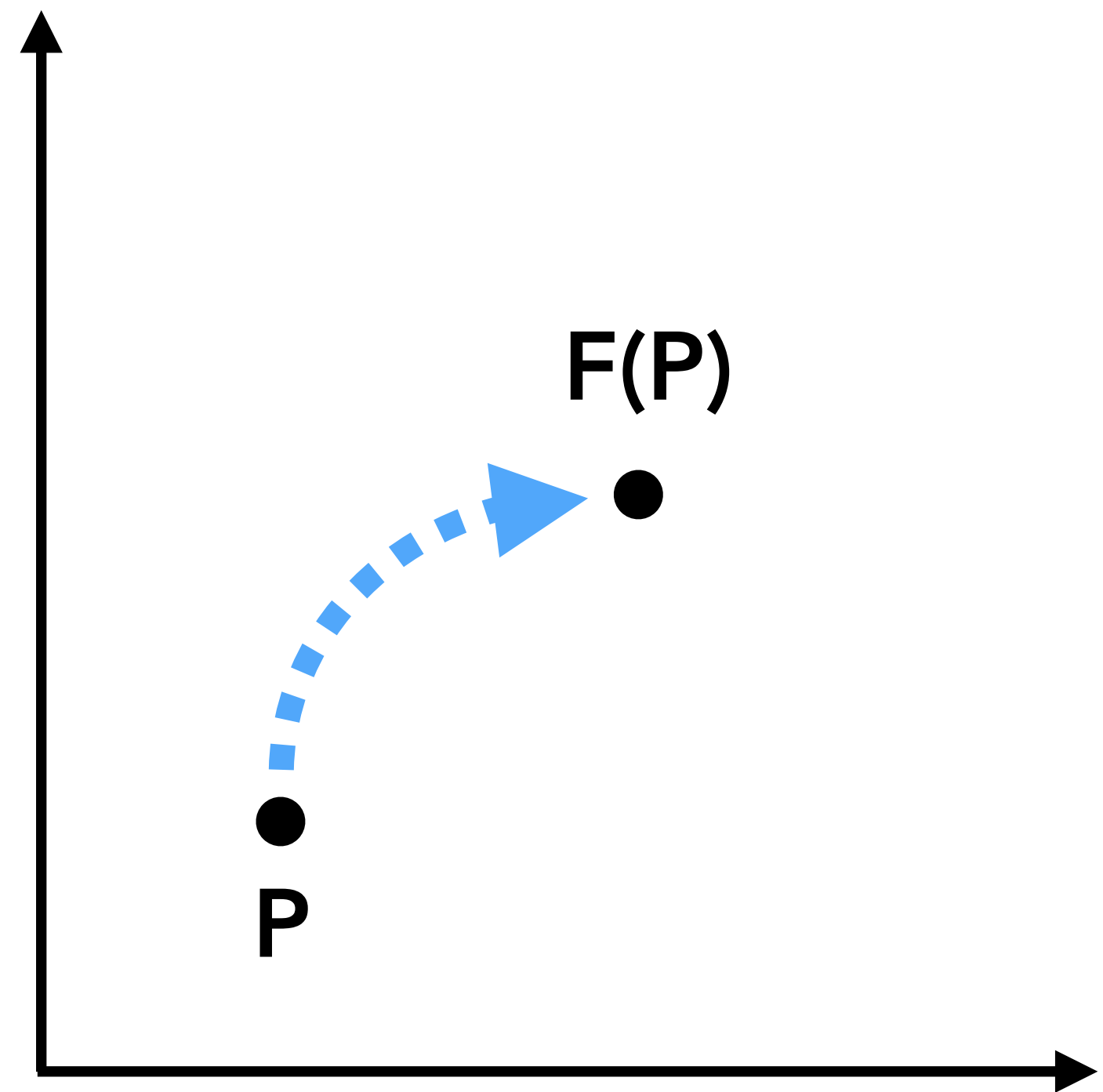
How to implement?

- Represent transforms as matrices
- Homogeneous coordinates

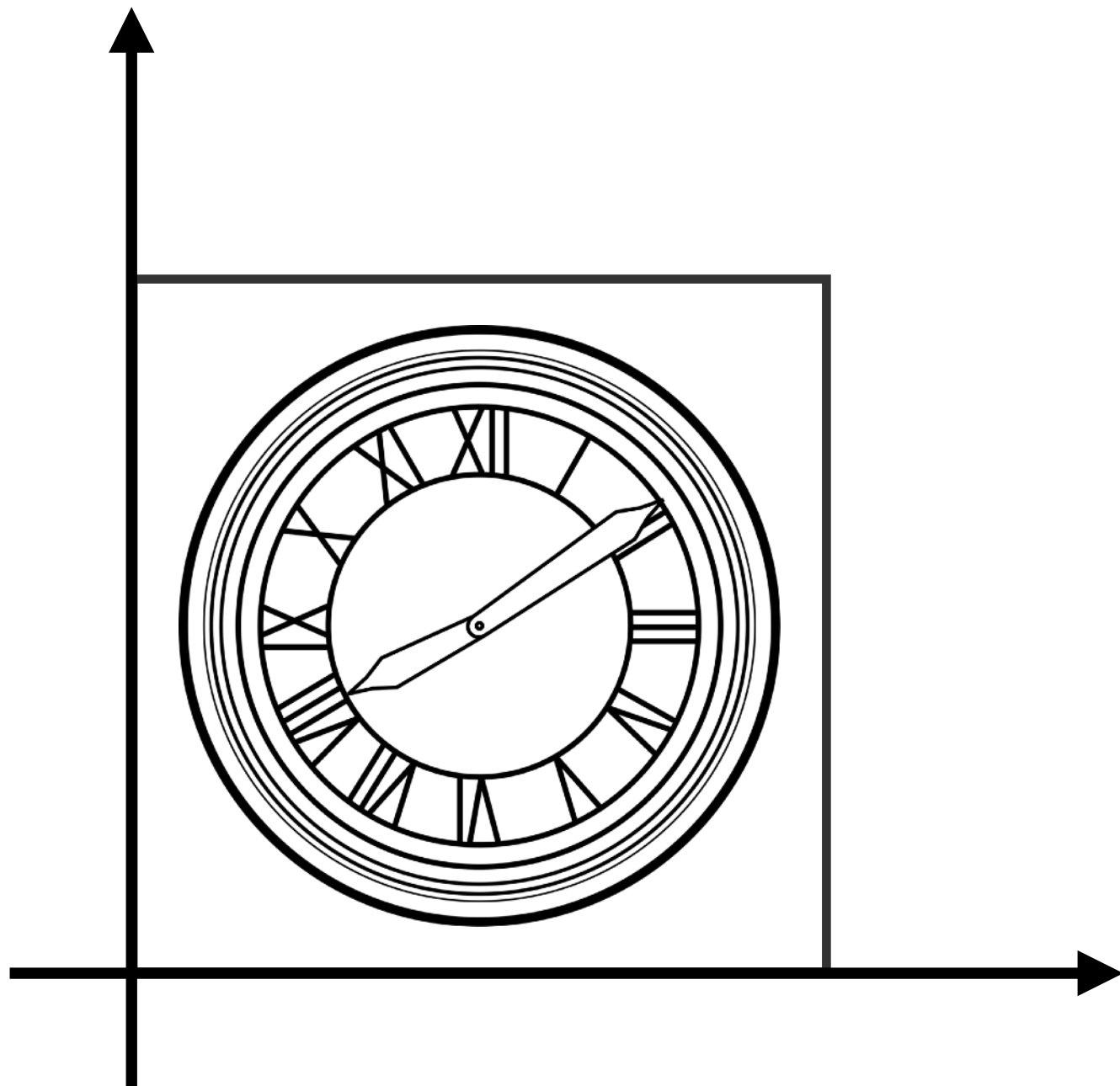
What Are Transforms?

Just functions acting on points

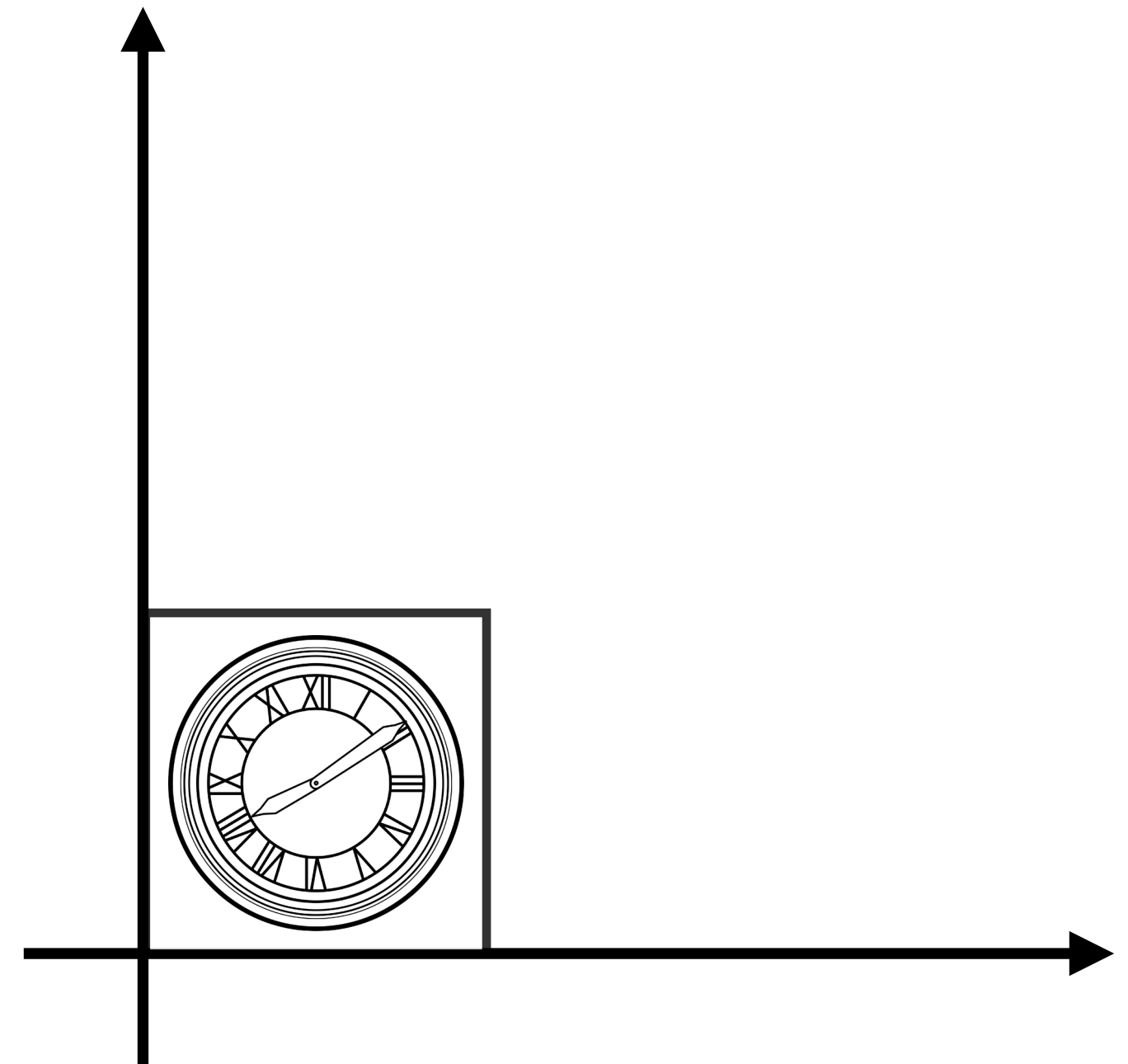
- $(x', y', z') = F(x, y, z)$
- $P' = F(P)$



Scale Transform



$S_{0.5}$

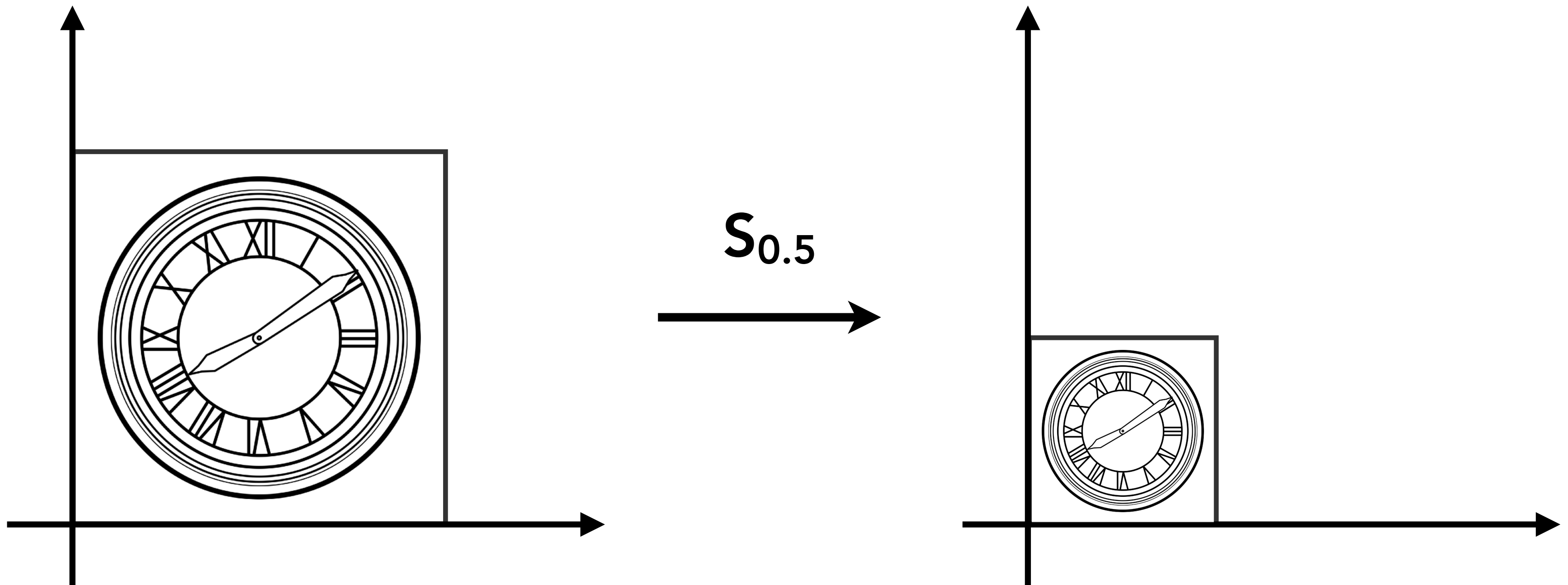


$$x' = sx$$

$$y' = sy$$

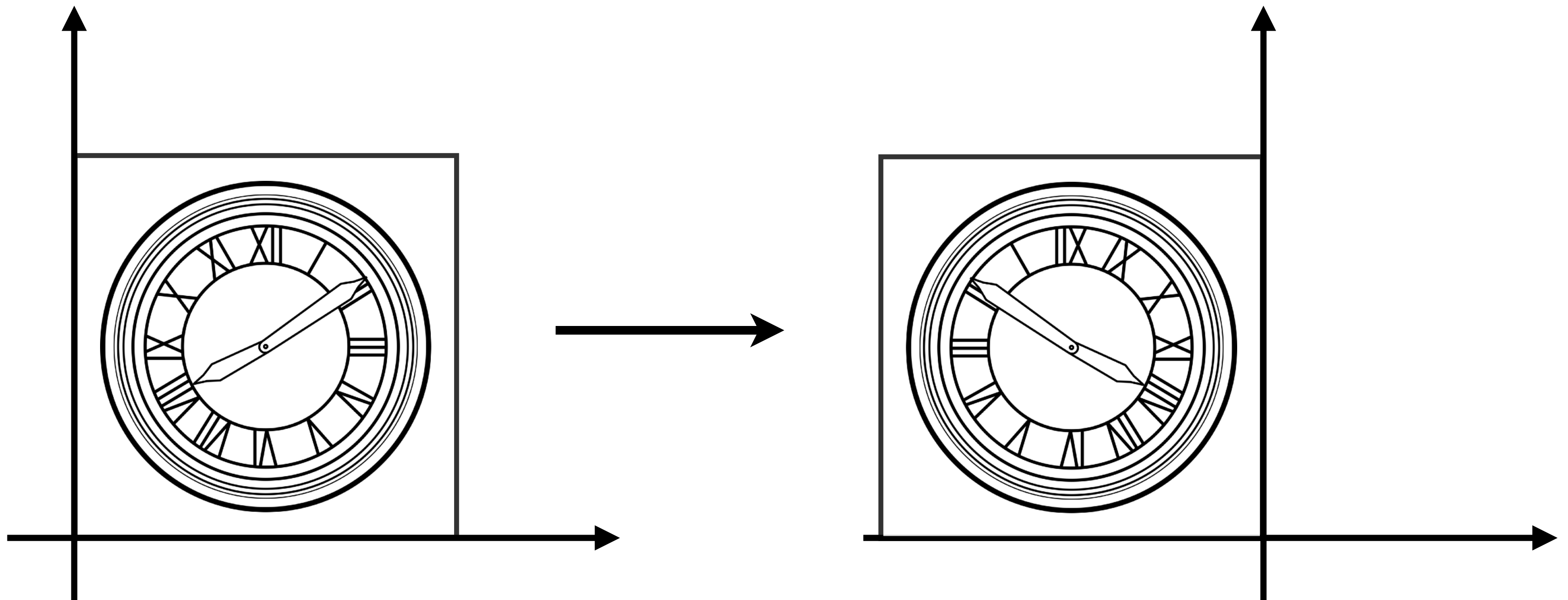
Linear Transforms = Matrices

Scale Matrix



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

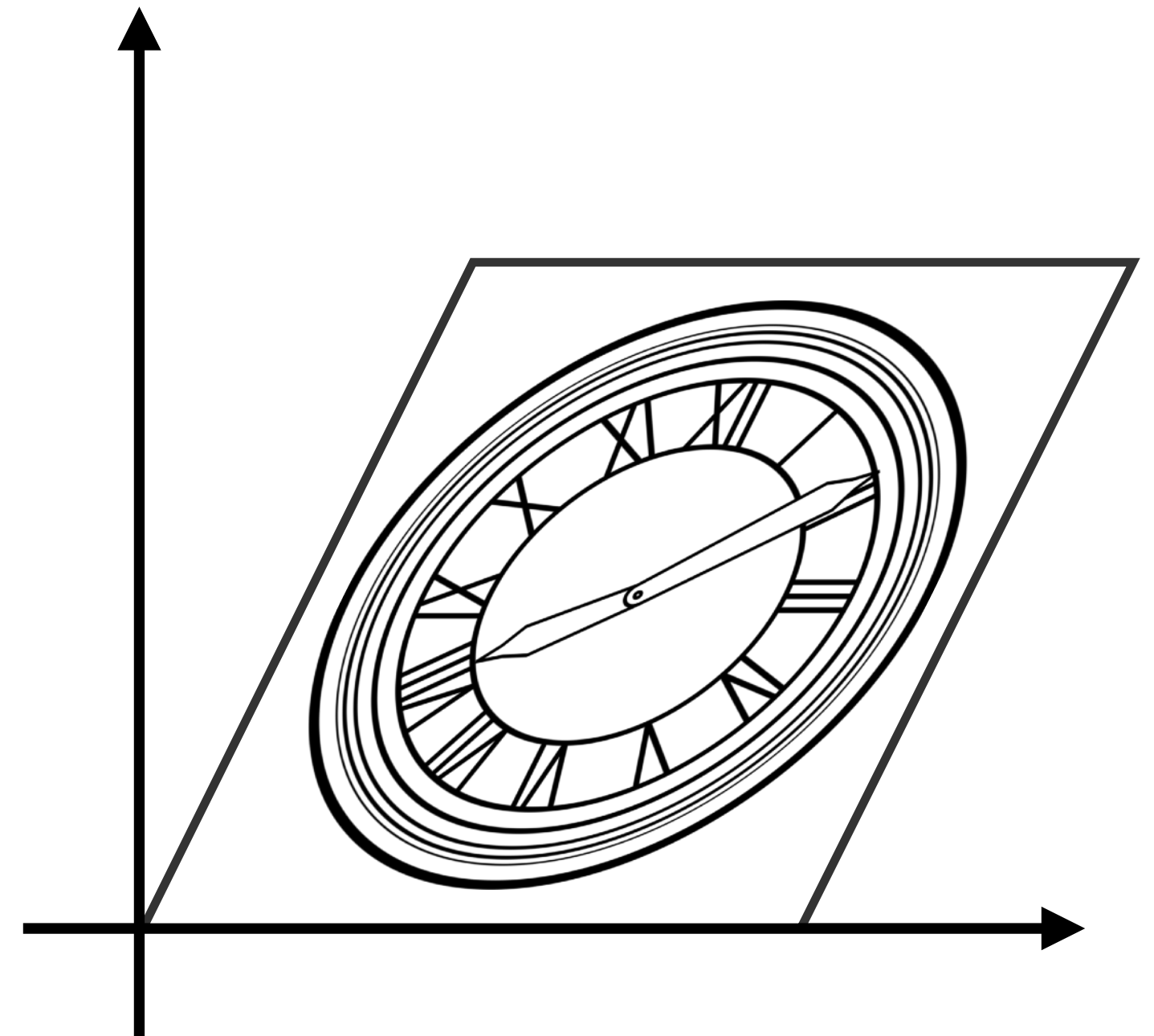
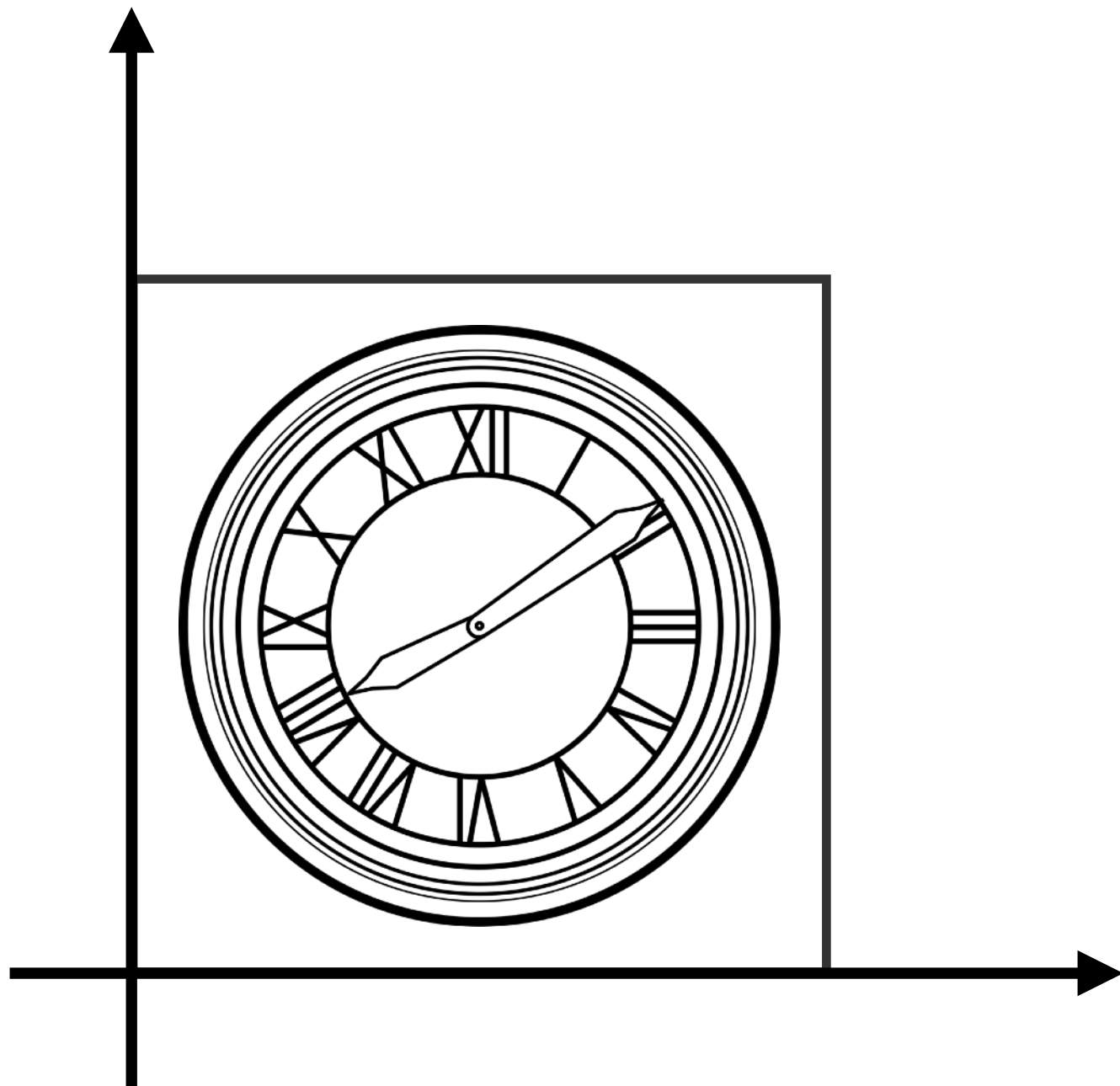
Reflection Matrix



$$x' = ??$$

$$y' = ??$$

Shear Matrix



$$x' = ??$$

$$y' = ??$$

Linear Transforms = Matrices

$$x' = a x + b y$$

$$y' = c x + d y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{M} \mathbf{x}$$

2D Coordinate Systems

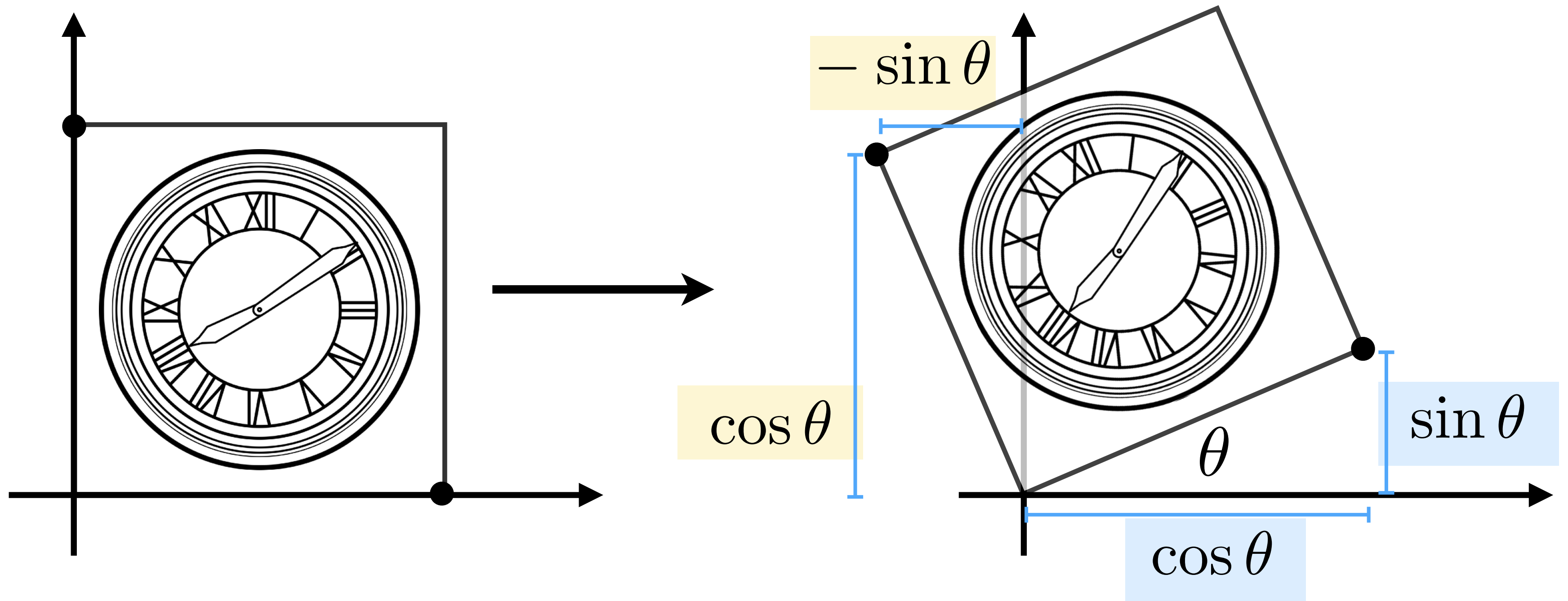
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Can interpret the columns of the matrix as the x and y axes of the coordinate frame

Rotation Matrix



$$\mathbf{R}_{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

2D Rotation Matrix: Another Way

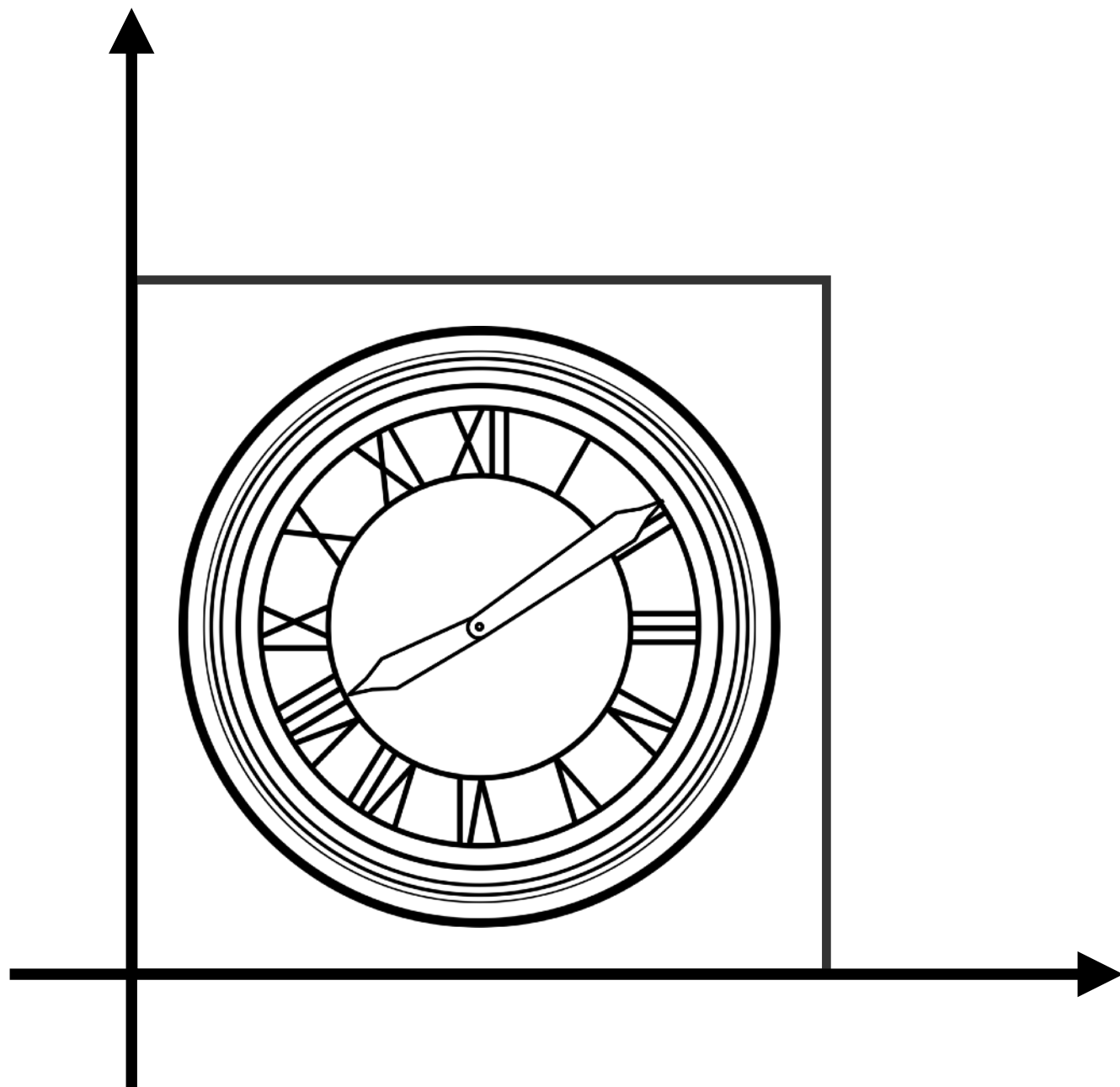


A *WEBCOMIC OF ROMANCE,
SARCASM, MATH, AND LANGUAGE.*

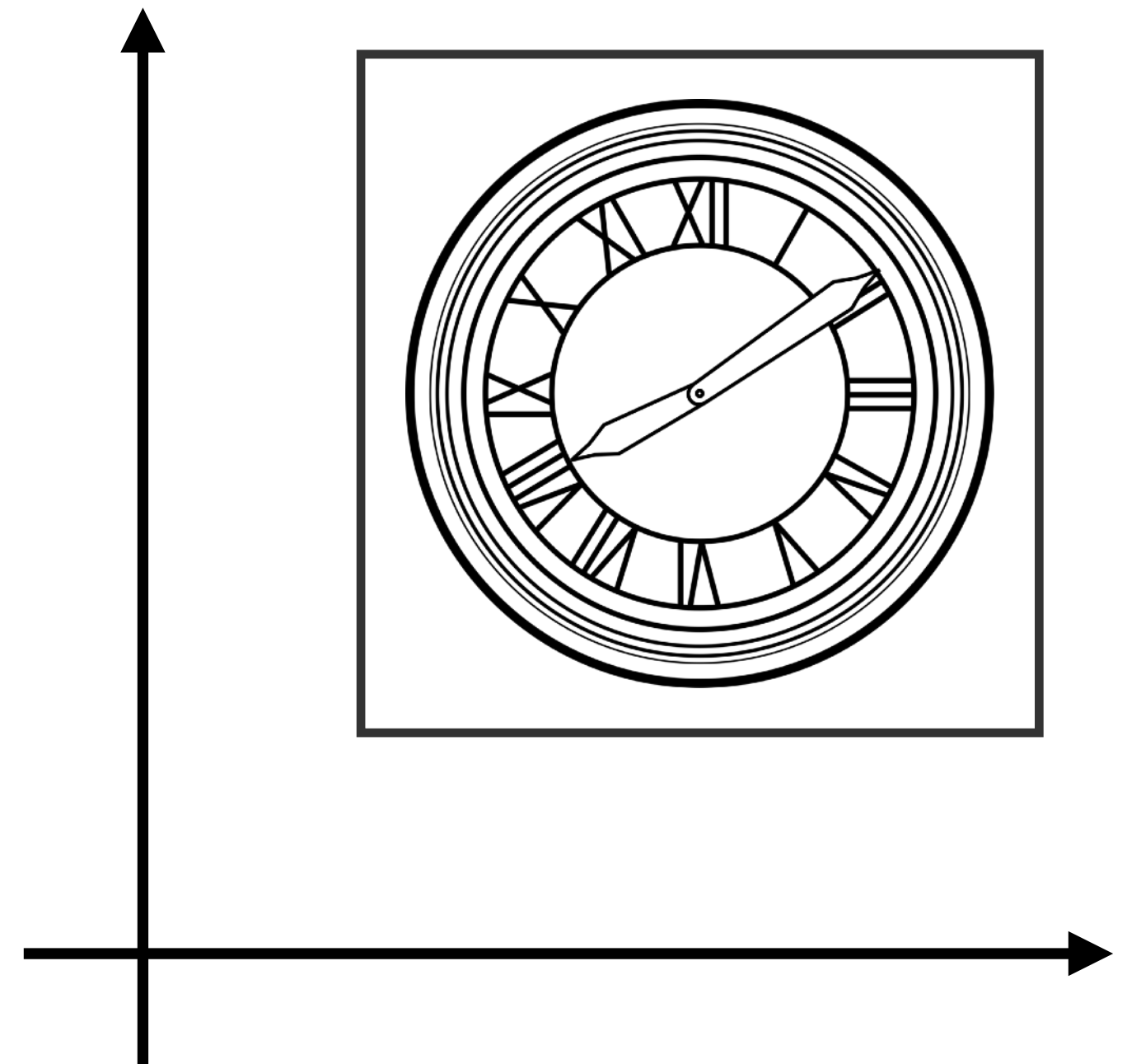
$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \end{bmatrix}$$

<http://xkcd.com/184/>

Translation??



$T_{1,1}$



$$x' = x + t_x$$

$$y' = y + t_y$$

Solution: Homogenous Coordinates

Add a third coordinate (*w-coordinate*)

- 2D point = $(x, y, 1)^T$
- 2D vector = $(x, y, 0)^T$

Now you can express translation as a matrix!!

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

Homogenous Coordinates

Valid operation if w-coordinate of result is 1 or 0

- $\text{vector} + \text{vector} = \text{vector}$
- $\text{point} - \text{point} = \text{vector}$
- $\text{point} + \text{vector} = \text{point}$
- $\text{point} + \text{point} = ??$

2D Transformations

Super set:
"Affine Transform" $Ax + b$

Scale

$$\mathbf{S}(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

"Similarity Transform"

Rotation

$$\mathbf{R}(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Translation

$$\mathbf{T}(t_x, t_y) = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

"Rigid Transform"

Affine Transformations

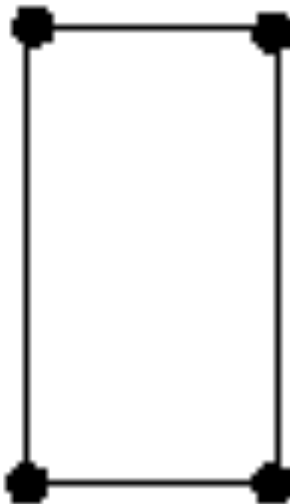
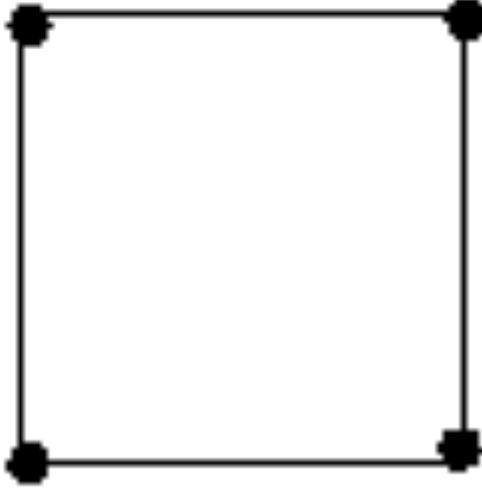
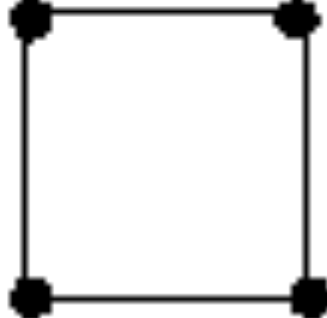
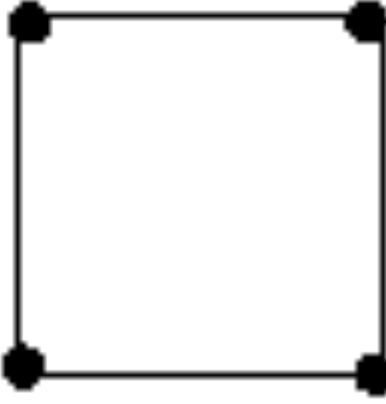
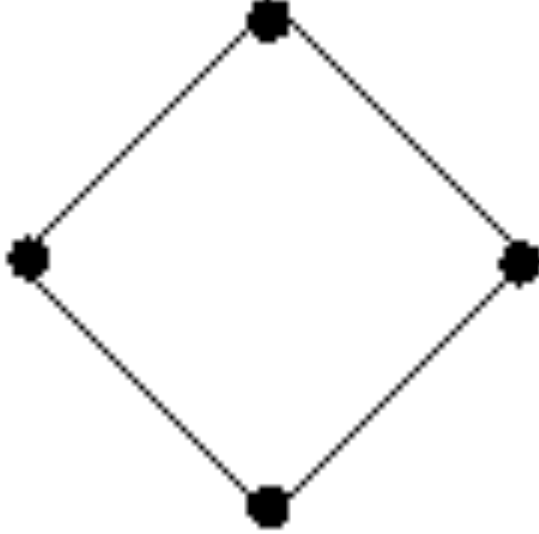
Affine map = linear map + translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Using homogenous coordinates:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

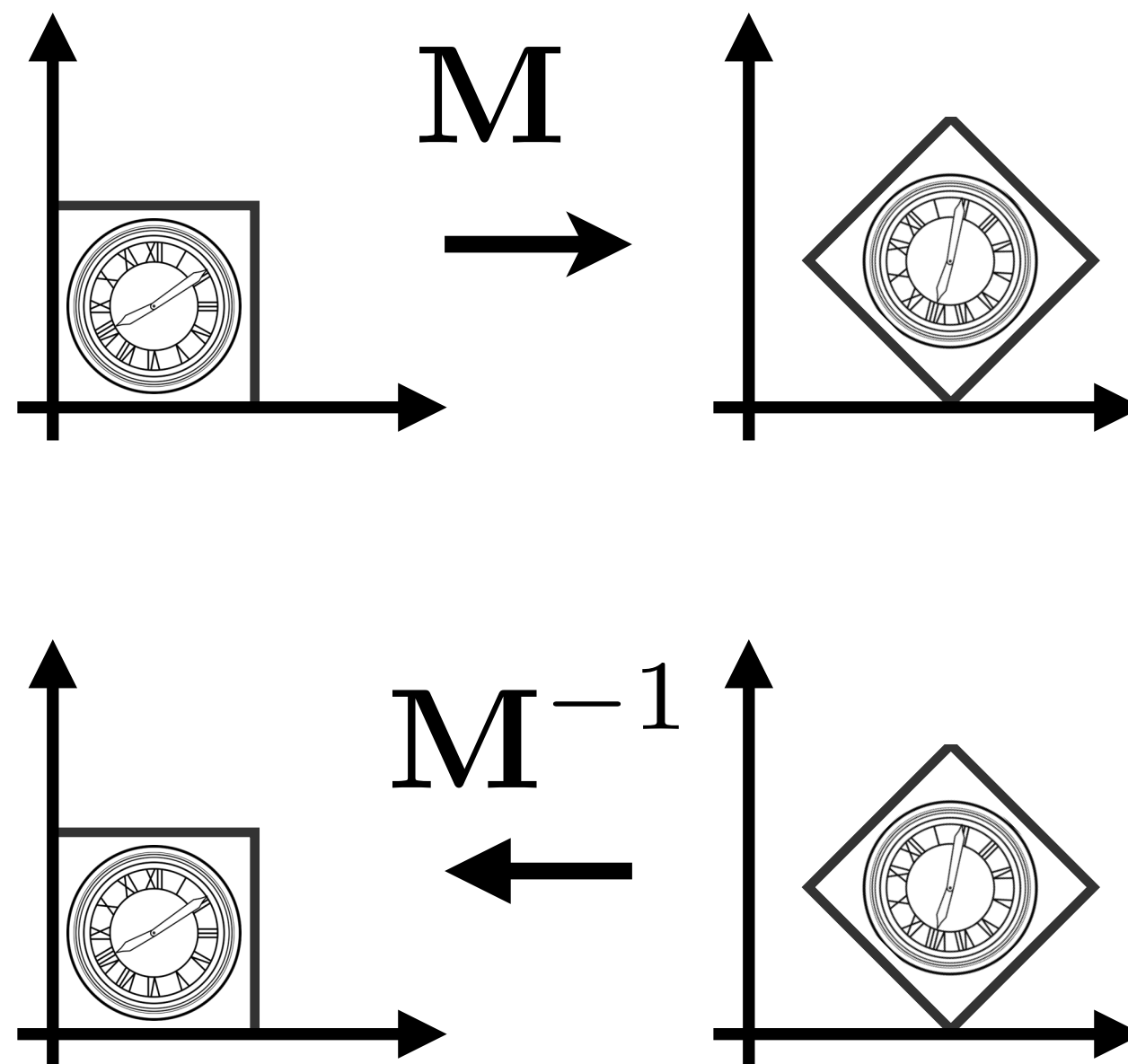
Geometric Intuition

Transformation	Before	After
Affine		
Similarity		
Euclidean		

Inverse Transform

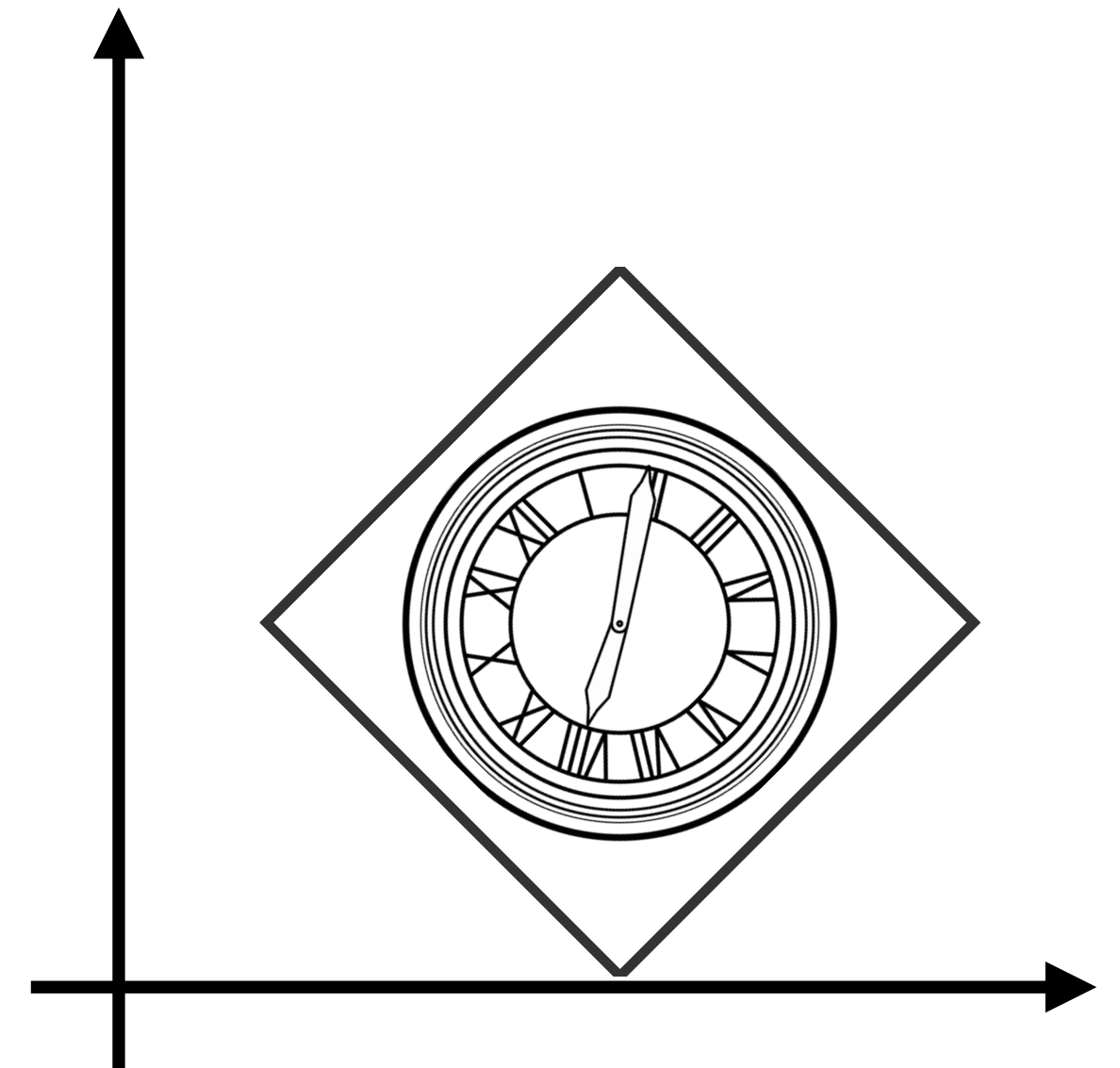
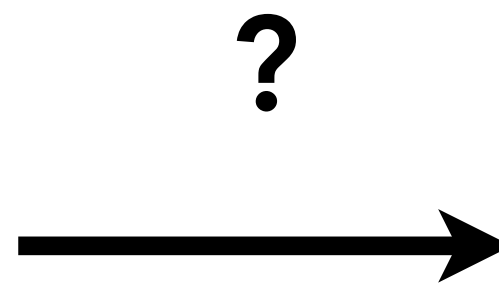
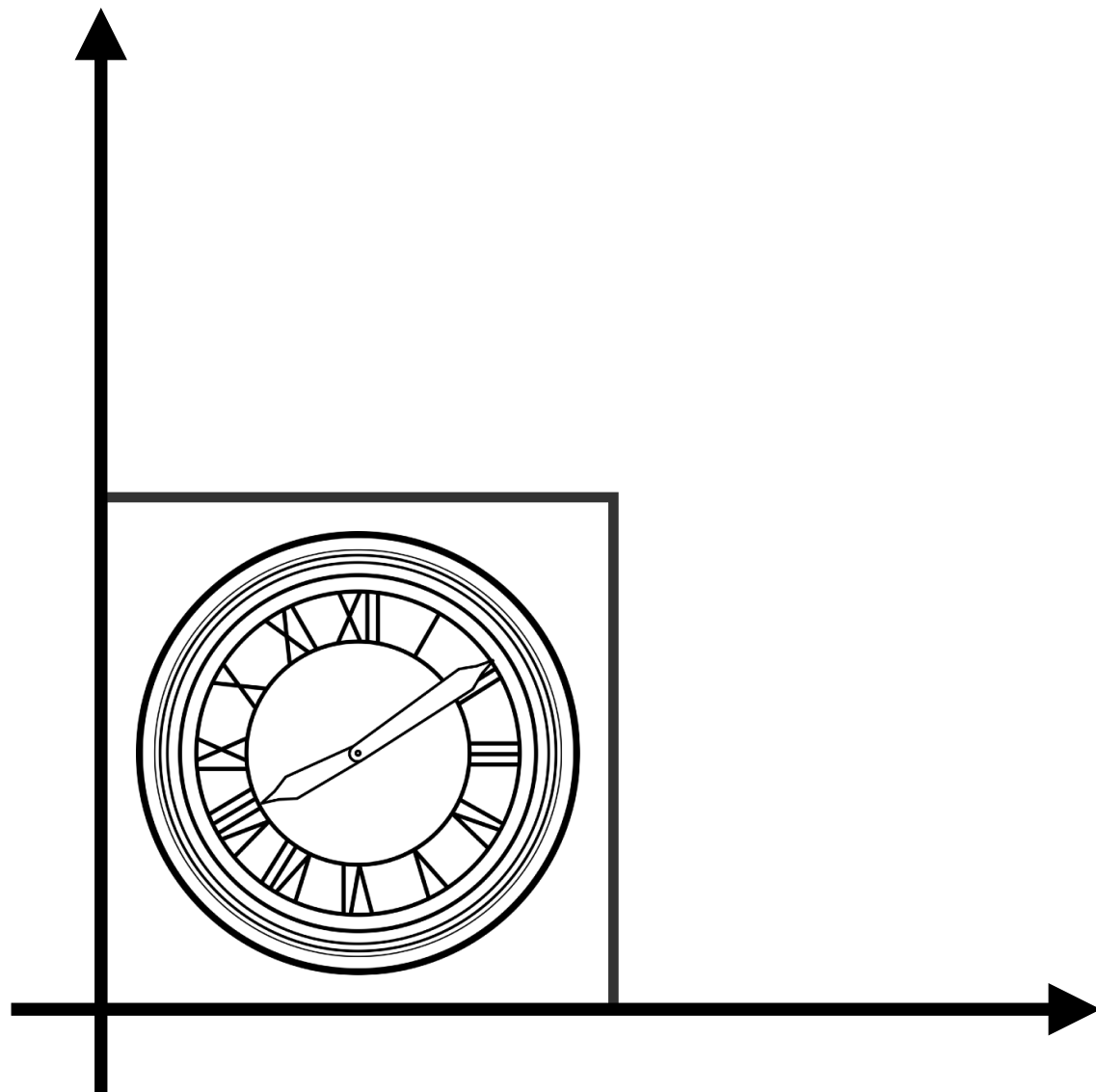
$$\mathbf{M}^{-1}$$

$\mathbf{M}^{-1}$ is the inverse of transform $\mathbf{M}$ in both a matrix and geometric sense

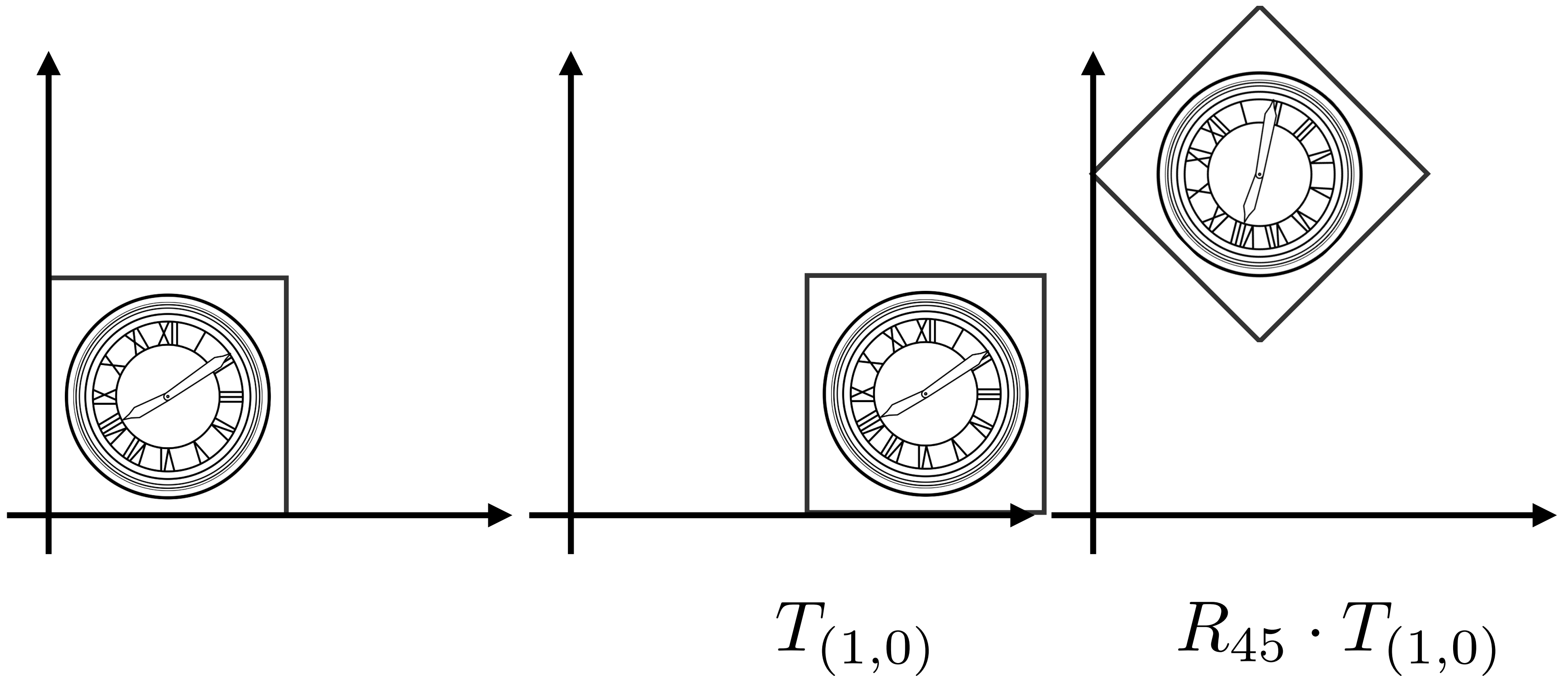


Composing Transforms

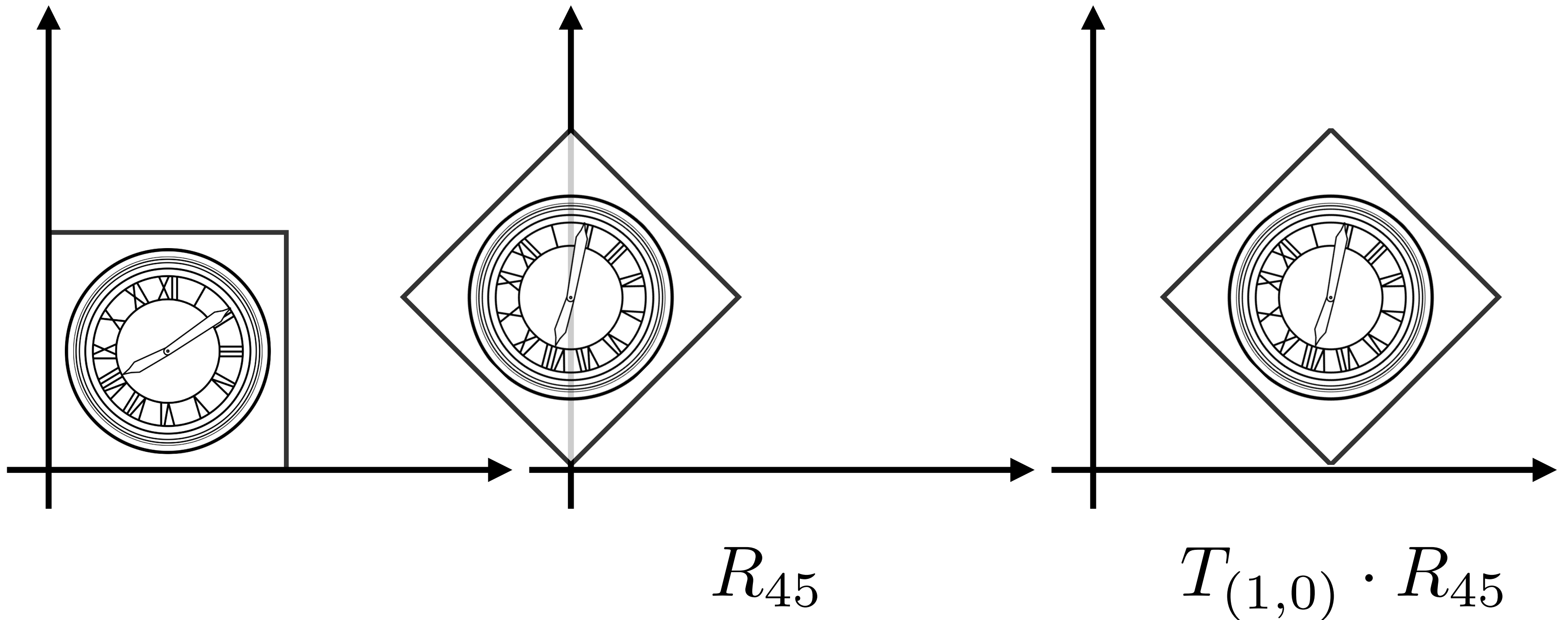
Composite Transform



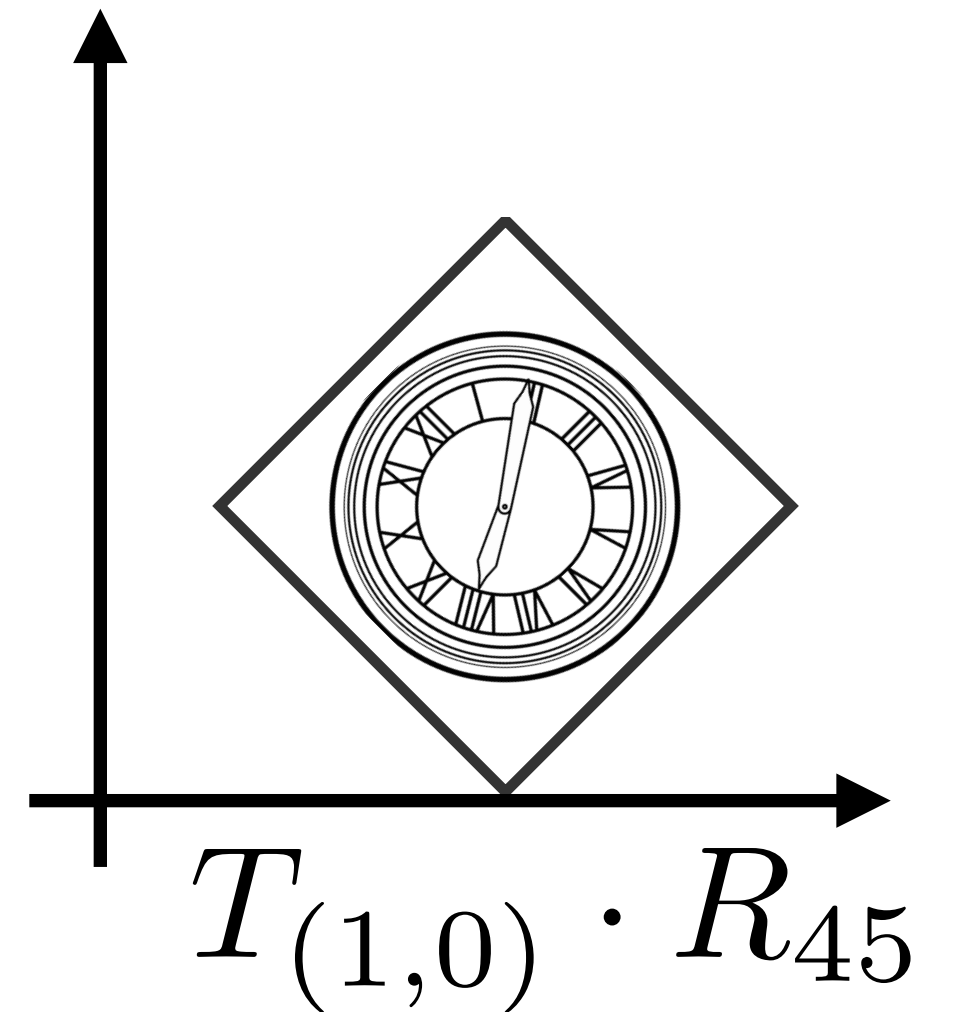
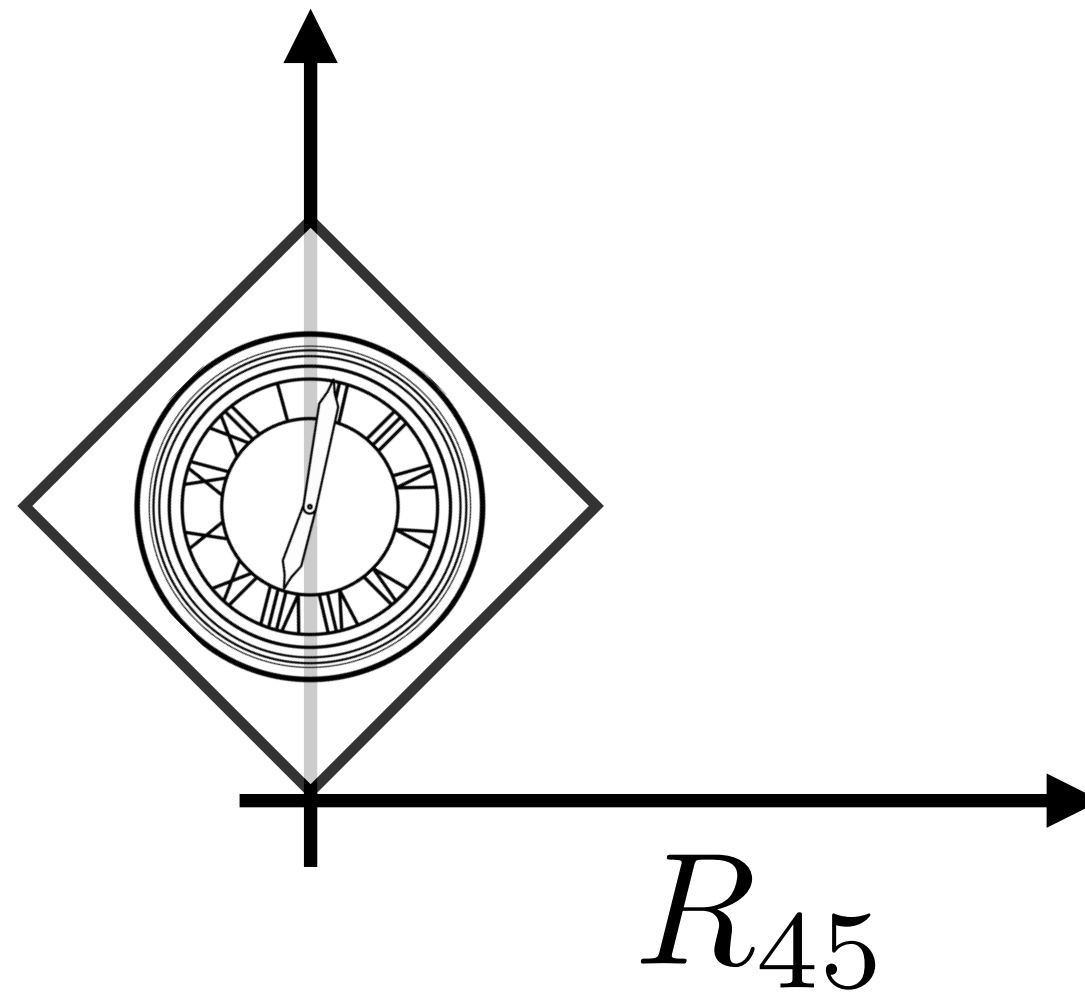
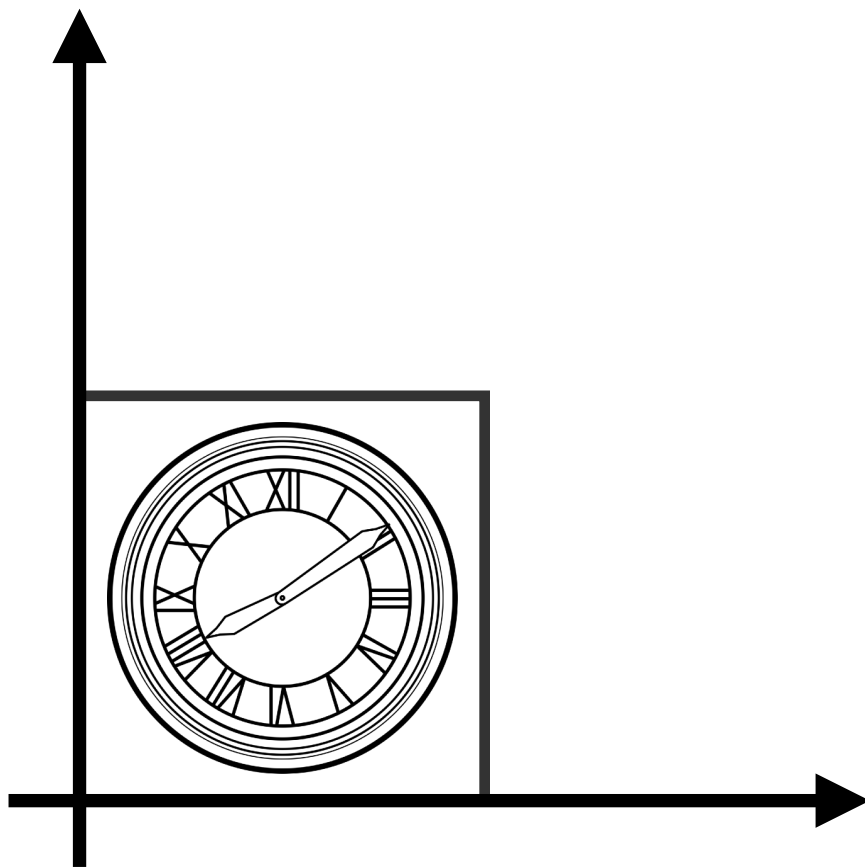
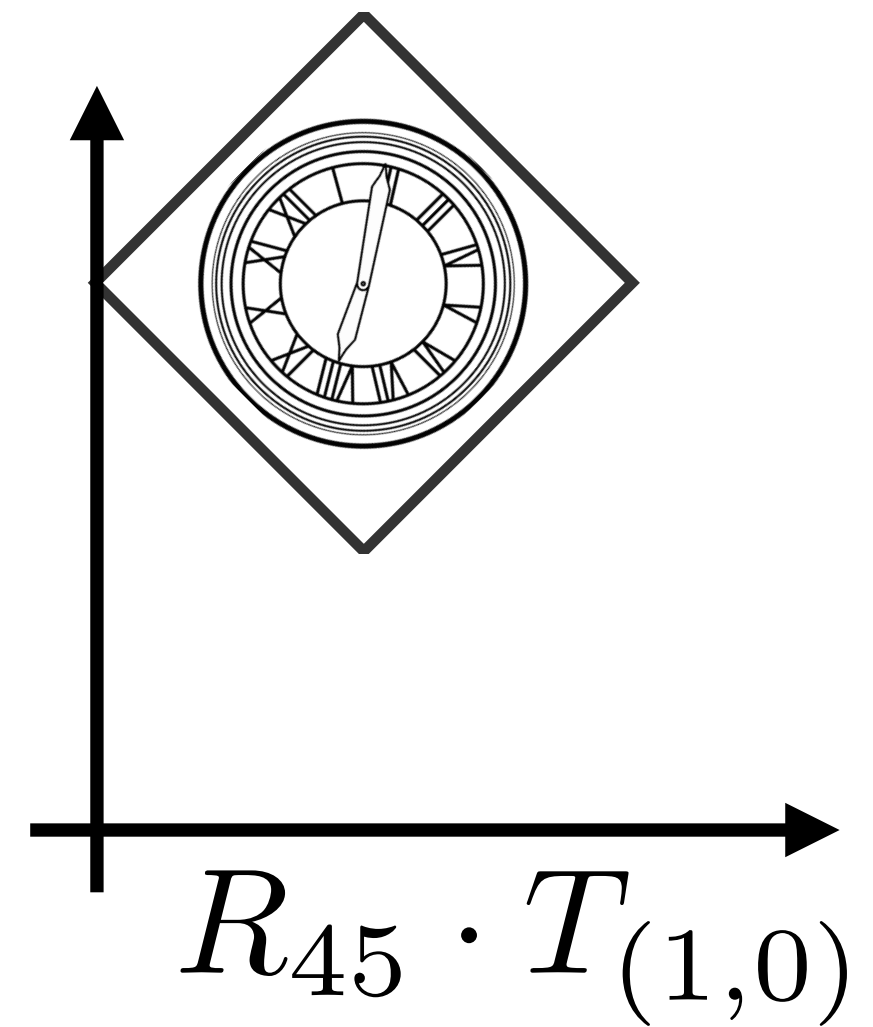
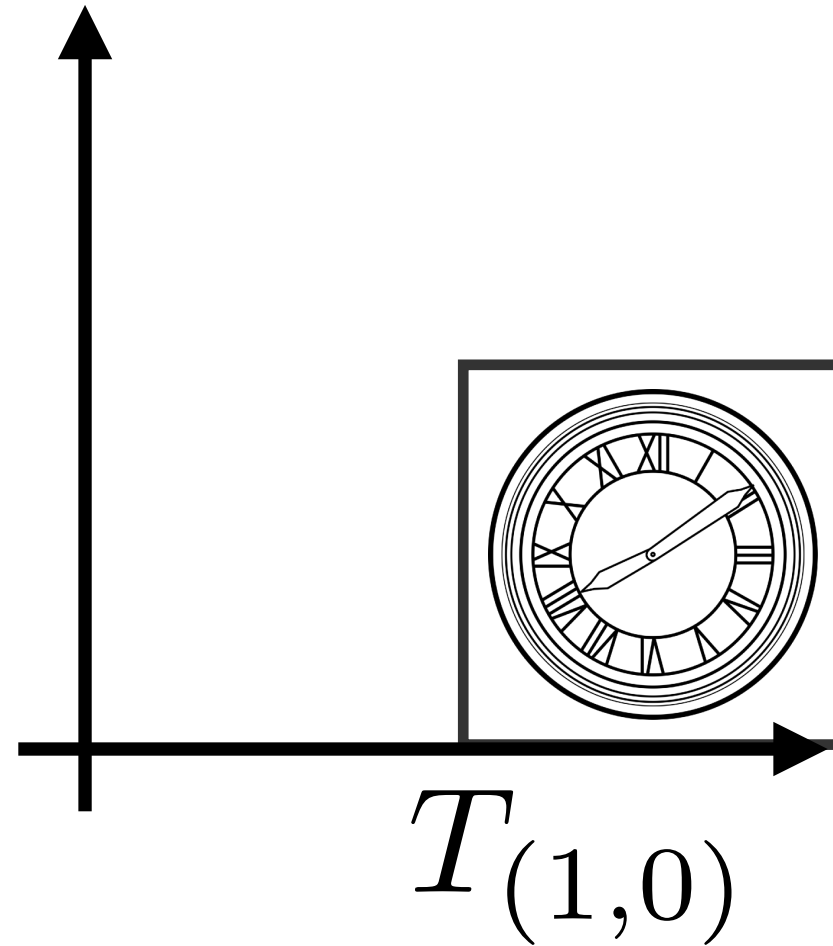
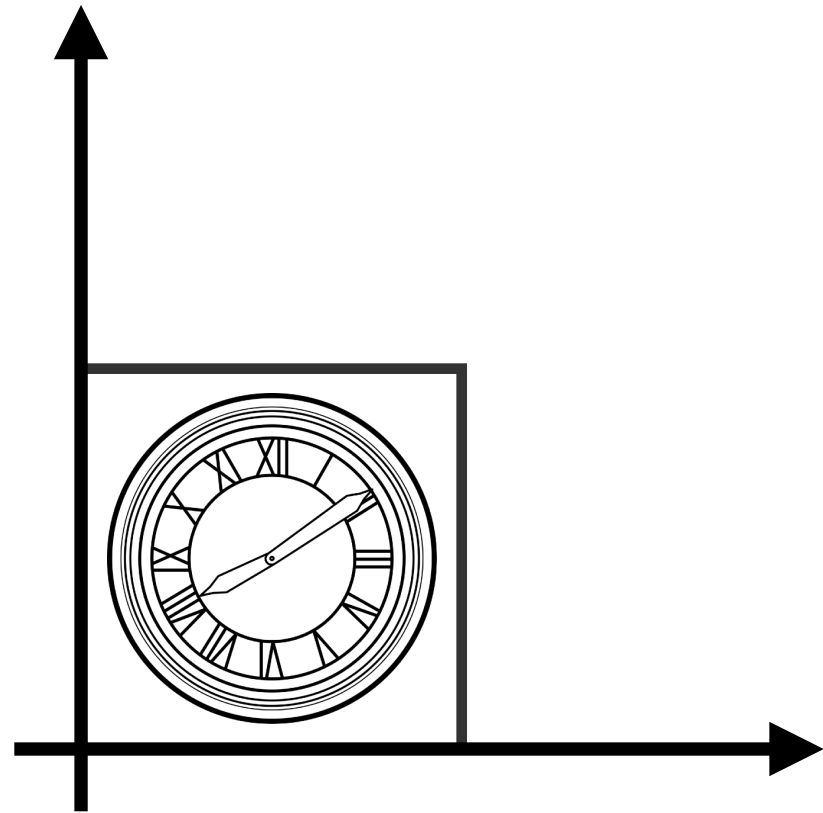
Translate Then Rotate?



Rotate Then Translate



Transform Ordering Matters!



Transform Ordering Matters!

Matrix multiplication is not commutative

$$R_{45} \cdot T_{(1,0)} \neq T_{(1,0)} \cdot R_{45}$$

Recall the matrix math represented by these symbols:

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that matrices are applied right to left:

$$T_{(1,0)} \cdot R_{45} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composing Transforms

Sequence of affine transforms $A_1, A_2, A_3, \dots$

- Compose by matrix multiplication
- Very important for performance!

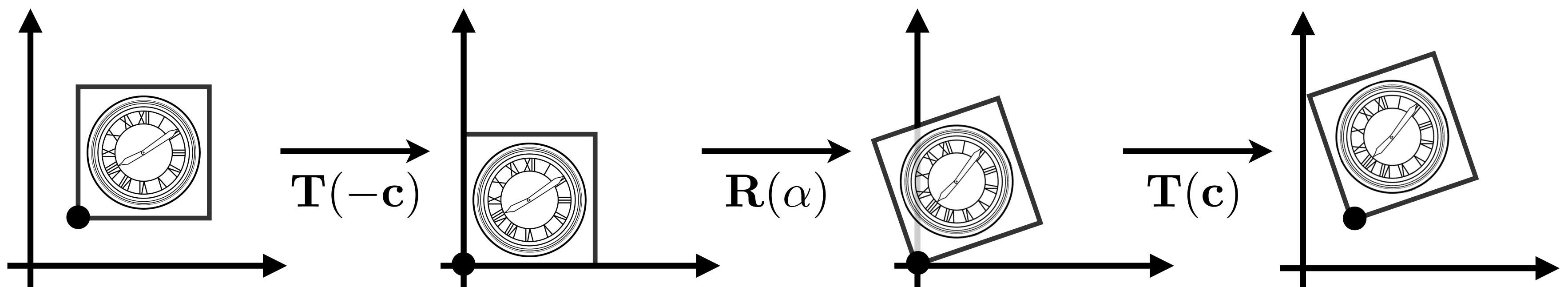
$$A_n(\dots A_2(A_1(\mathbf{x}))) = \underbrace{\mathbf{A}_n \cdots \mathbf{A}_2 \cdot \mathbf{A}_1}_{\text{Pre-multiply } n \text{ matrices to obtain a single matrix representing combined transform}} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Pre-multiply n matrices to obtain a single matrix representing combined transform

Decomposing Complex Transforms

How to rotate around a given point c ?

1. Translate center to origin
2. Rotate
3. Translate back



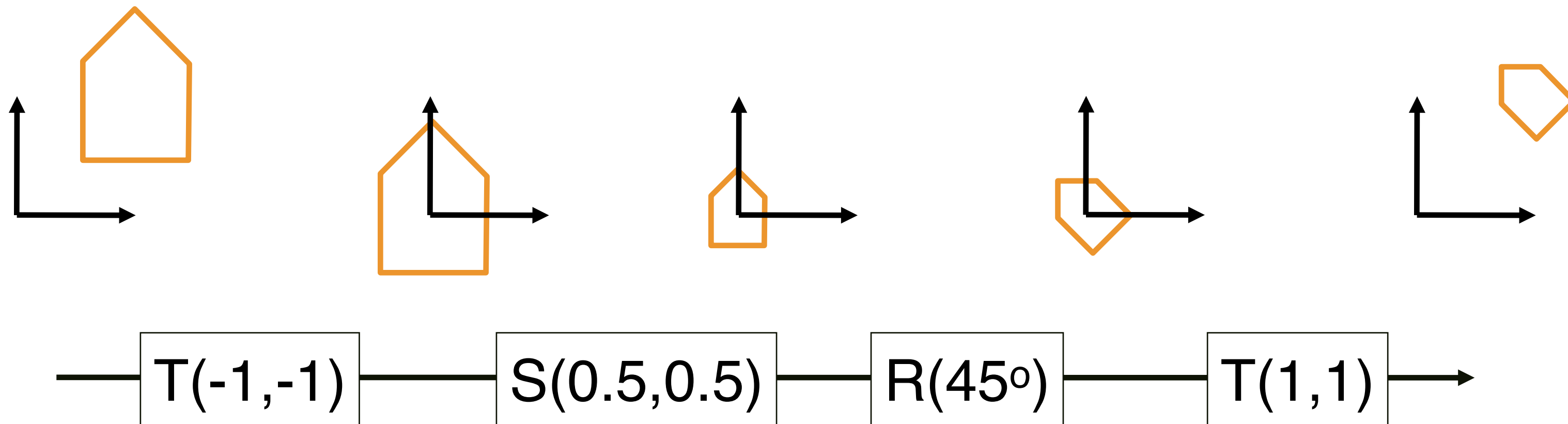
Matrix representation?

$$T(c) \cdot R(\alpha) \cdot T(-c)$$

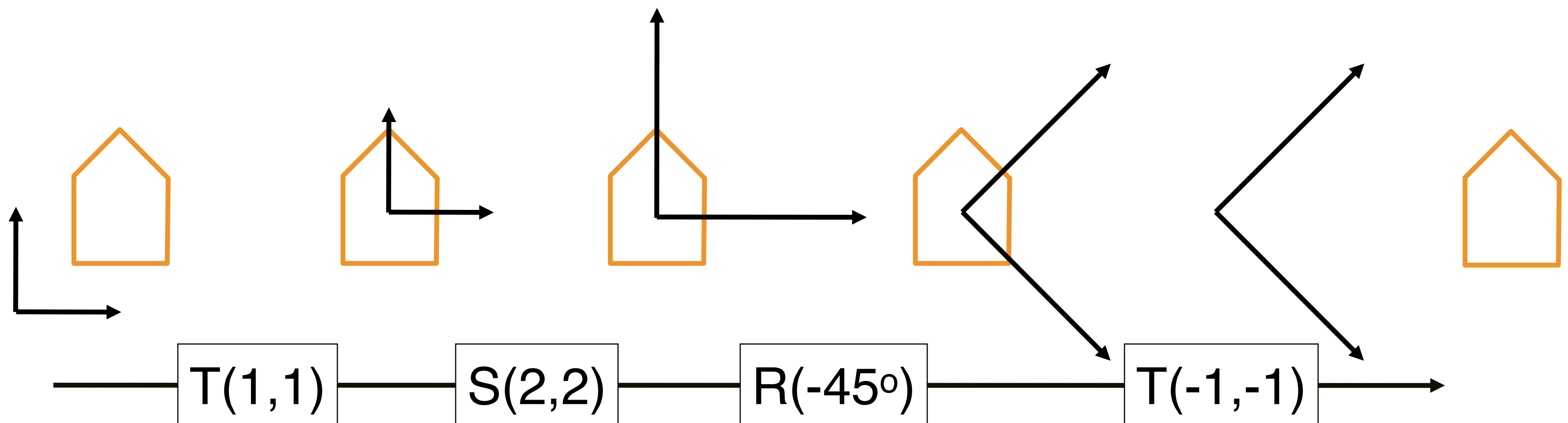
Coordinate Systems

Two Interpretations of A Transform

Interpretation 1: Transforms object points

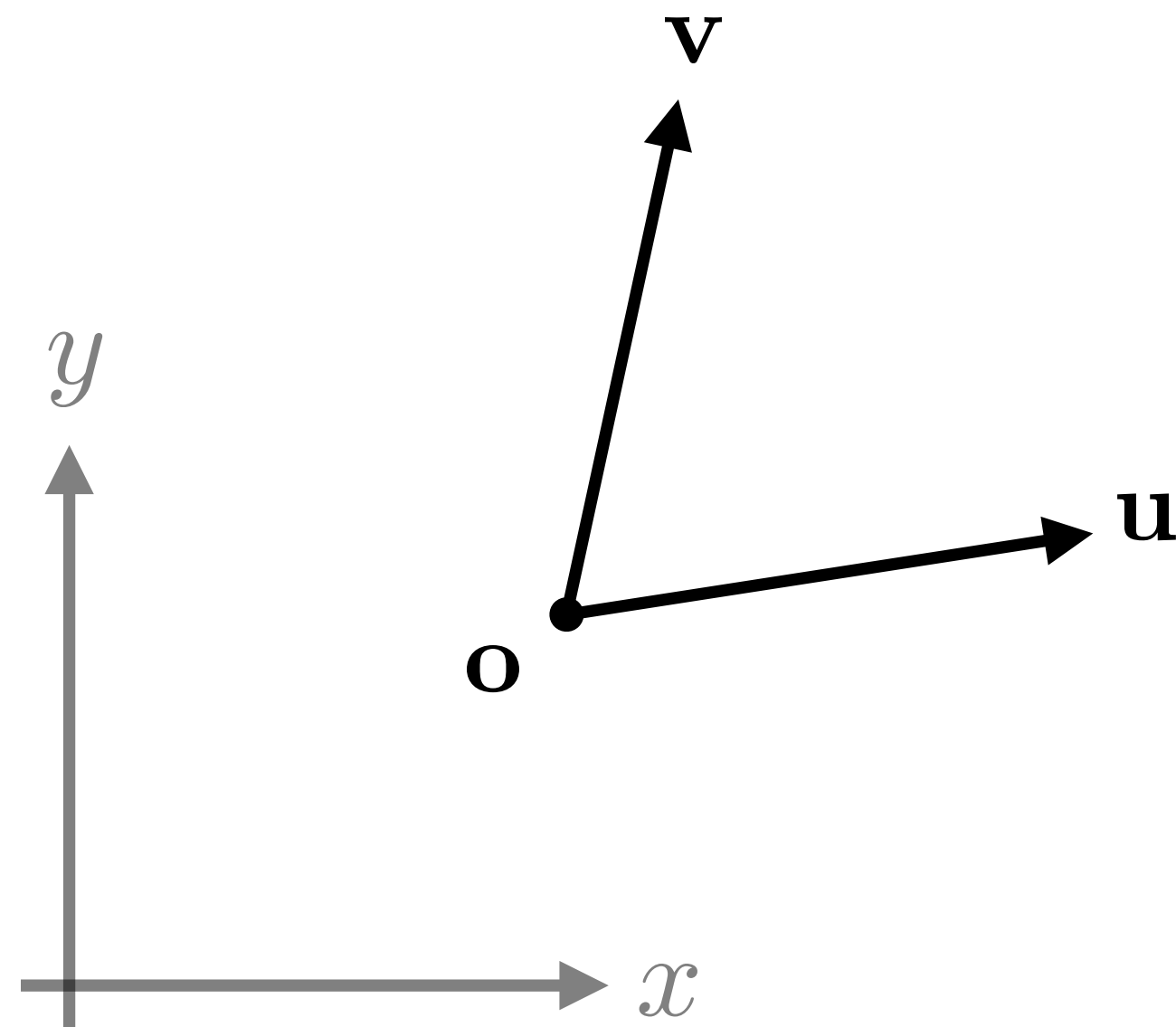


Interpretation 2: Transforms coordinate system



Coordinate System Transform

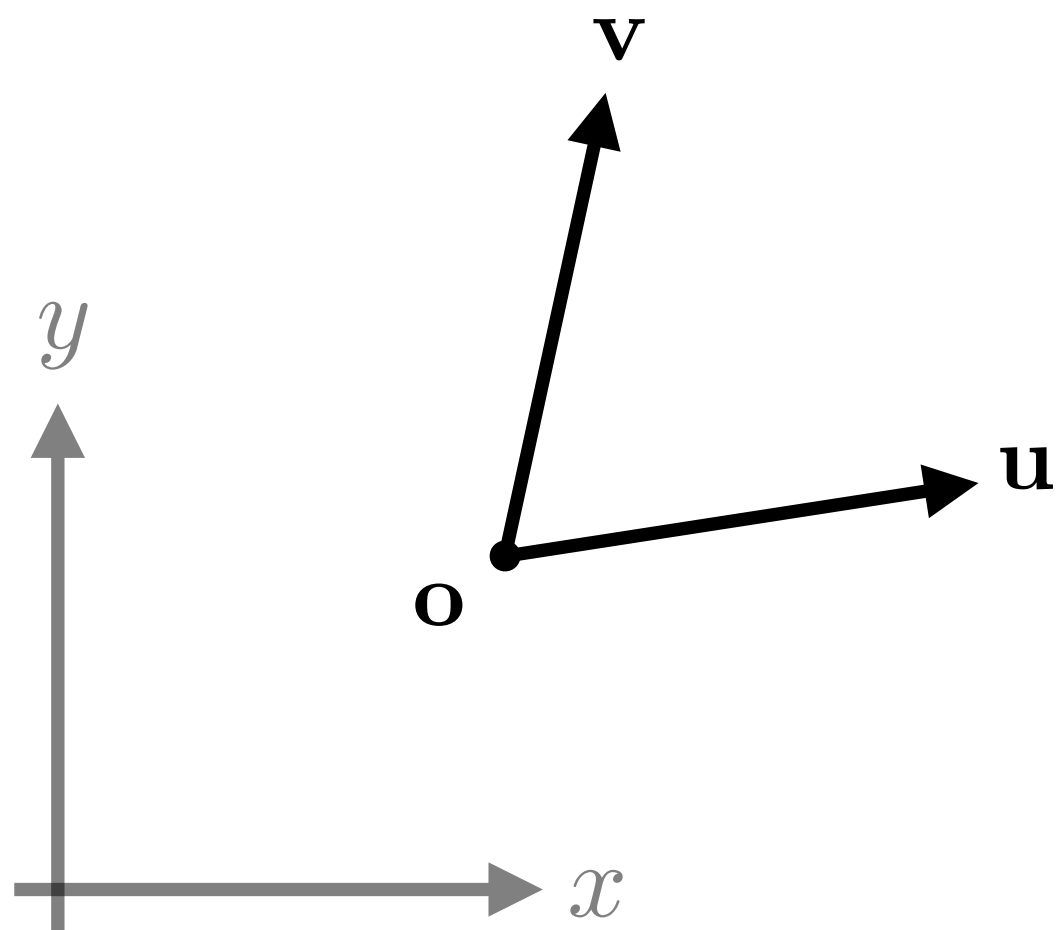
In general, a new coordinate frame is defined by an origin (point) and two unit axes (vectors)



Given coordinates in the (o,u,v) reference frame, what is the transform to coordinates in the (x,y) frame?

Coordinate System Transform Matrix

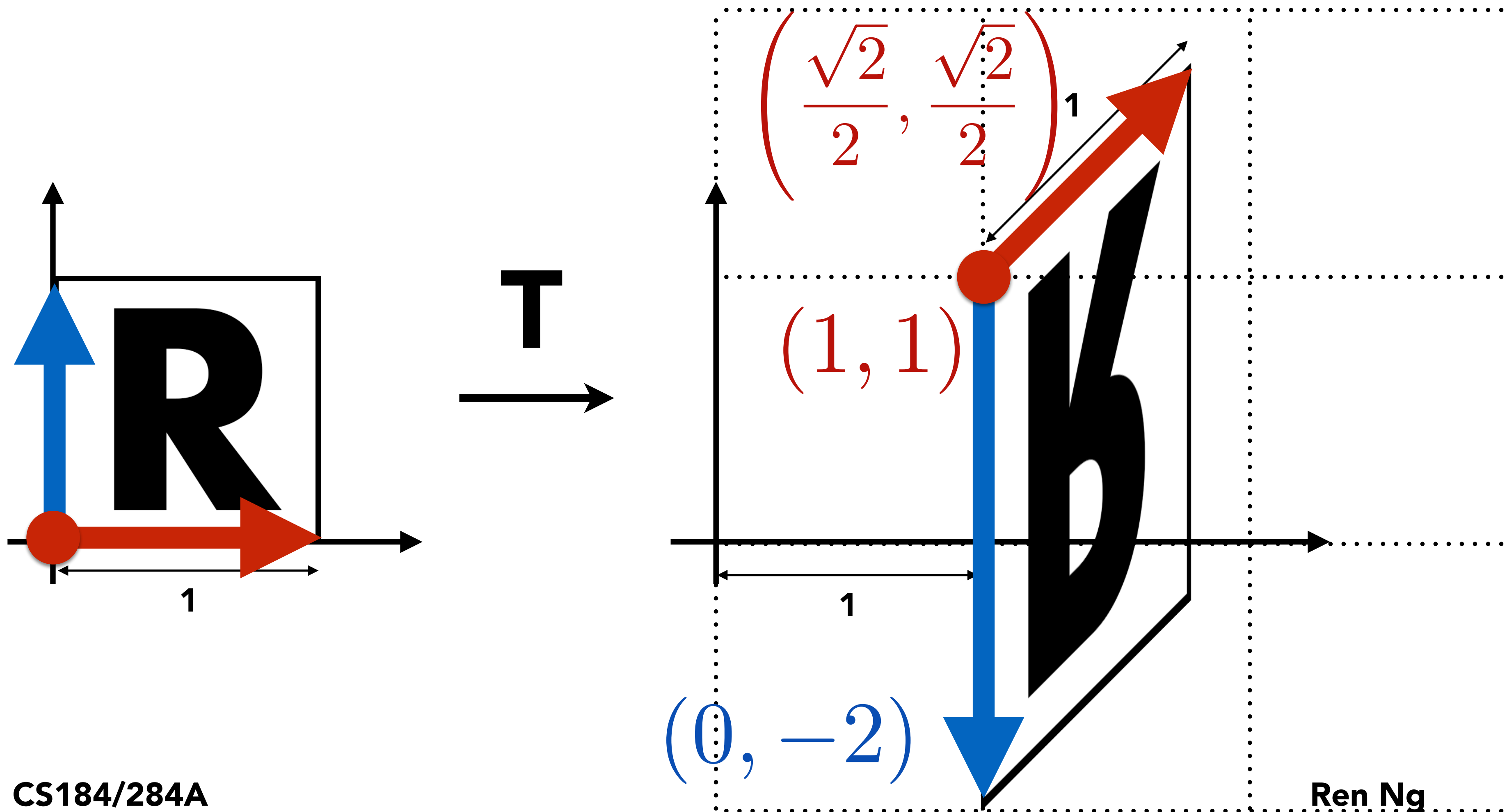
$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{o} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} u_x & v_x & o_x \\ u_y & v_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$



- Columns of matrix are defined by the reference frame's coordinates in the world
- Gives a new way to "read off" columns of matrix

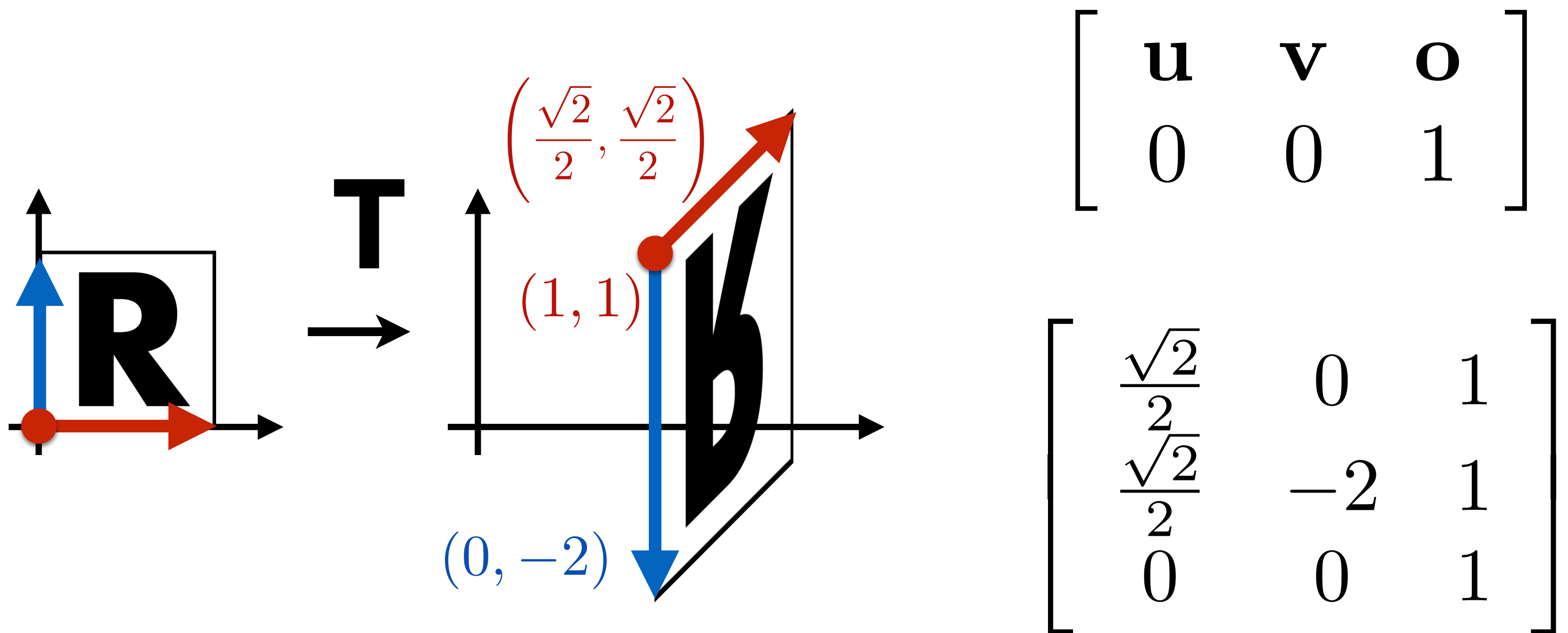
Coordinate System Transform - Example

Write down a matrix T representing this transform:



Coordinate System Transform - Example

Write down a matrix T representing this transform:



3D Transforms

3D Transformations

Use homogeneous coordinates again:

- 3D point = $(x, y, z, 1)^T$
- 3D vector = $(x, y, z, 0)^T$

Use 4×4 matrices for affine transformations

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

3D Transformations

Scale

$$\mathbf{S}(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation

$$\mathbf{T}(t_x, t_y, t_z) = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Coordinate Change (Frame-to-world)

$$\mathbf{F}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{o}) = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{o} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

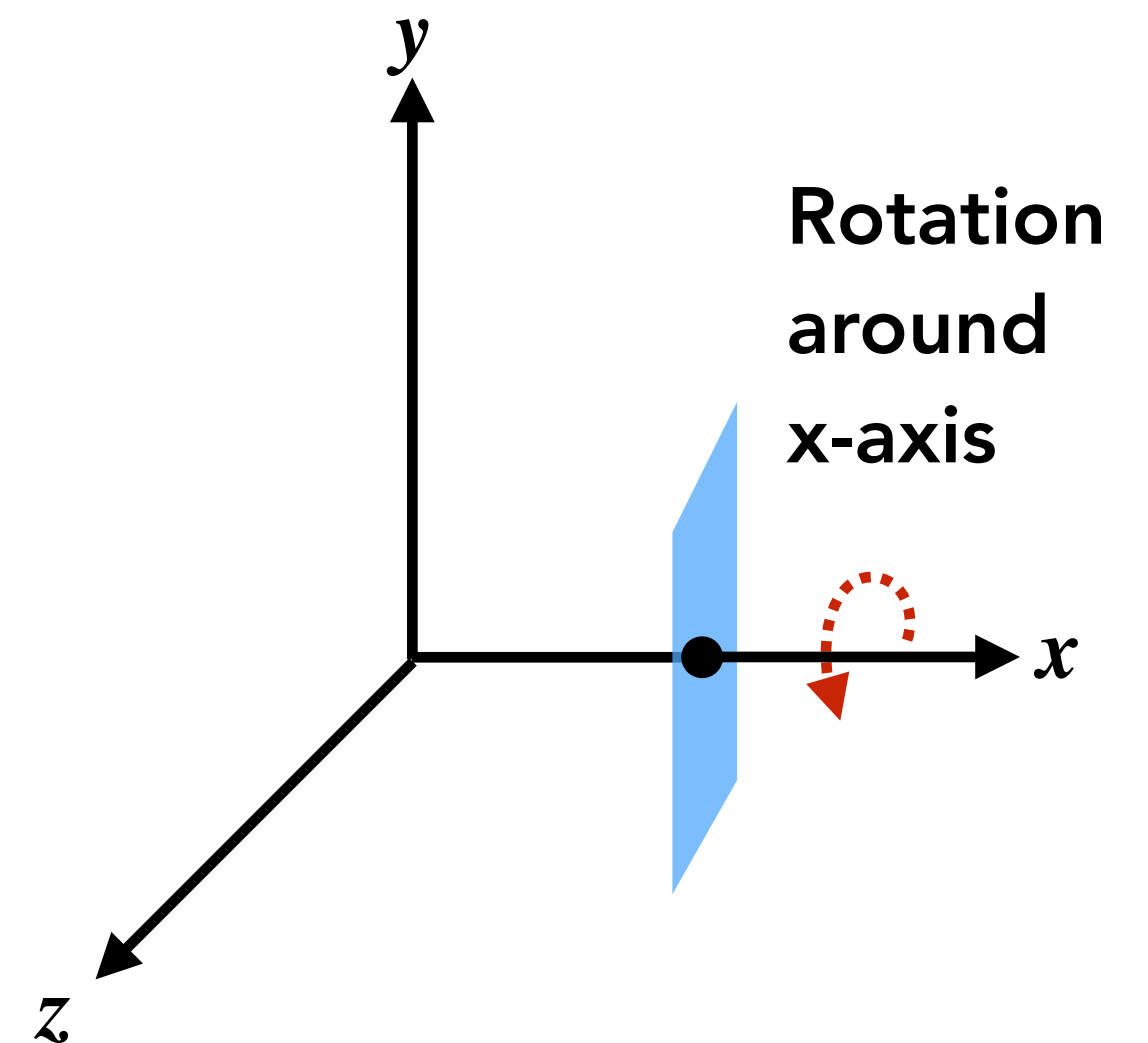
3D Transformations

Rotation around x-, y-, or z-axis

$$\mathbf{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_y(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \alpha & 0 & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

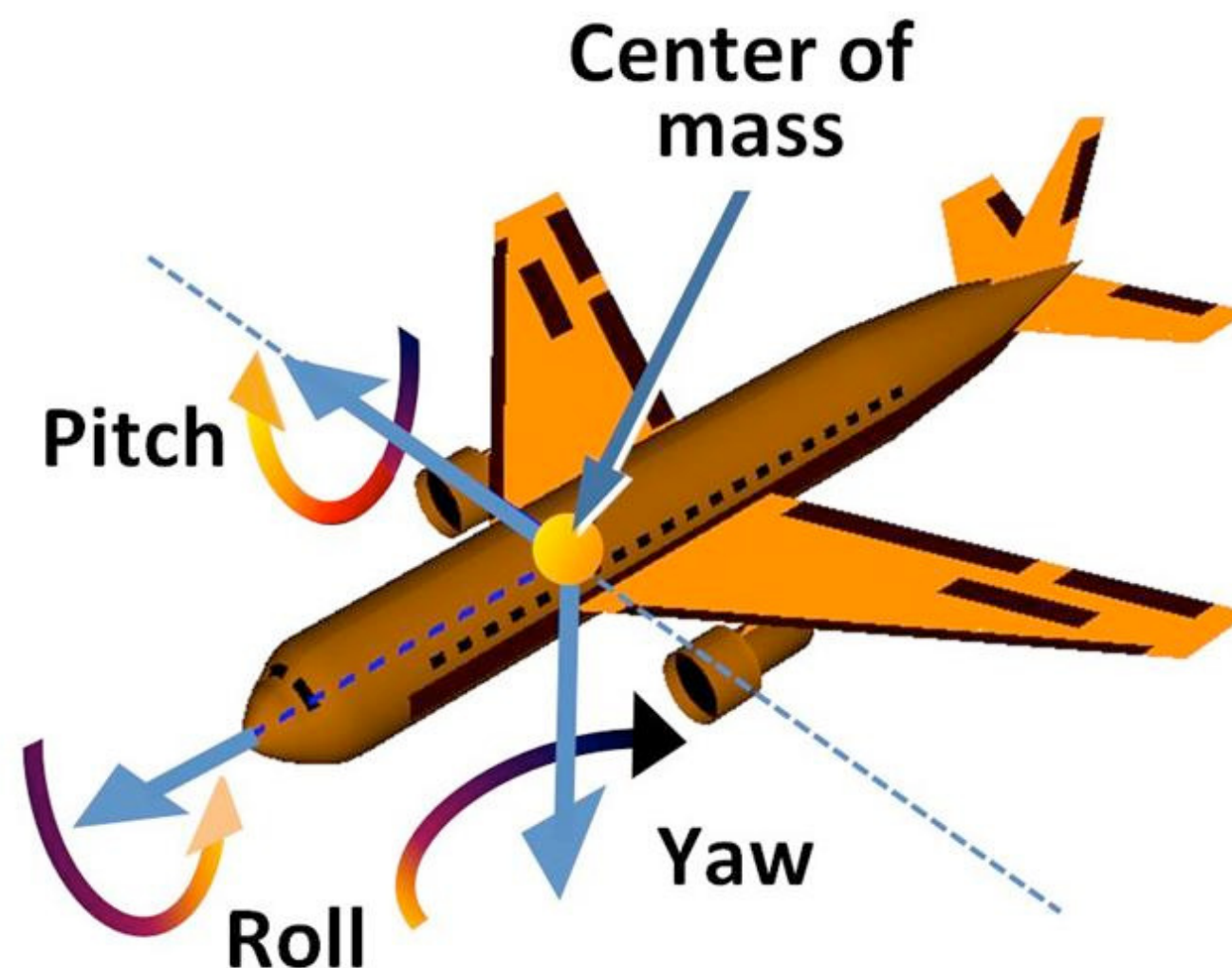


3D Rotations

Compose any 3D rotation from $\mathbf{R}_x$, $\mathbf{R}_y$, $\mathbf{R}_z$?

$$\mathbf{R}_{xyz}(\alpha, \beta, \gamma) = \mathbf{R}_x(\alpha) \mathbf{R}_y(\beta) \mathbf{R}_z(\gamma)$$

- So-called *Euler angles*
- Often used in flight simulators: roll, pitch, yaw



3D Rotations

Compose any 3D rotation from $\mathbf{R}_x$, $\mathbf{R}_y$, $\mathbf{R}_z$?

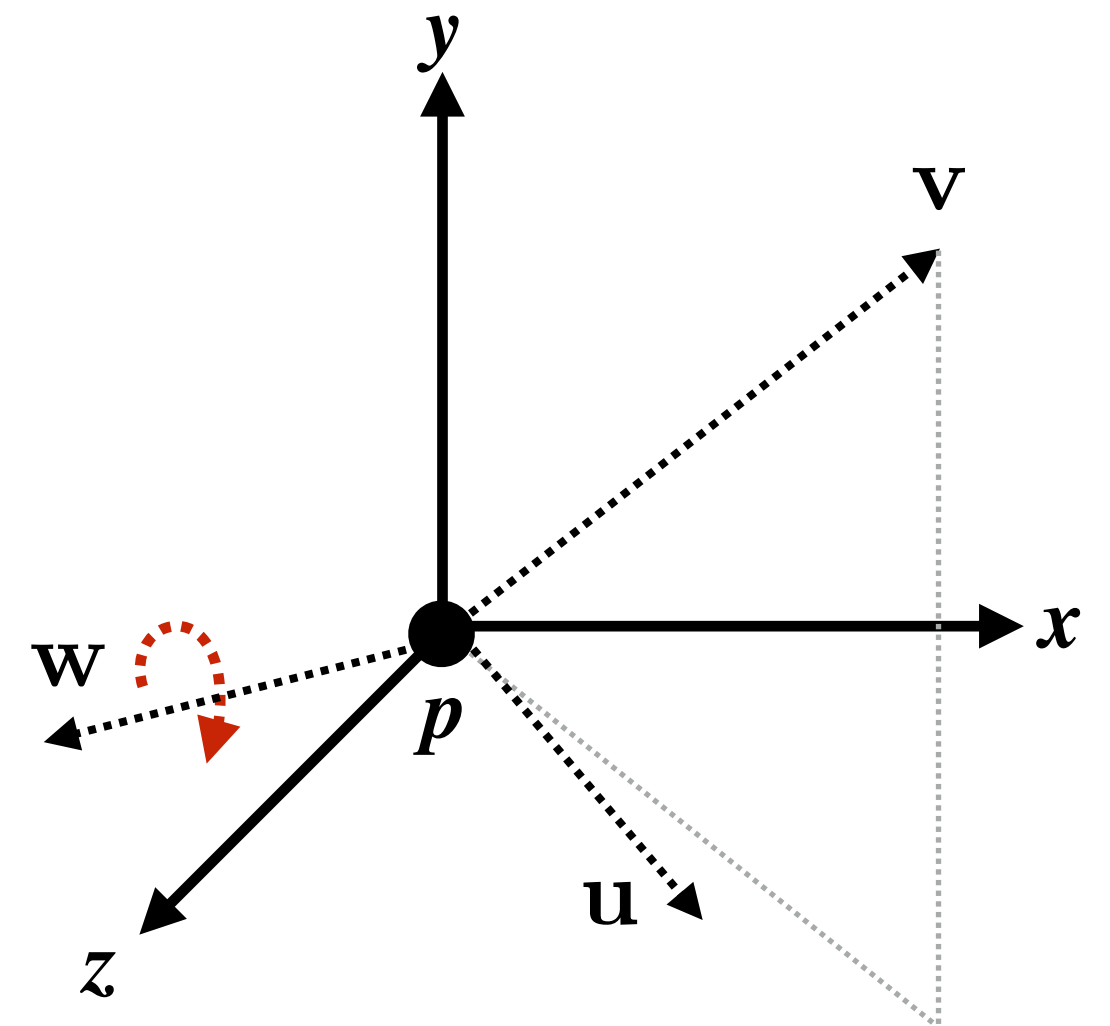
$$\mathbf{R}_{xyz}(\alpha, \beta, \gamma) = \mathbf{R}_x(\alpha) \mathbf{R}_y(\beta) \mathbf{R}_z(\gamma)$$

- So-called *Euler angles*
- Often used in flight simulators: roll, pitch, yaw

3D Rotation Around Arbitrary Axis

Construct orthonormal frame transformation F with p, u, v, w , where p and w match the rotation axis

Apply the transform $(F R_z(\theta) F^{-1})$



Interpretation:

- Move to Z axis, rotate, then move back

Rodrigues' Rotation Formula

Rotation by angle α around axis $\mathbf{n}$

$$\mathbf{R}(\mathbf{n}, \alpha) = \cos(\alpha) \mathbf{I} + (1 - \cos(\alpha)) \mathbf{n} \mathbf{n}^T + \sin(\alpha) \underbrace{\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}}_{\mathbf{N}}$$

How to prove this magic formula?

- Matrix $\mathbf{N}$ computes a cross-product: $\mathbf{N} \mathbf{x} = \mathbf{n} \times \mathbf{x}$
- Assume orthonormal system $\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$

$$\mathbf{R} \mathbf{n} = \mathbf{n}$$

$$\mathbf{R} \mathbf{e}_1 = \cos \alpha \mathbf{e}_1 + \sin \alpha \mathbf{e}_2$$

$$\mathbf{R} \mathbf{e}_2 = -\sin \alpha \mathbf{e}_1 + \cos \alpha \mathbf{e}_2$$

Many Other Representations of Rotations

Quaternions

Exponential map

...

Hierarchical Transforms

Skeleton - Linear Representation

head

torso

right upper arm

right lower arm

right hand

left upper arm

left lower arm

left hand

right upper leg

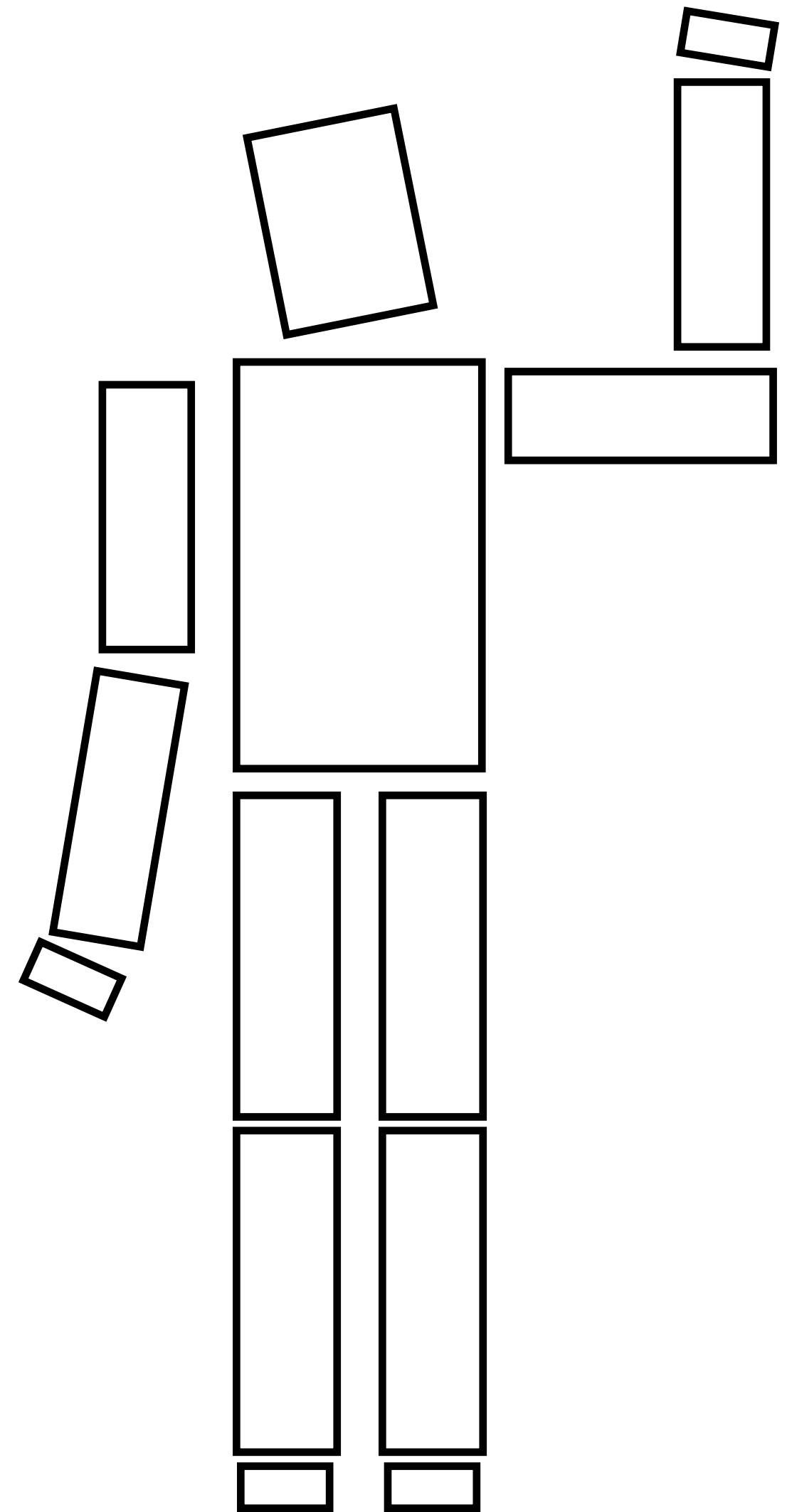
right lower leg

right foot

left upper leg

left lower leg

left foot



Linear Representation

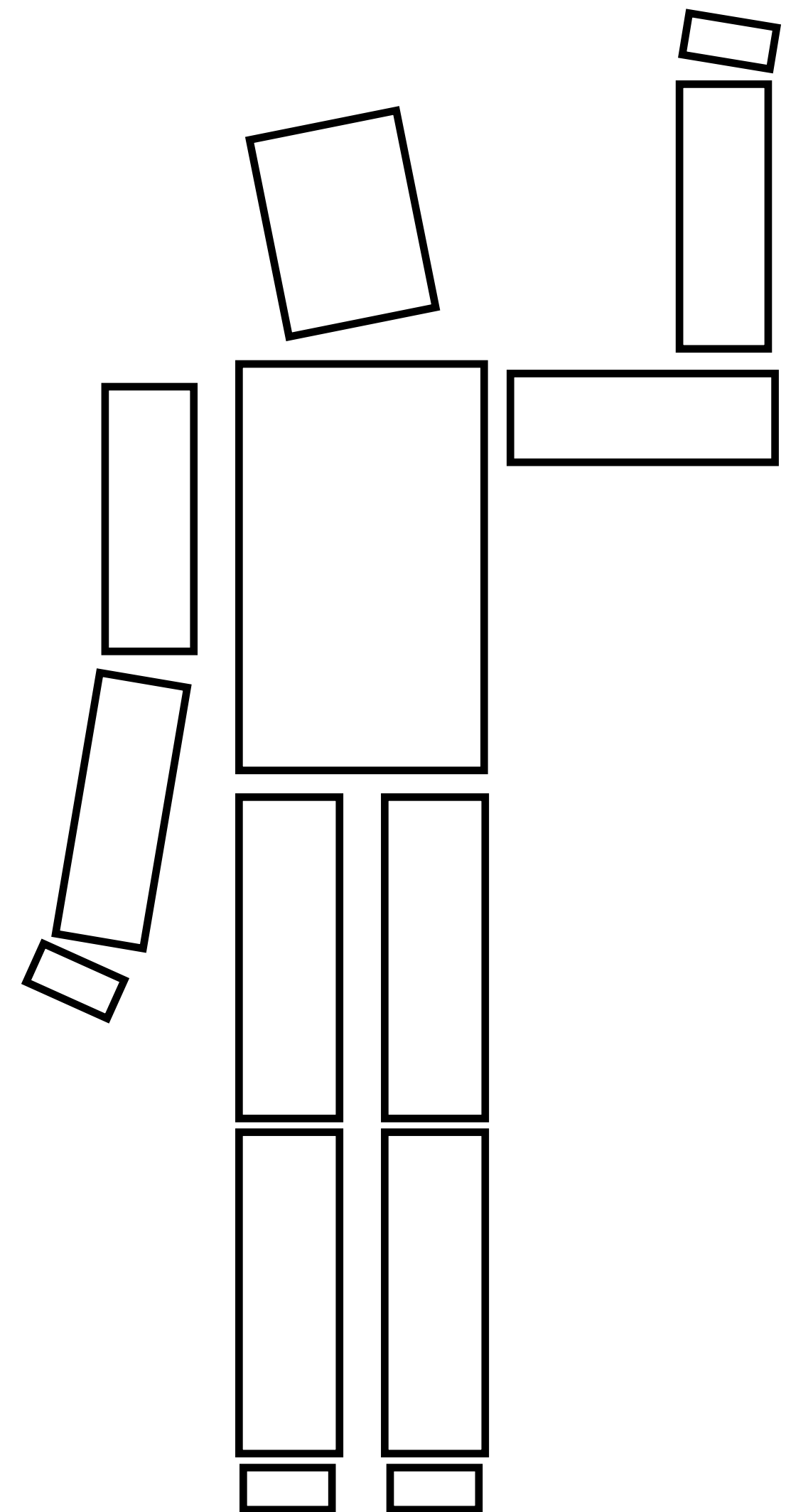
Each shape associated with its own transform

A single edit can require updating many transforms

- E.g. raising arm requires updating transforms for all arm parts

Skeleton - Hierarchical Representation

torso
head
right arm
 upper arm
 lower arm
 hand
left arm
 upper arm
 lower arm
 hand
right leg
 upper leg
 lower leg
 foot
left leg
 upper leg
 lower leg
 foot



Hierarchical Representation

Grouped representation (tree)

- Each group contains subgroups and/or shapes
- Each group is associated with a transform relative to parent group
- Transform on leaf-node shape is concatenation of all transforms on path from root node to leaf
- Changing a group's transform affects all parts
 - Allows high level editing by changing only one node
 - E.g. raising left arm requires changing only one transform for that group

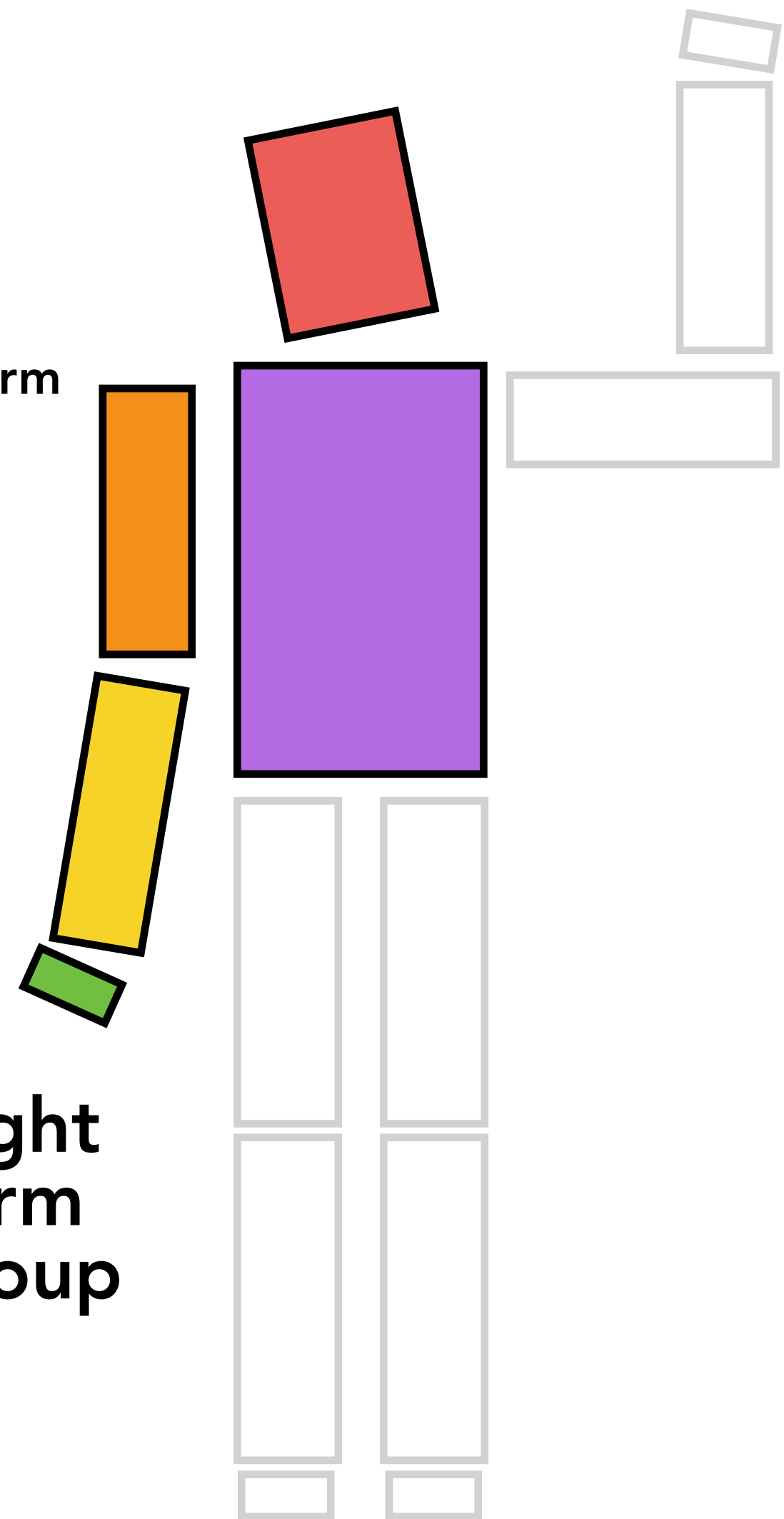
Skeleton - Hierarchical Representation

```
translate(0, 10);
drawTorso();
pushmatrix(); // push a copy of transform onto stack
  translate(0, 5); // right-multiply onto current transform
  rotate(headRotation); // right-multiply onto current transform
  drawHead();
popmatrix(); // pop current transform off stack
pushmatrix(); -----
  translate(-2, 3);
  rotate(rightShoulderRotation);
  drawUpperArm();
  pushmatrix(); -----
    translate(0, -3);
    rotate(elbowRotation);
    drawLowerArm();
    pushmatrix(); -----
      translate(0, -3);
      rotate(wristRotation);
      drawHand();
      popmatrix(); -----
    popmatrix(); -----
  popmatrix(); -----
....
```

right
hand

right
lower
arm
group

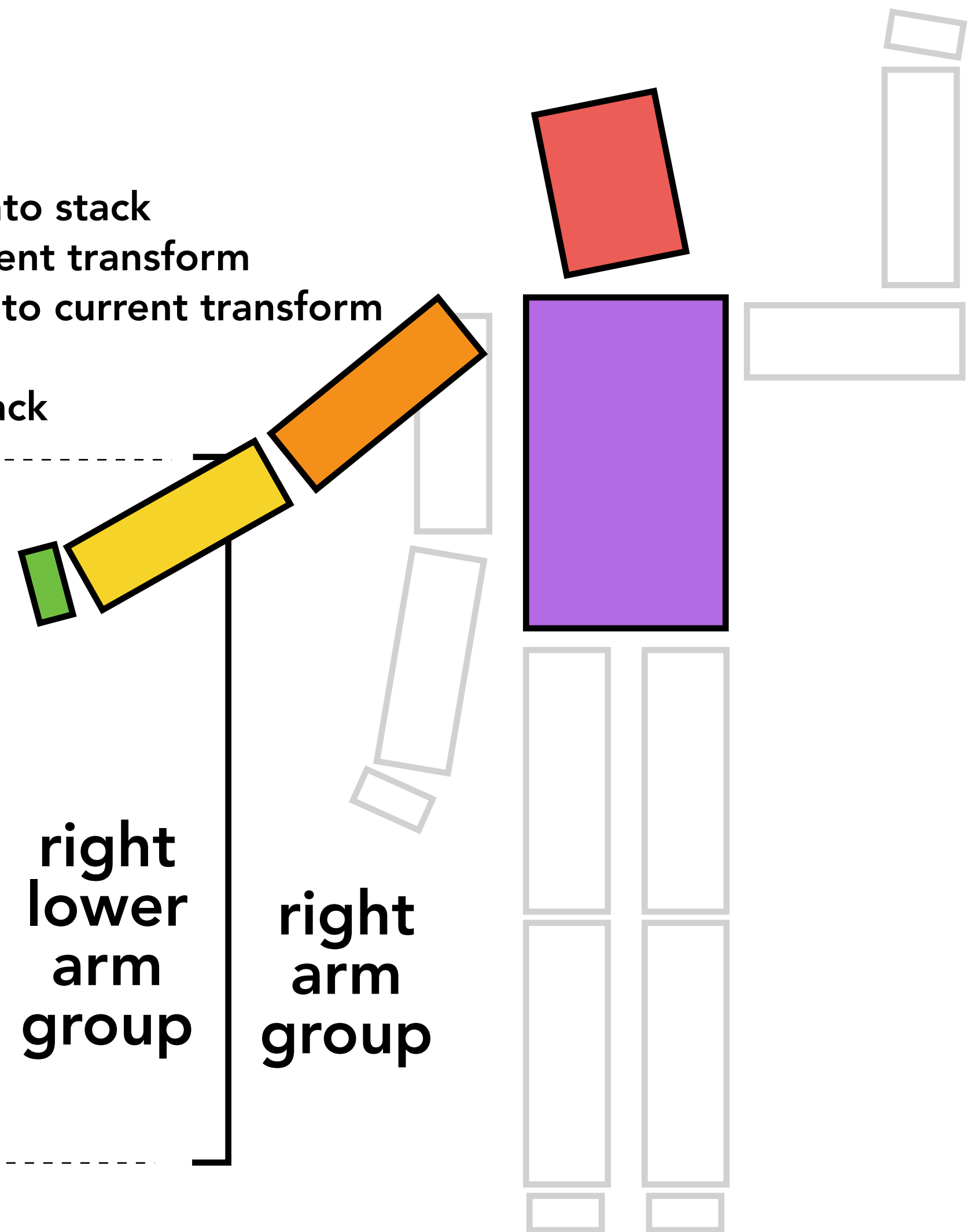
right
arm
group

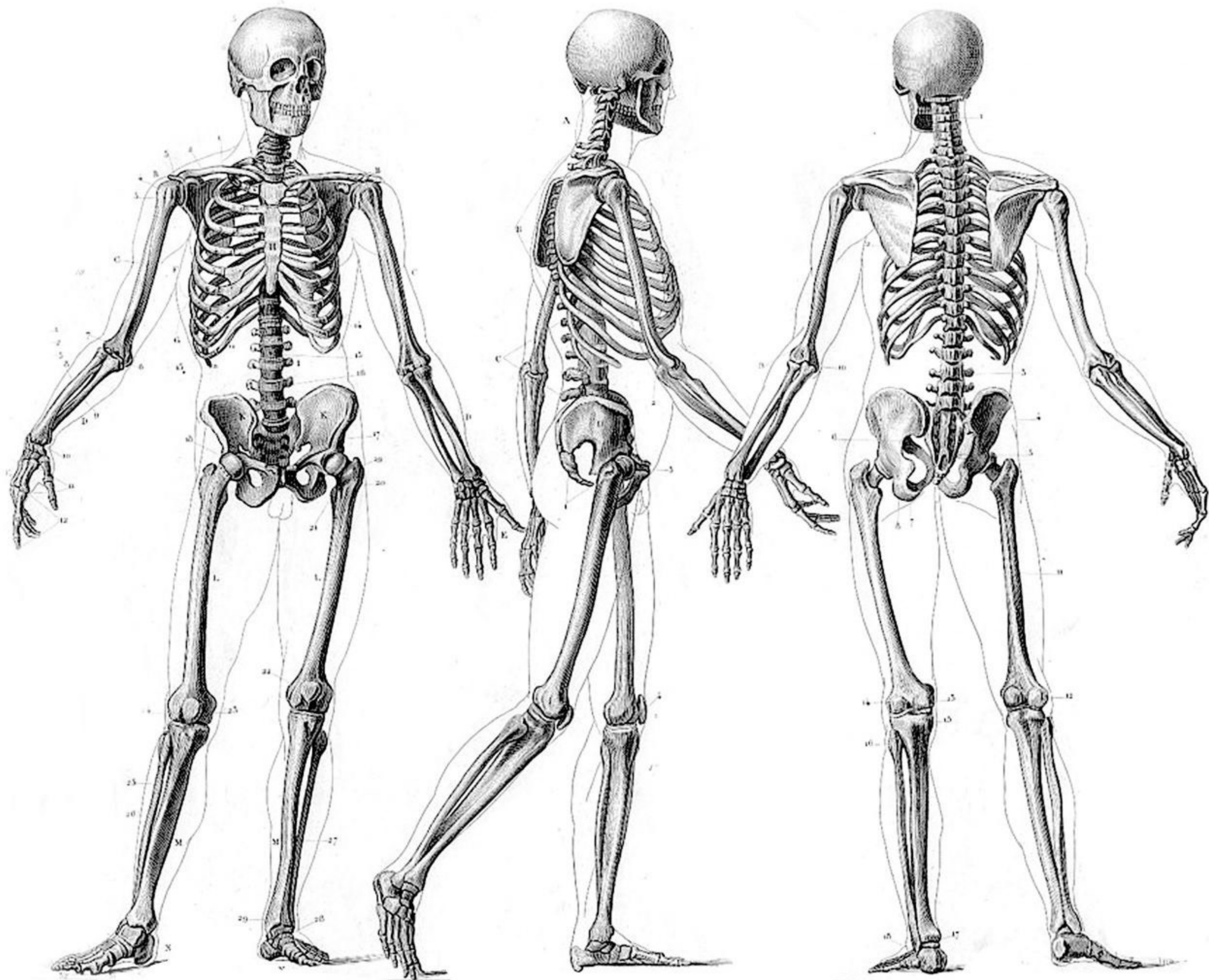


Skeleton - Hierarchical Representation

```
translate(0, 10);
drawTorso();
pushmatrix(); // push a copy of transform onto stack
  translate(0, 5); // right-multiply onto current transform
  rotate(headRotation); // right-multiply onto current transform
  drawHead();
popmatrix(); // pop current transform off stack
pushmatrix(); -----
  translate(-2, 3);
  rotate(rightShoulderRotation);
  drawUpperArm();
  pushmatrix(); -----
    translate(0, -3);
    rotate(elbowRotation);
    drawLowerArm();
    pushmatrix(); -----
      translate(0, -3);
      rotate(wristRotation);
      drawHand();
      popmatrix(); -----
    popmatrix(); -----
  popmatrix(); -----
....
```

→

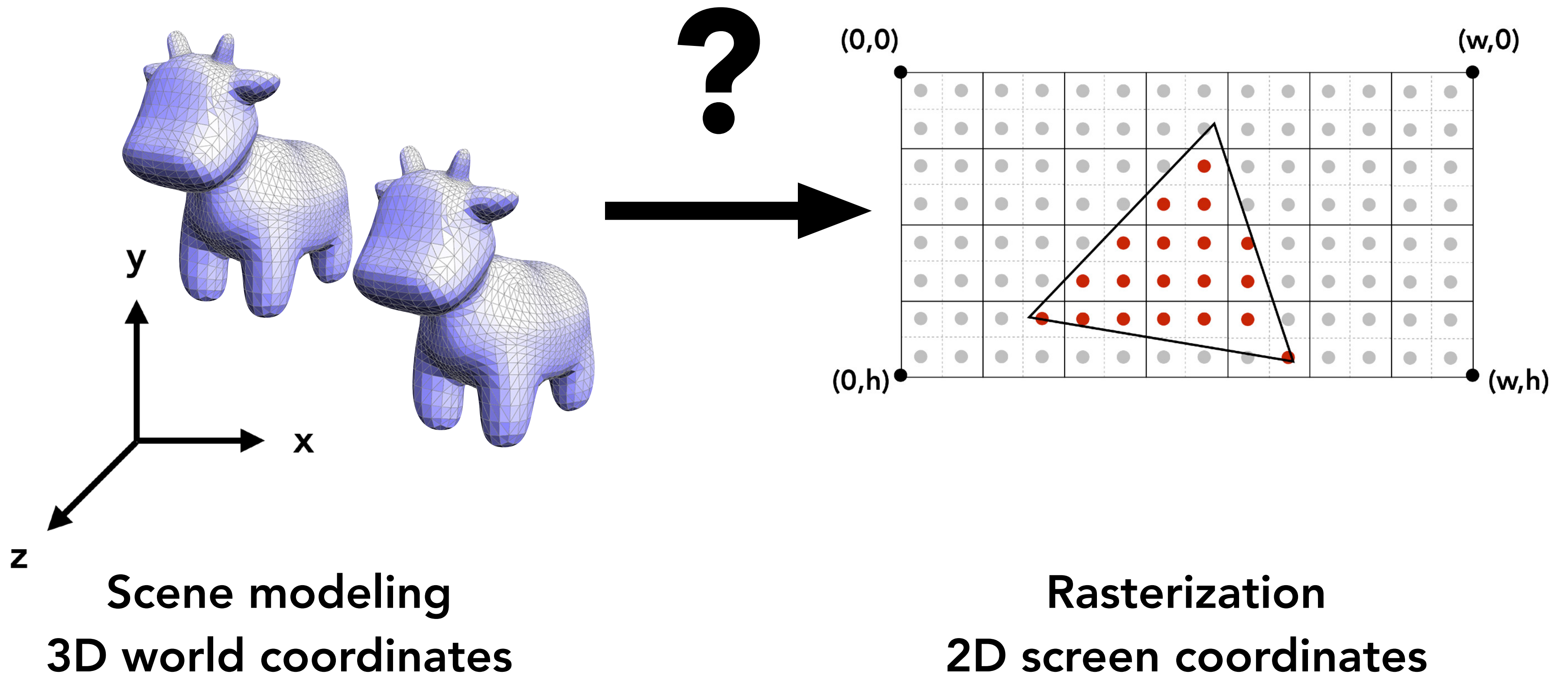




Viewing and Perspective



Viewing and Perspective Transforms



Camera Space

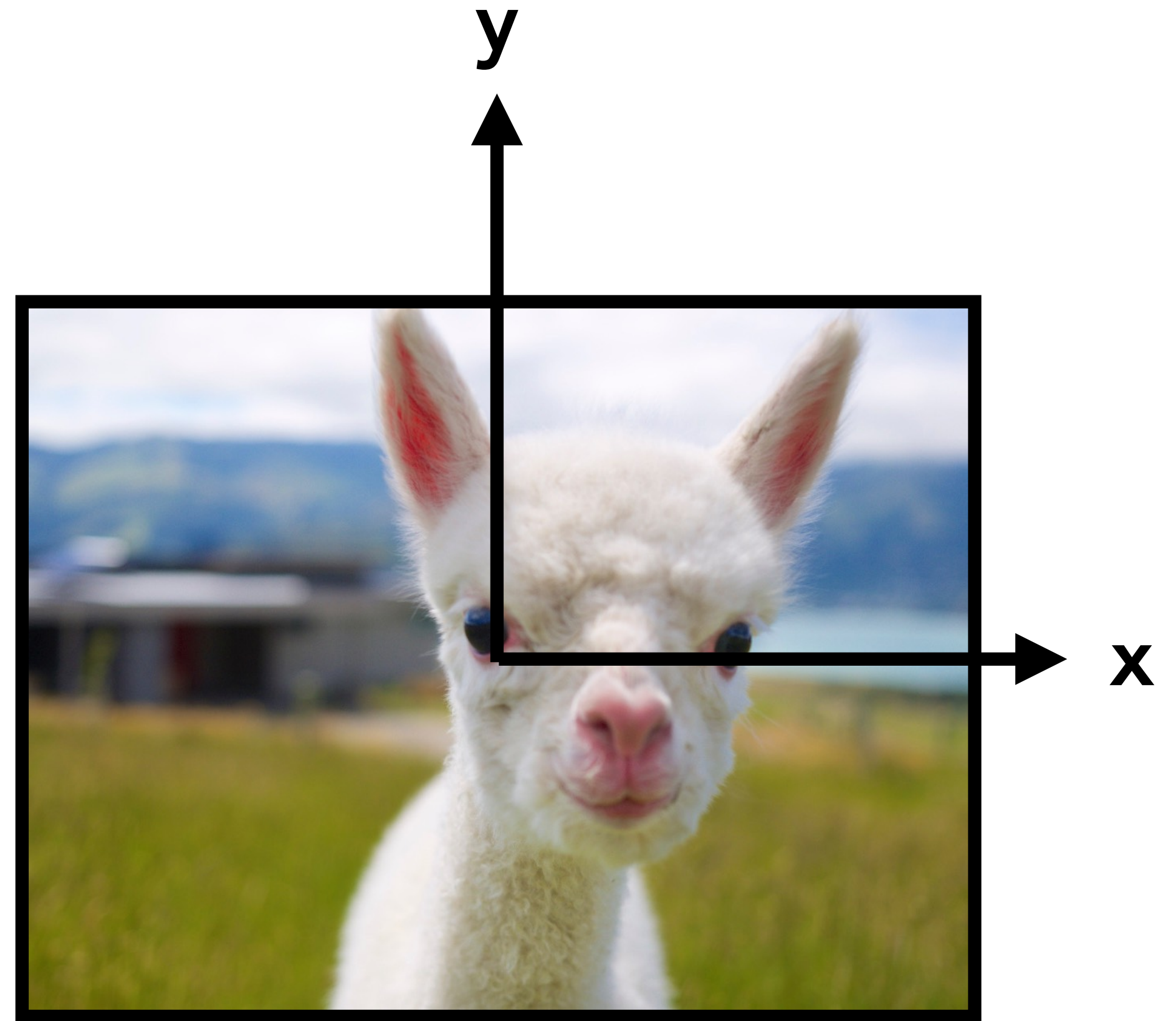
"Standard" Camera Space



We will use this convention for "standard" camera coordinates:

- camera located at the origin
- looking down *negative* z-axis
- vertical vector is y-axis
- (x-axis) orthogonal to y & z

"Standard" Camera Coordinates

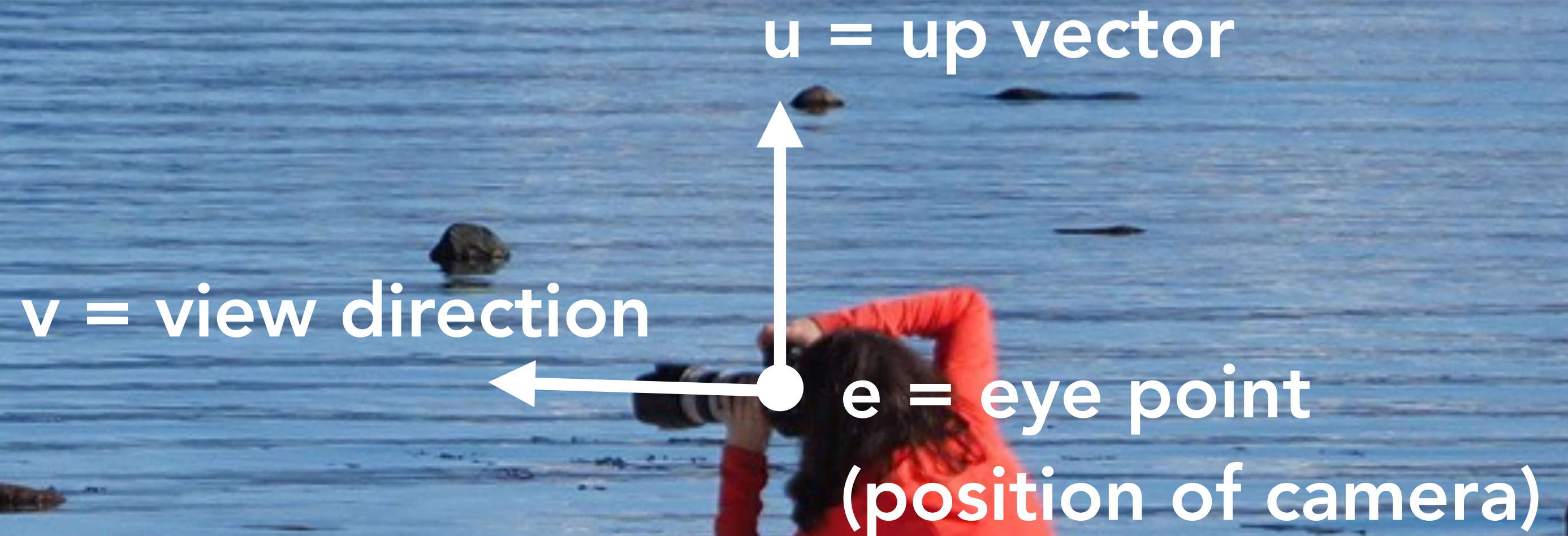


Resulting image
(z-axis pointing away from scene)

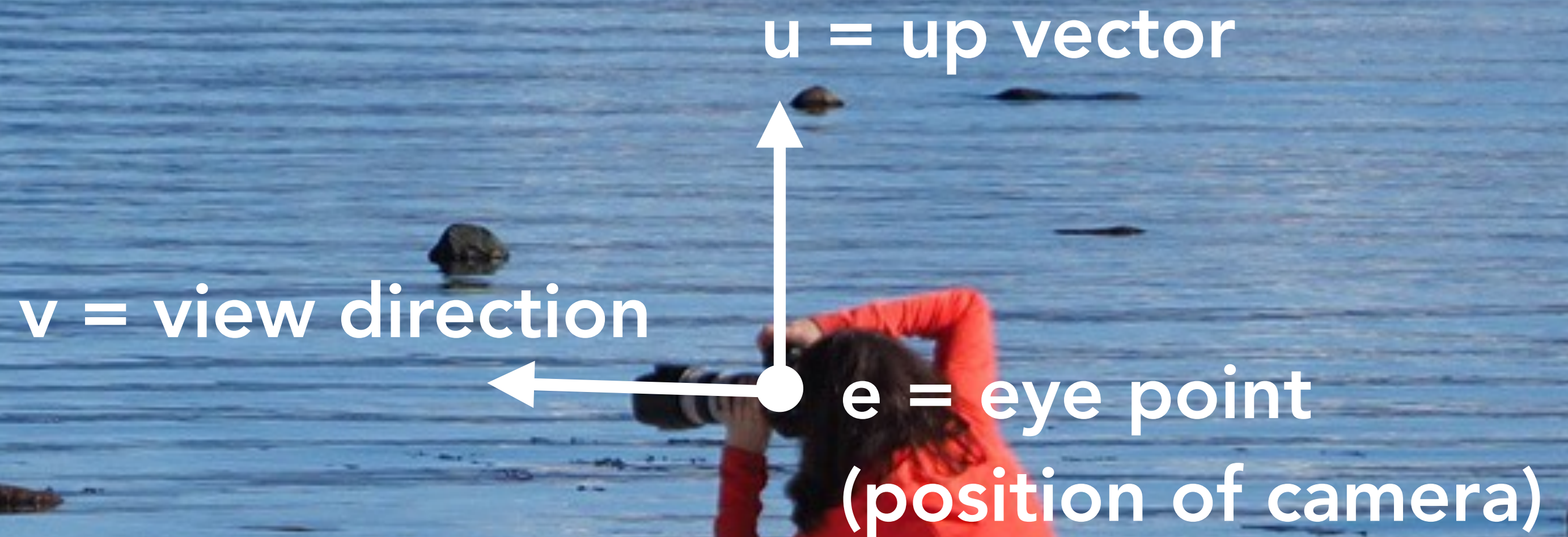
Consider A Camera Pointing in The World



Consider A Camera Pointing in The World



Camera "Look-At" Transformation



Input: e , u & v given in world space coordinates
Output: transform matrix from world space
to standard camera space

Camera “Look-At” Transformation

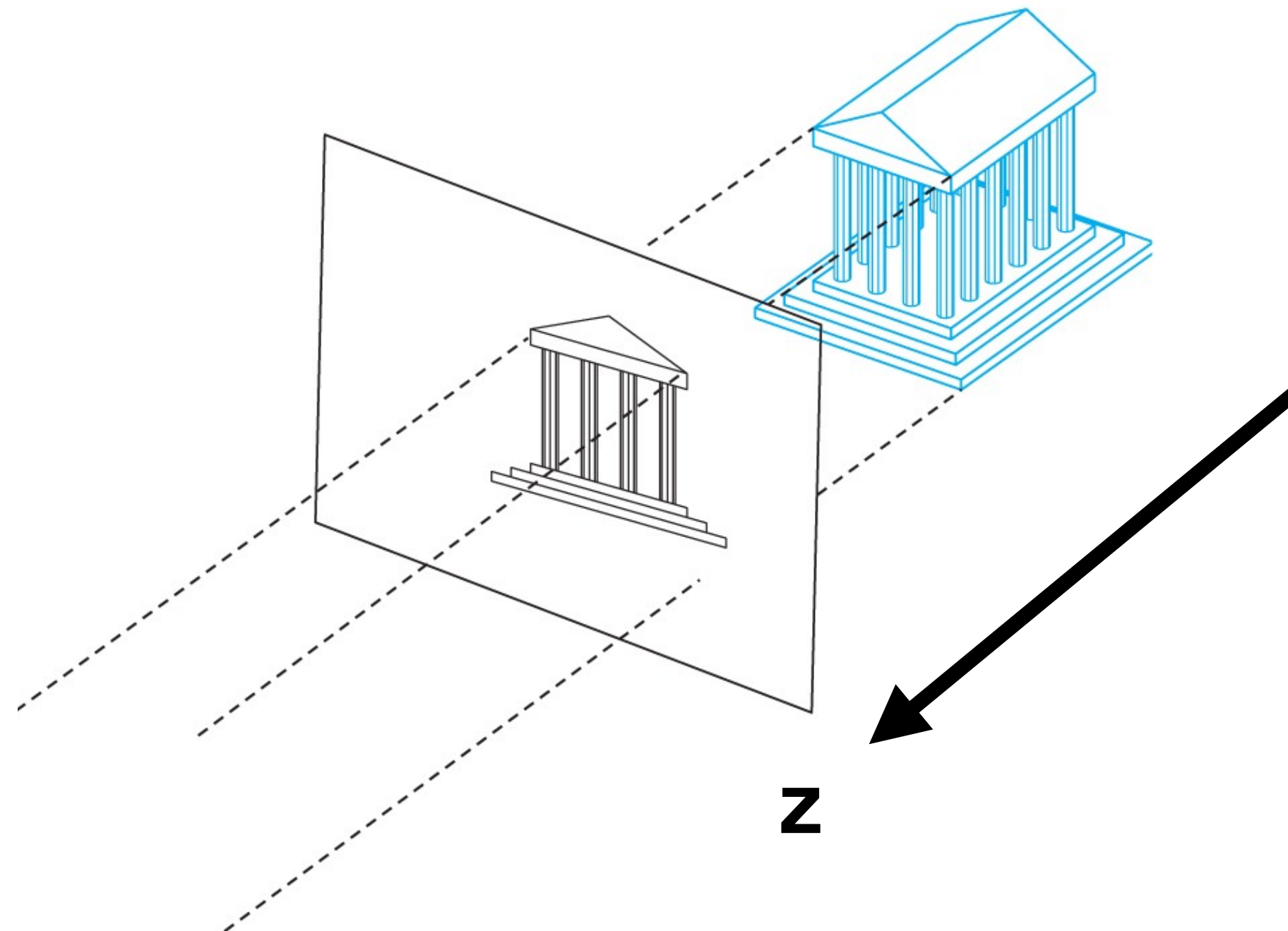
Inverse: Matrix from standard camera to world space
(Why? This is a coordinate frame transform to (e,r,u,-v))

$$\begin{pmatrix} r_x & u_x & -v_x & e_x \\ r_y & u_y & -v_y & e_y \\ r_z & u_z & -v_z & e_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

“Look-at” (world->camera) transform is the inverse of above matrix:

$$\begin{pmatrix} r_x & u_x & -v_x & e_x \\ r_y & u_y & -v_y & e_y \\ r_z & u_z & -v_z & e_z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} r_x & r_y & r_z & 0 \\ u_x & u_y & u_z & 0 \\ -v_x & -v_y & -v_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transform Camera Space to Image Plane?



How to transform from 3D camera space to 2D image plane?

- One option: orthographic projection (just delete z)
- Useful, e.g. for engineering drawings
- But is this the whole story?

Perspective



Credit: jefflynchphoto.com

Perspective in Art



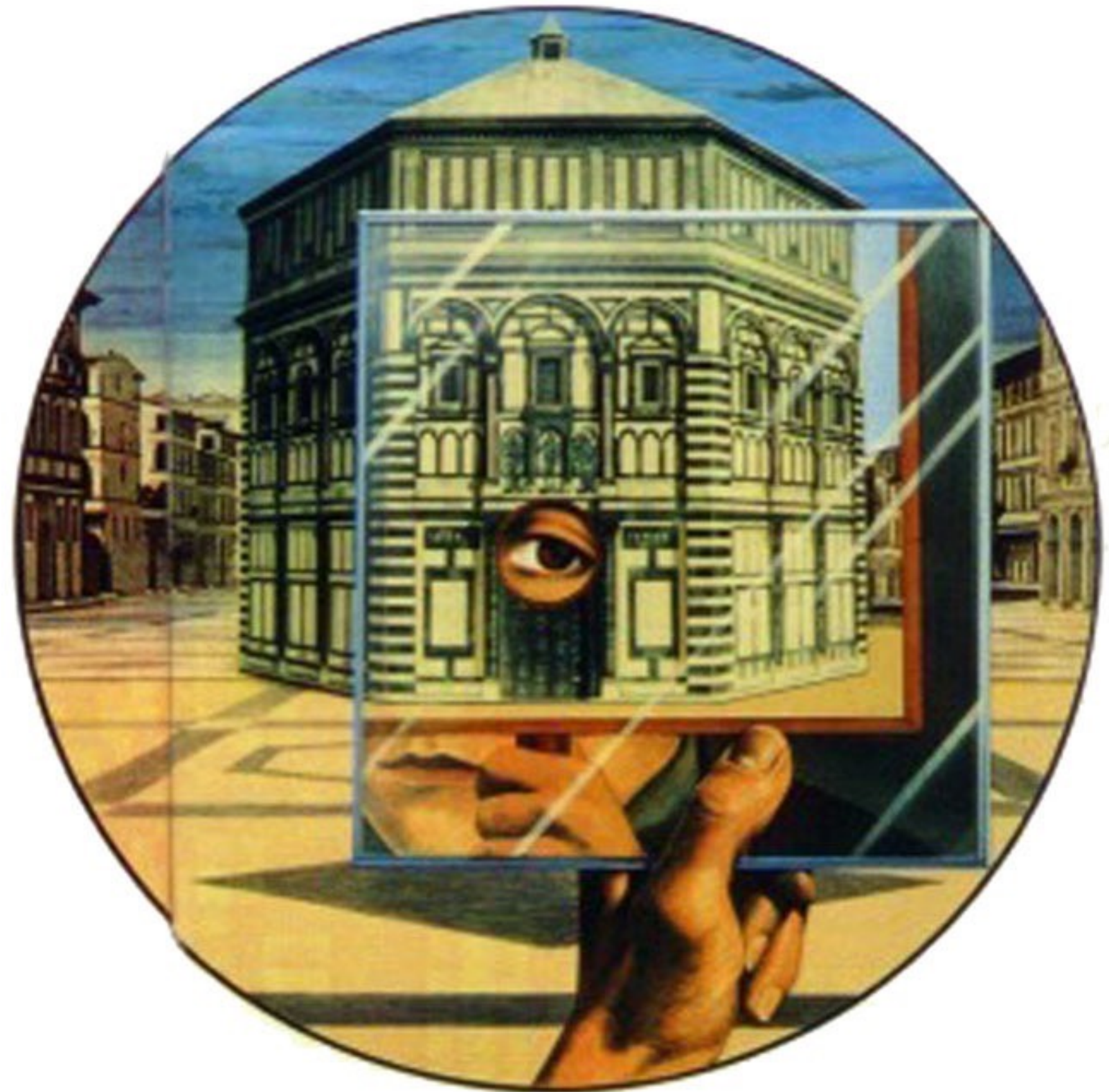
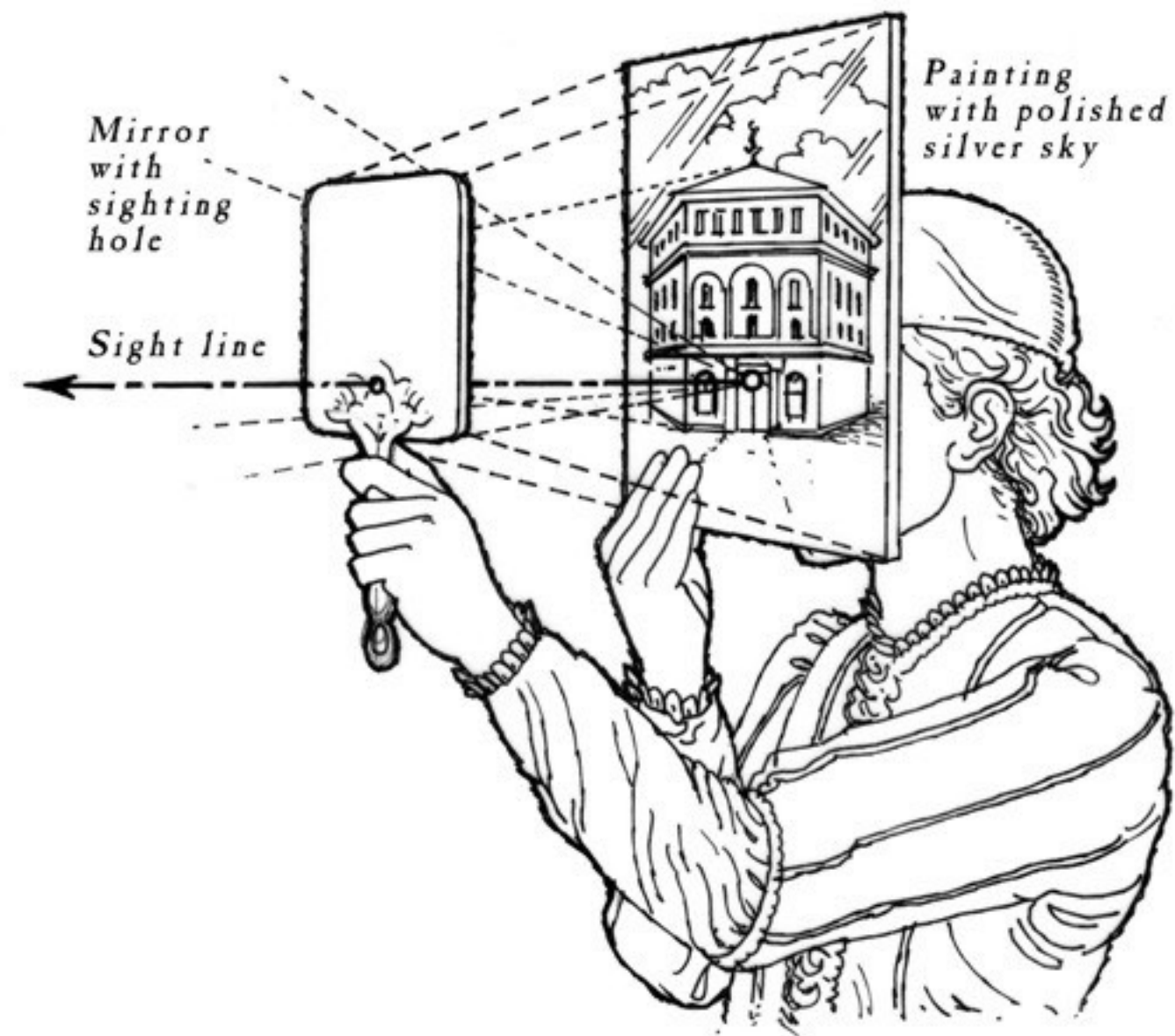
Perspective in Art



Perspective in Art



Perspective in Art



Brunelleschi experiment c. 1413

Florence Cathedral



North Door (1403~)



East Door (1425~)

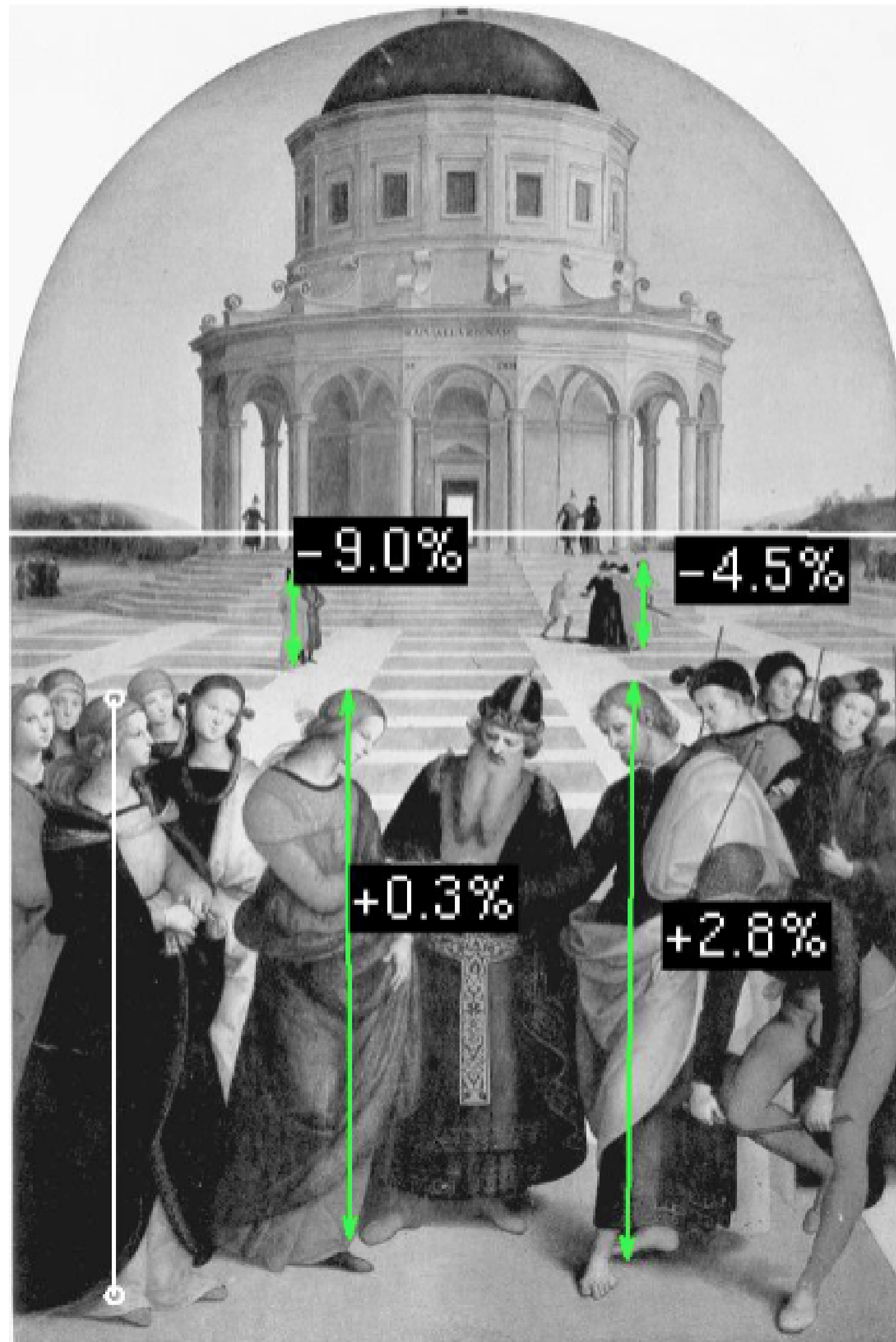
Both works by Lorenzo Ghiberti

Perspective in Art



Delivery of the Keys (Sistine Chapel), Perugino, 1482

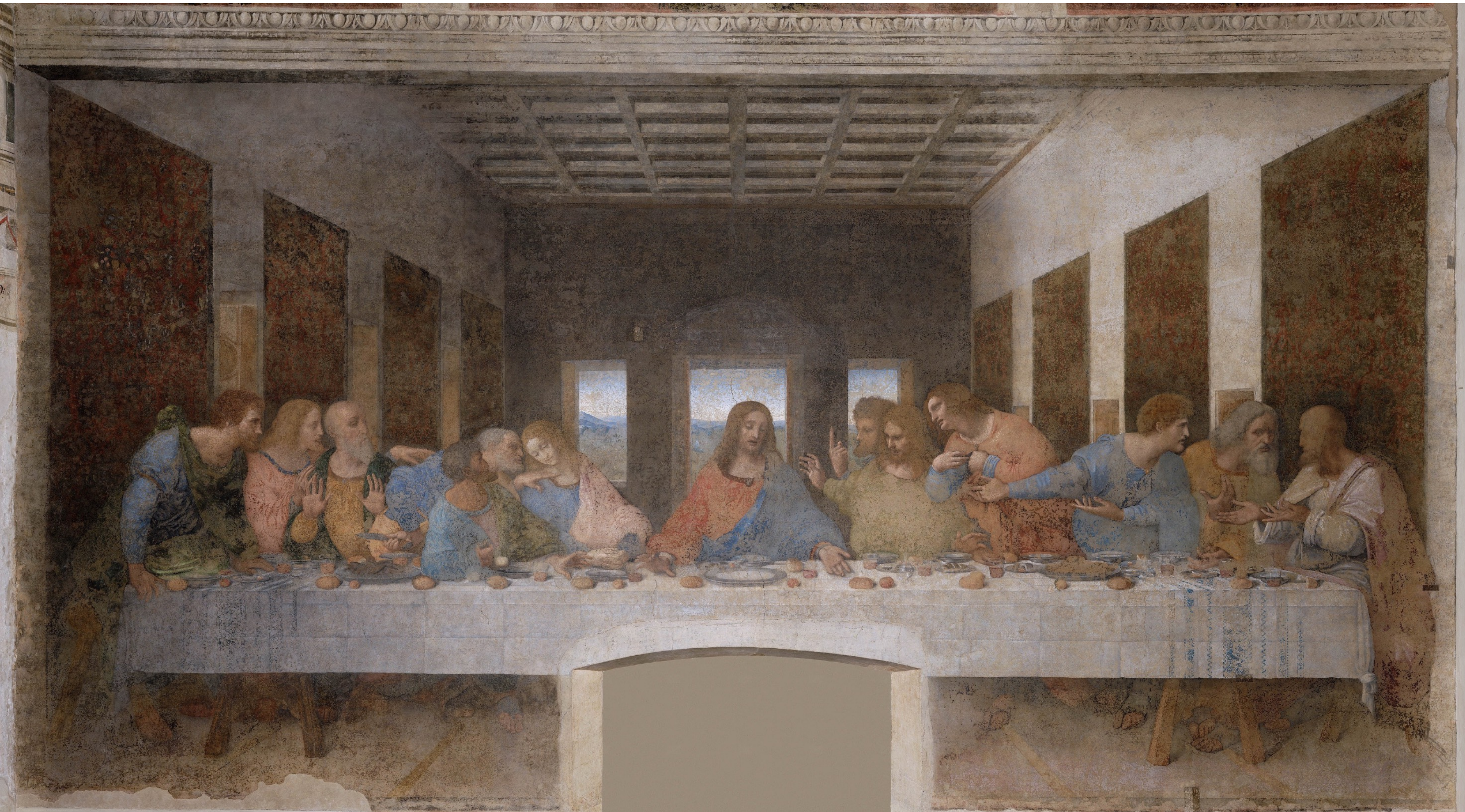
Perspective in Art



Too bad Rafael didn't understand the math...

Credit: Alyosha Efros

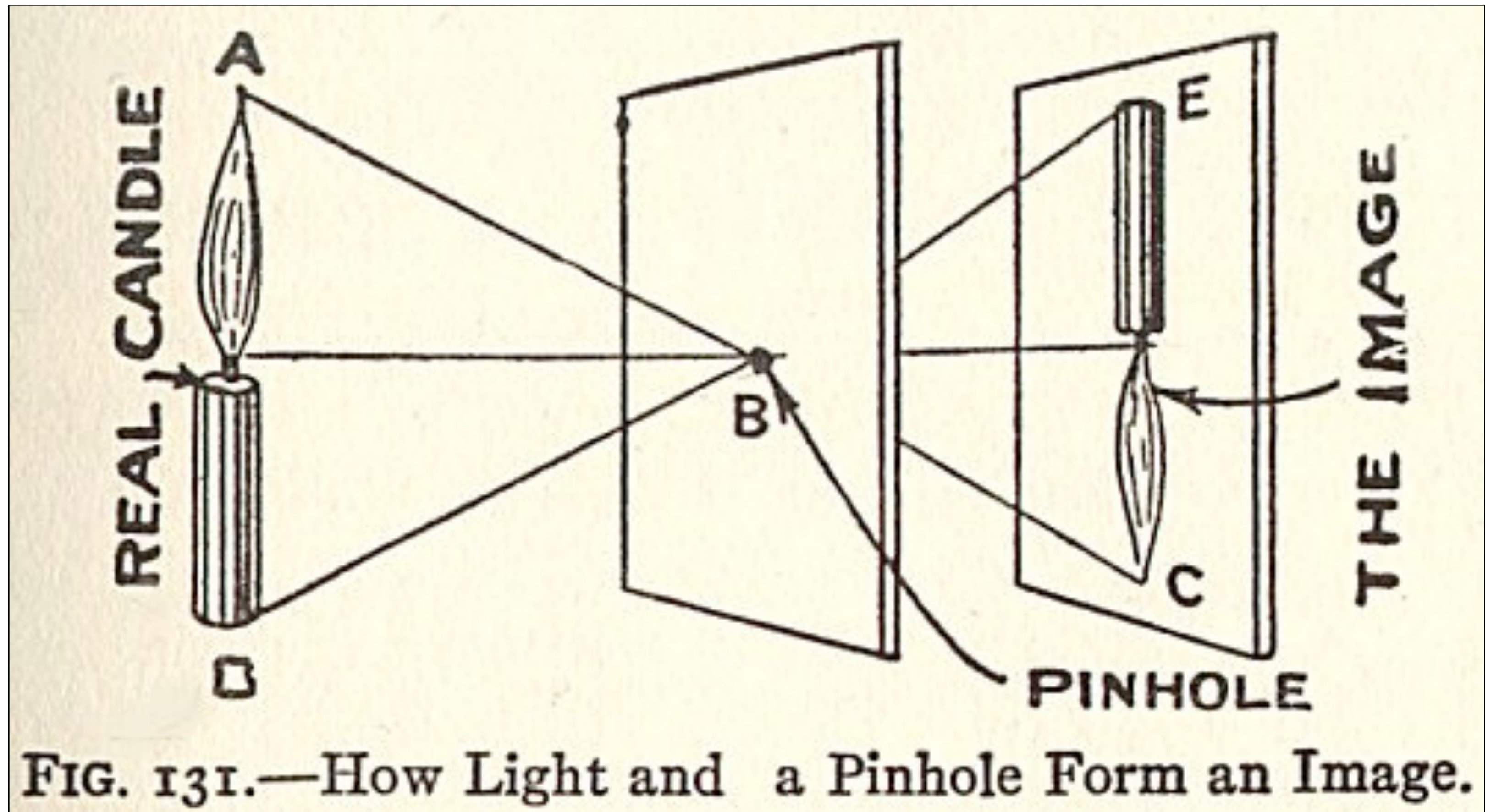
Perspective in Art



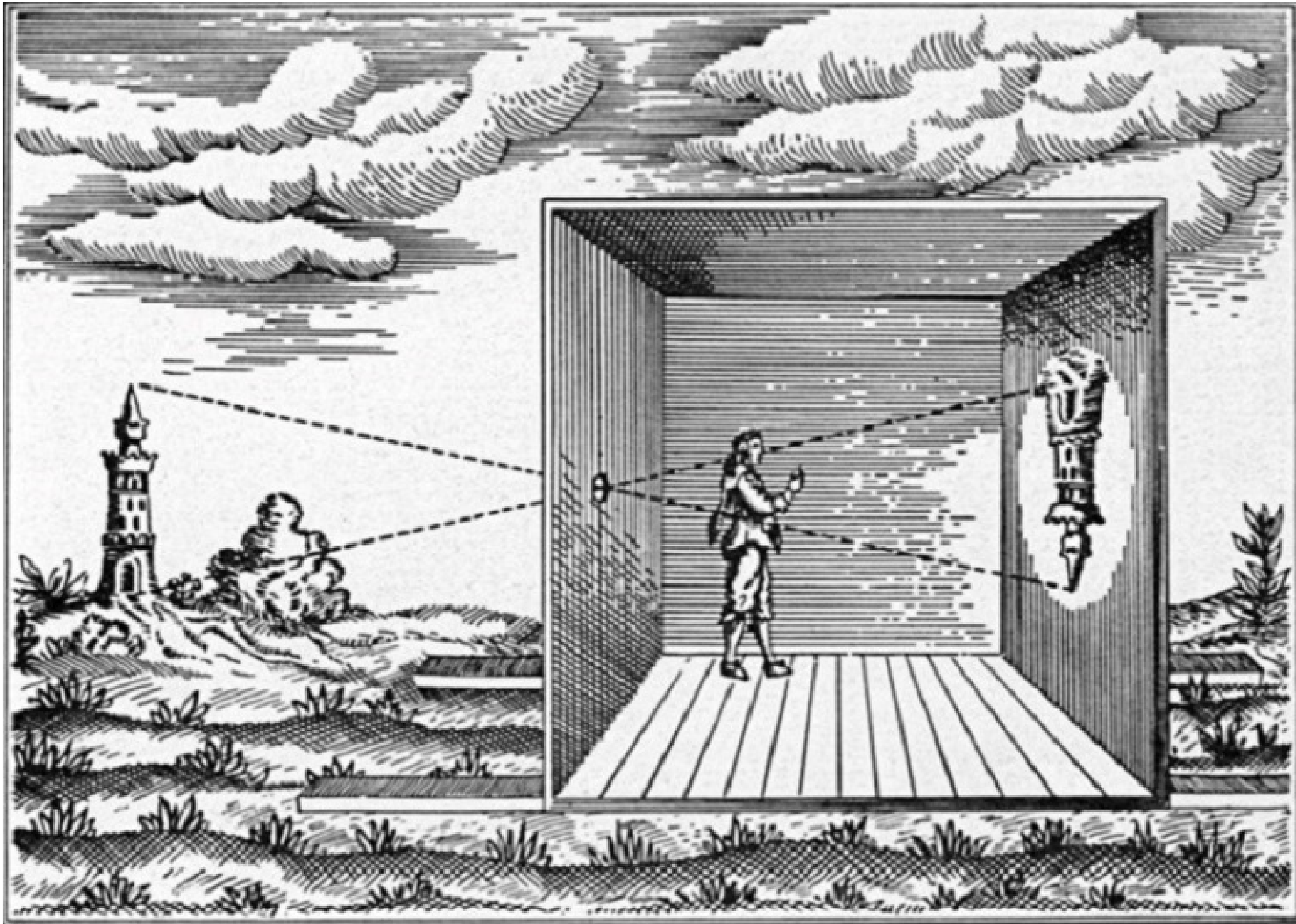
The Last Supper, Leonardo da Vinci, 1499

Pinhole Camera Model

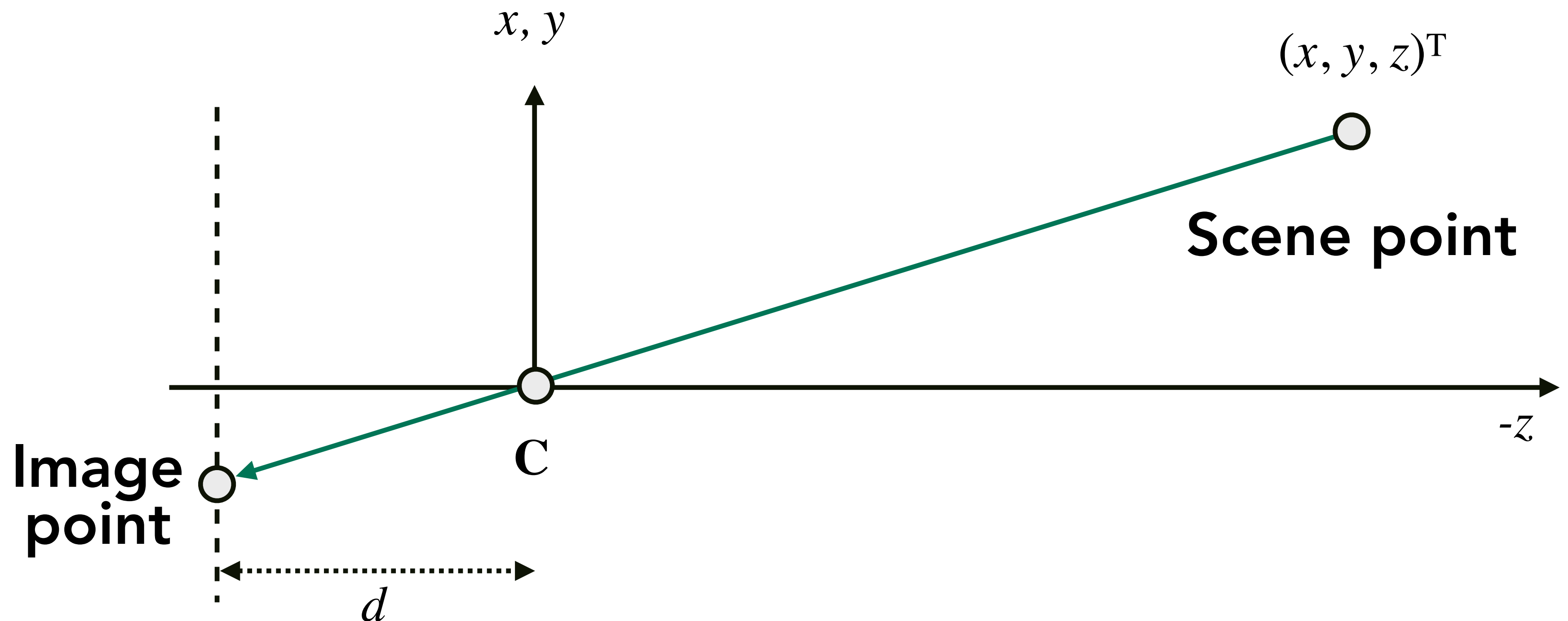
Pinhole Camera



Pinhole Camera

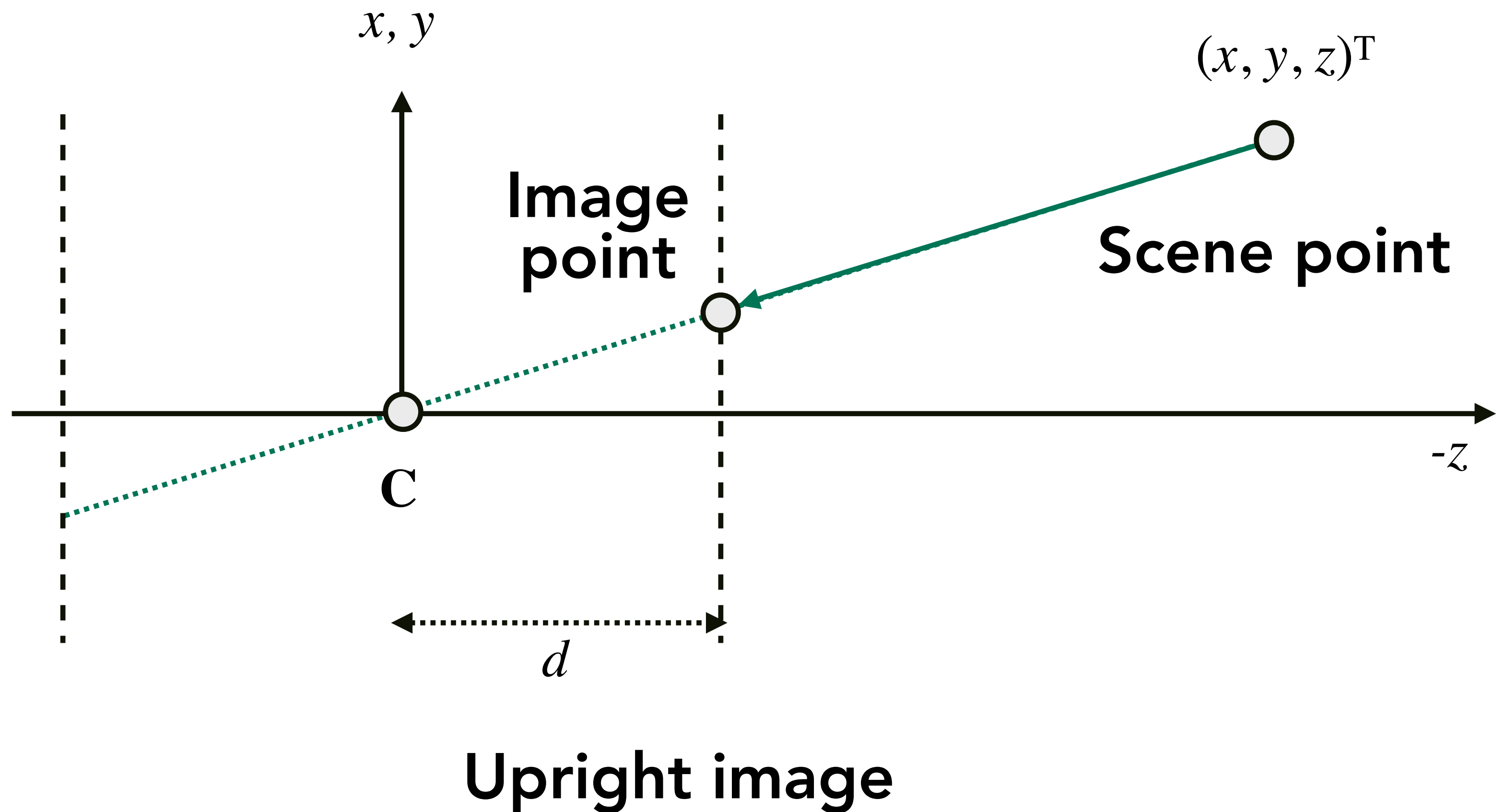


Projective Transform



Inverted image (as in real pinhole camera)

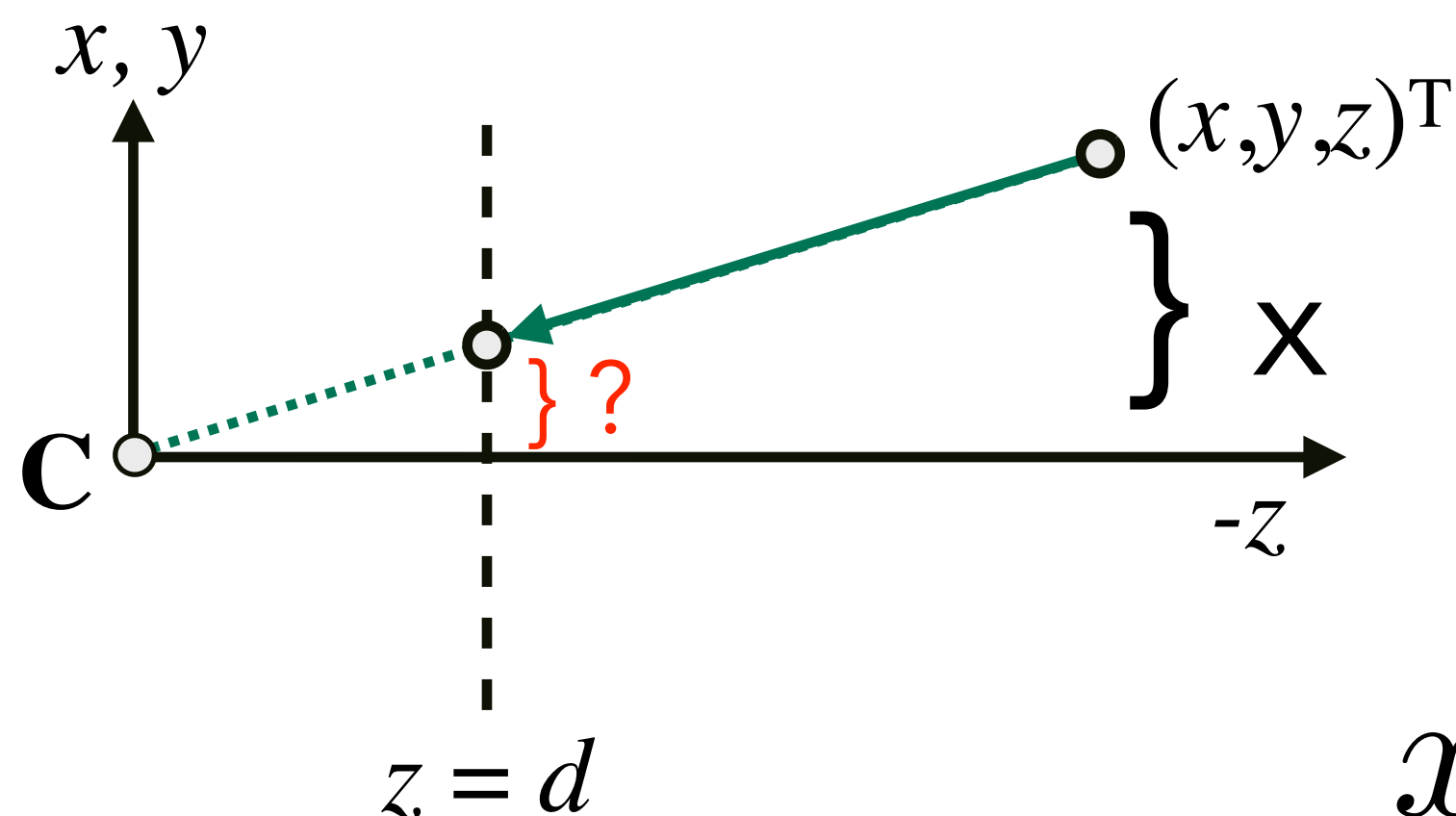
Pinhole Camera Projective Transform



Projective Transforms

Standard perspective projection

- Center of projection: $(0, 0, 0)^T$
- Image plane at $z = d$



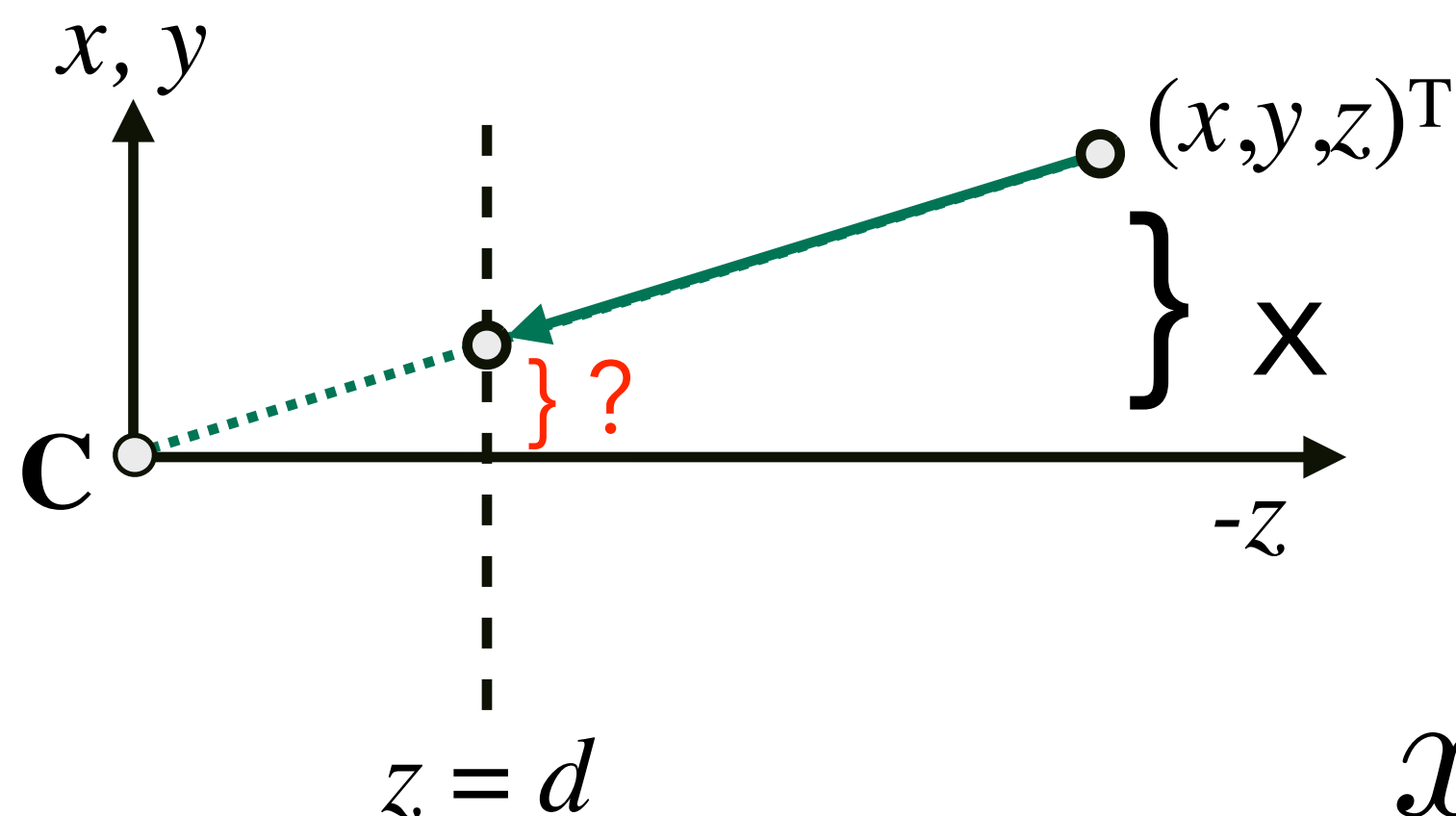
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \\ \\ d \end{pmatrix}$$

$$\frac{x}{z} = \frac{?}{d}$$

Projective Transforms

Standard perspective projection

- Center of projection: $(0, 0, 0)^T$
- Image plane at $z = d$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \cdot d/z \\ y \cdot d/z \\ d \end{pmatrix}$$

$$\frac{x}{z} = \frac{?}{d}$$

Projective Transforms

Standard perspective projection

- Center of projection: $(0, 0, 0)^T$
- Image plane at $z = d$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \cdot d/z \\ y \cdot d/z \\ d \end{pmatrix}$$

Perspective foreshortening

- Need division by z
 - Matrix representation?
- ➡ Homogeneous coordinates!

Exciting New Study!!



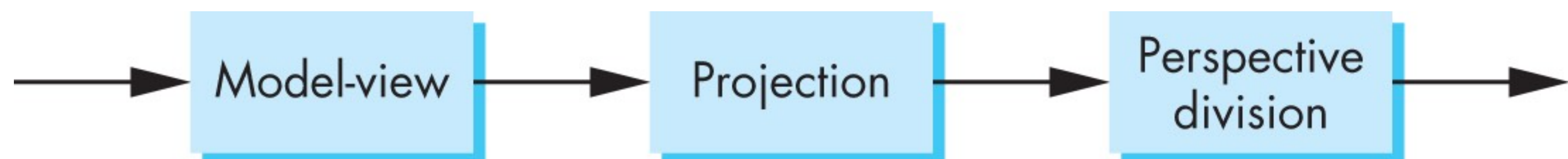
Homogenous Coordinates (3D)

$$\mathbf{p} = \begin{pmatrix} wx \\ wy \\ wz \\ w \end{pmatrix} \longleftrightarrow \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{pmatrix}$$

$$\mathbf{q} = \mathbf{M} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ z/d \end{pmatrix} \longleftrightarrow \begin{pmatrix} xd/z \\ yd/z \\ d \\ 1 \end{pmatrix}$$

**Note non-zero term in final row.
First time we have seen this.**



Pinhole Camera Model

This mathematical model produces all linear perspective effects!

- Converging lines + vanishing points
- Closer objects appear larger in images
- ...

Specifying Real Camera Perspectives

Perspective Composition = Camera Position + Angle of View

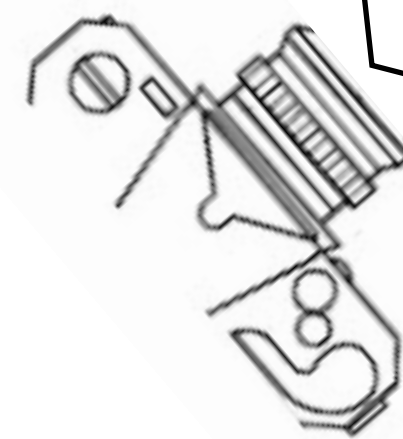
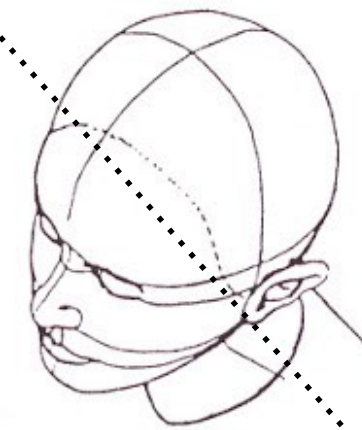


In this sequence, angle of view decreases as distance from subject increases, to size of human subject in image.

Notice the dramatic change in background perspective.

From Canon EF Lens Work III

Perspective Composition



16 mm (110°)

Up close and zoomed wide
with short focal length

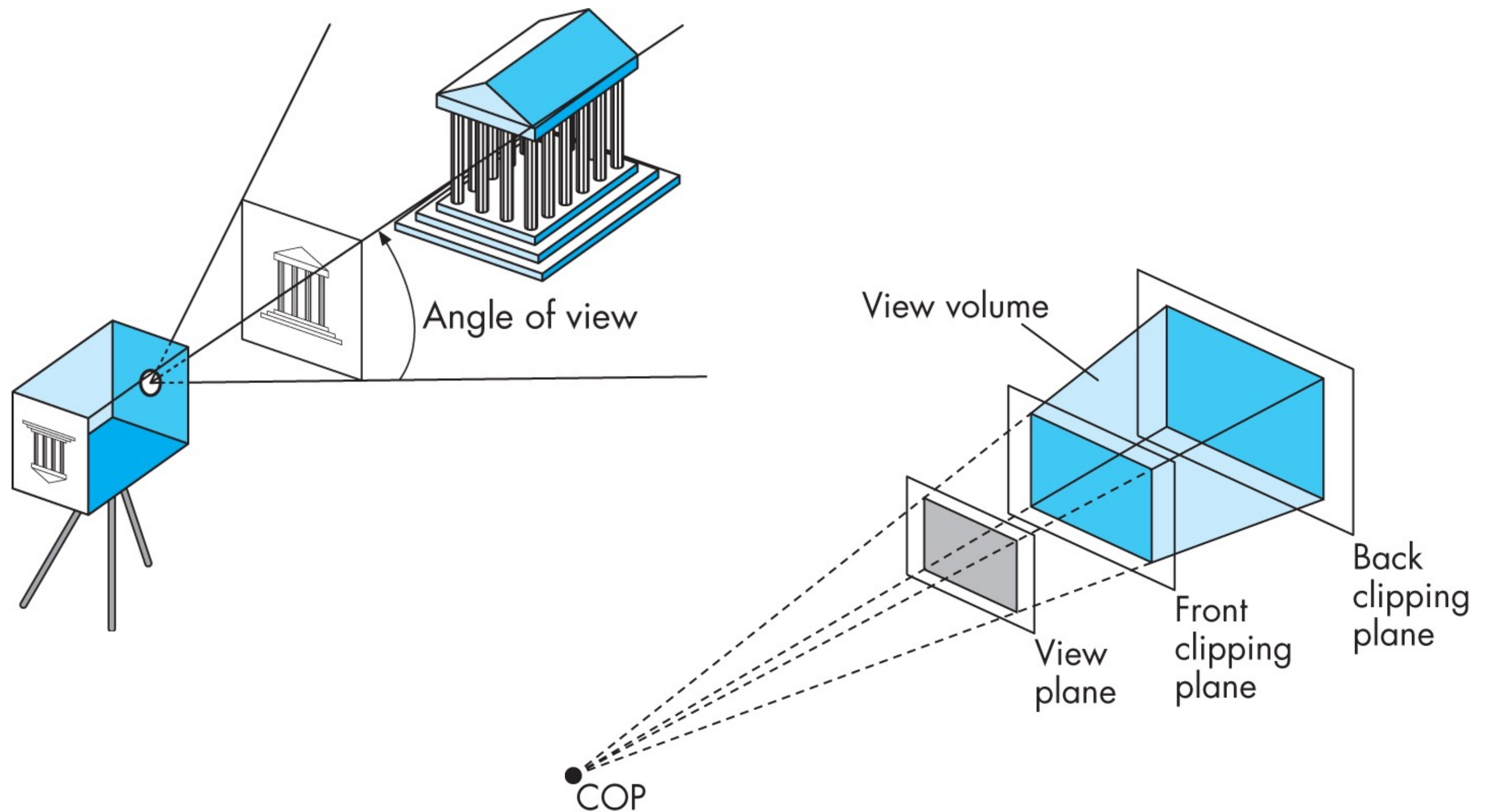
Perspective Composition



200 mm (12°)

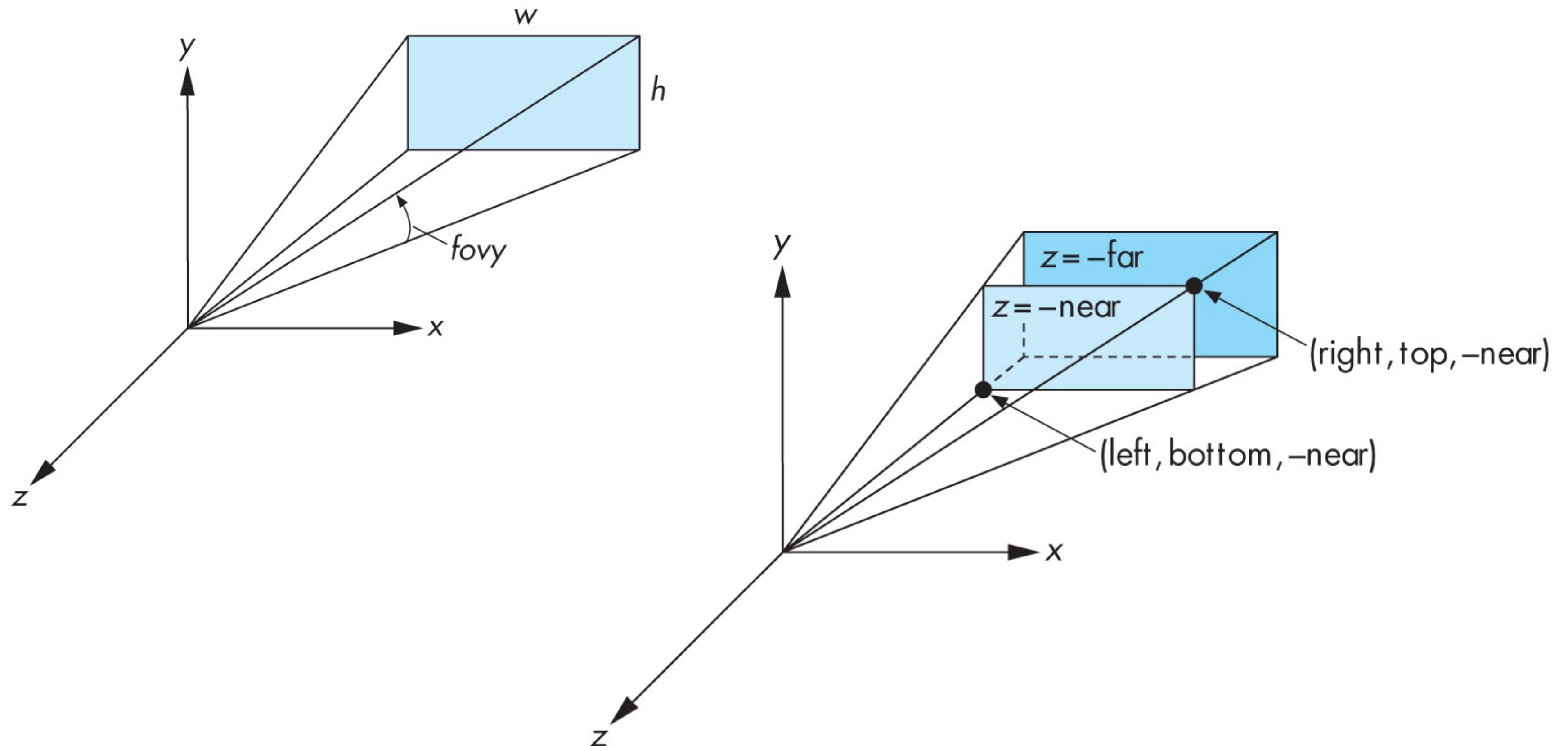
Walk back and zoom in
with long focal length

Specifying Perspective Projection



From Angel and Shreiner, Interactive Computer Graphics

Specifying Perspective Viewing Volume



From Angel and Shreiner, Interactive Computer Graphics

Specifying Perspective Viewing Volume

Parameterized by

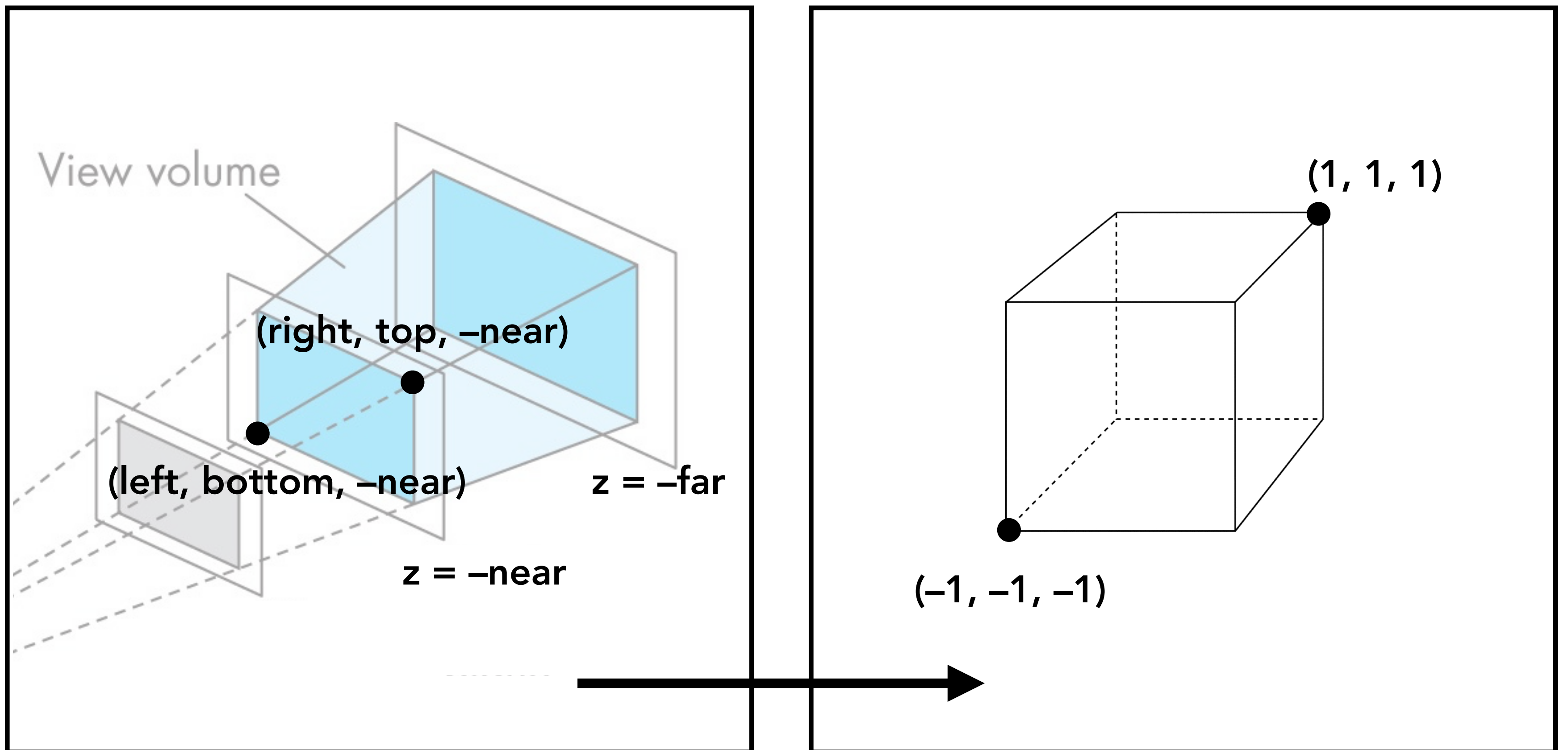
- **fovy** : vertical angular field of view
- **aspect ratio** : width / height of field of view
- **near** : depth of near clipping plane
- **far** : depth of far clipping plane

Derived quantities

- **top** = $\text{near} * \tan(\text{fovy})$
- **bottom** = $-\text{top}$
- **right** = $\text{top} * \text{aspect}$
- **left** = $-\text{right}$

Perspective Projection Implementation

Perspective Projection Transform



Camera Coordinates

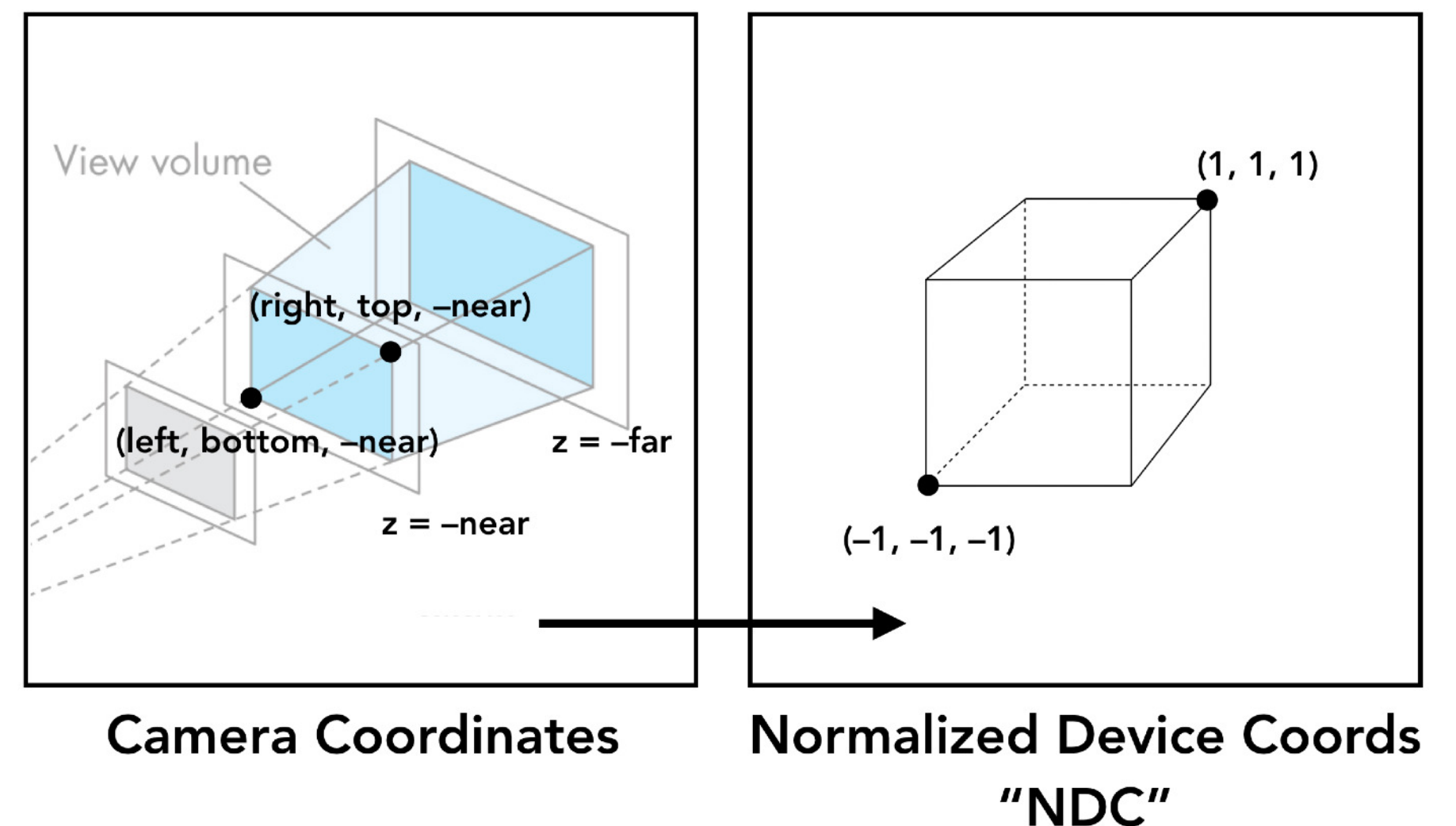
Normalized Device Coords
"NDC"

Later we will "flatten and scale " NDC to get framebuffer coordinates

Perspective Projection Transform

Notes:

- Need not be symmetric about z-axis, but for simplicity here we assume so
- This transform will preserve depth information (ordering) in NDC



Perspective Transform Matrix

$$\mathbf{P} = \begin{bmatrix} \frac{near}{right} & 0 & 0 & 0 \\ 0 & \frac{near}{top} & 0 & 0 \\ 0 & 0 & -\frac{far+near}{far-near} & \frac{-2far*near}{far-near} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

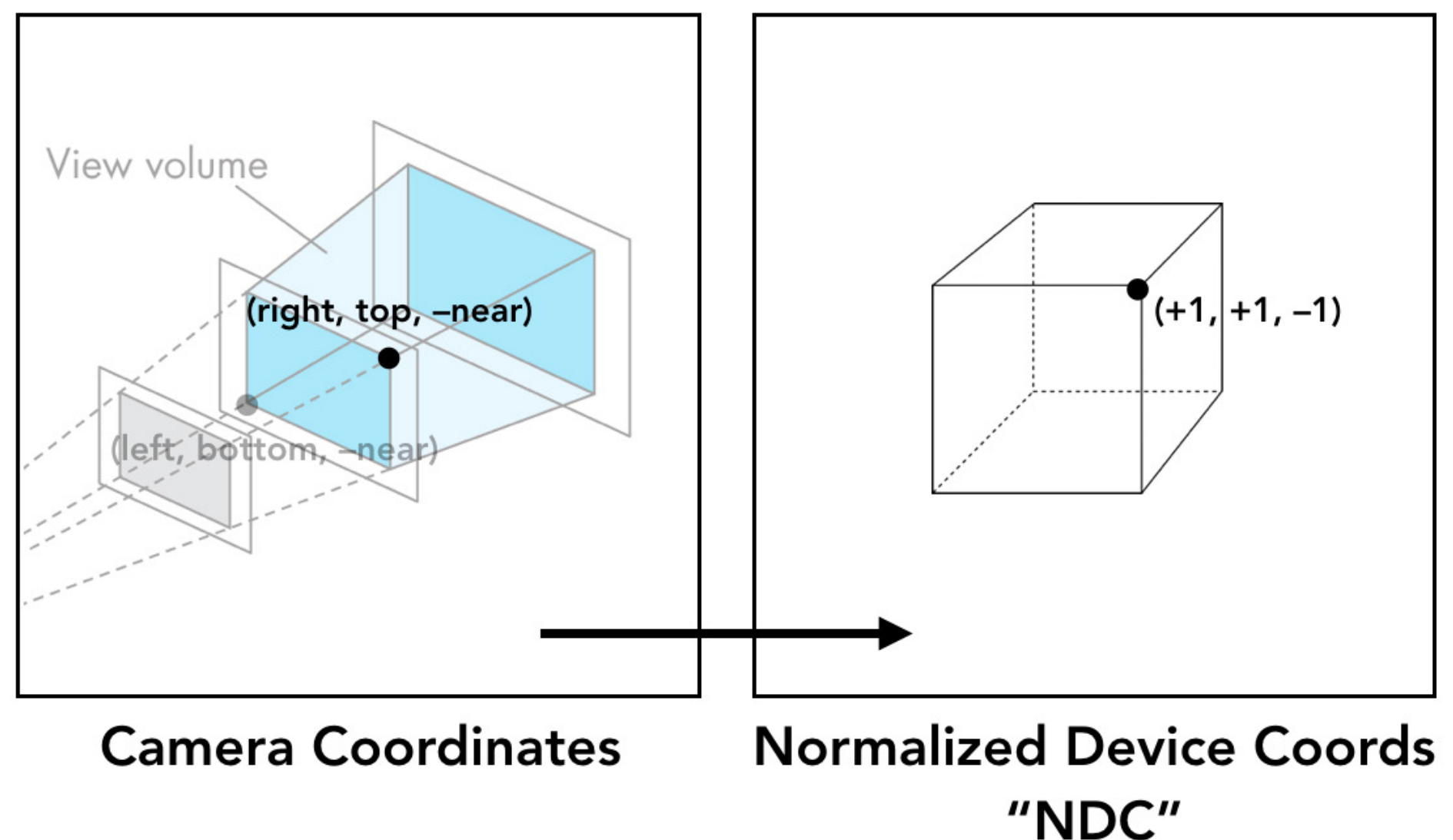
Perspective Transform Matrix Example

$$\mathbf{P} = \begin{bmatrix} \frac{\text{near}}{\text{right}} & 0 & 0 & 0 \\ 0 & \frac{\text{near}}{\text{top}} & 0 & 0 \\ 0 & 0 & -\frac{\text{far}+\text{near}}{\text{far}-\text{near}} & \frac{-2\text{far}*\text{near}}{\text{far}-\text{near}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \text{right} \\ \text{top} \\ -\text{near} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \text{near} \\ \text{near} \\ \text{near} * \frac{\text{far}+\text{near}}{\text{far}-\text{near}} - \frac{2\text{far}*\text{near}}{\text{far}-\text{near}} \\ \text{near} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ \frac{-\text{far}+\text{near}}{\text{far}-\text{near}} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$



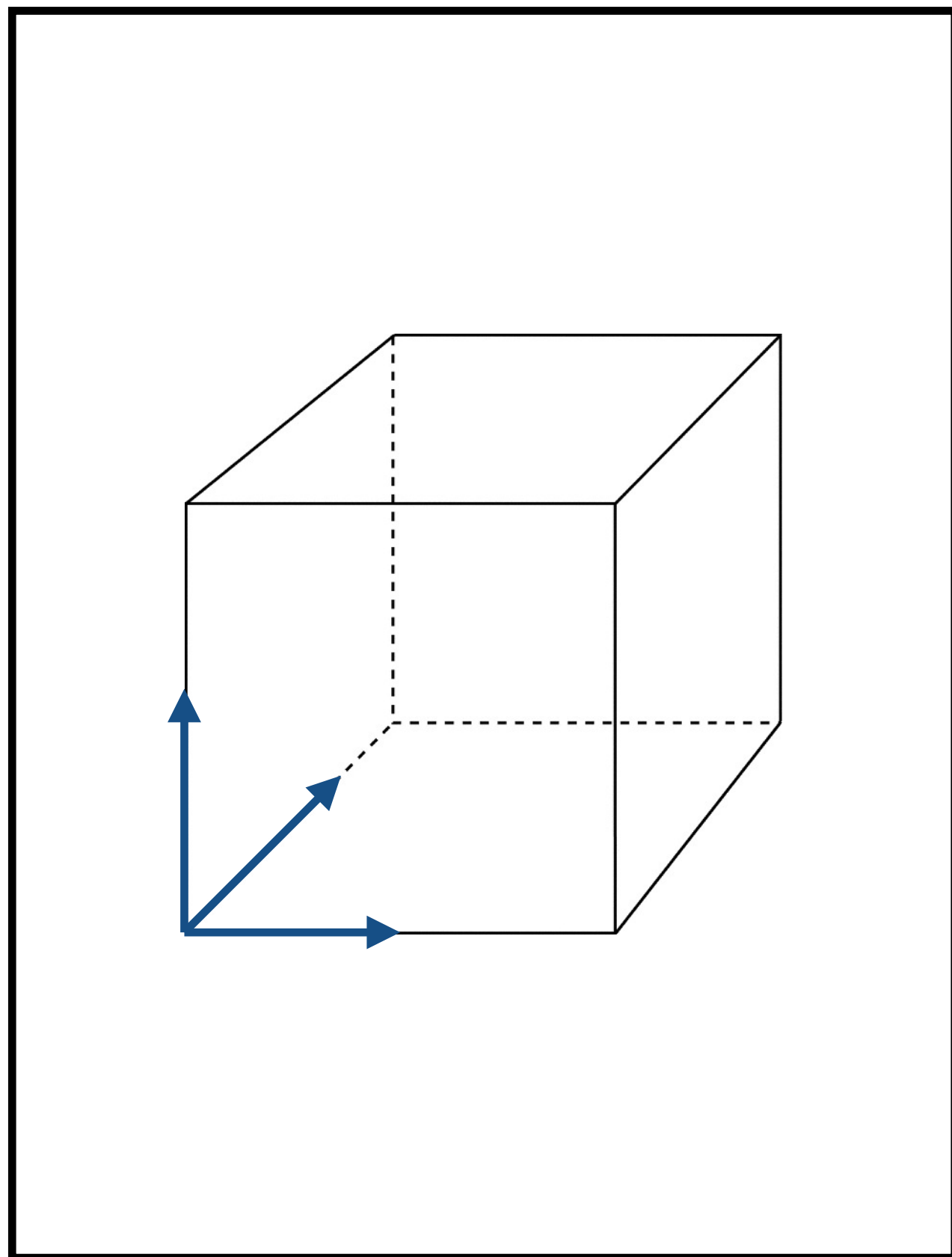
Transforms Recap

Transforms Recap

Coordinate Systems

- Object coordinates
 - Apply modeling transforms...
- World (scene) coordinates
 - Apply viewing transform...
- Camera (eye) coordinates
 - Apply perspective transform + homog. divide...
- Normalized device coordinates
 - Apply 2D screen transform...
- Screen coordinates

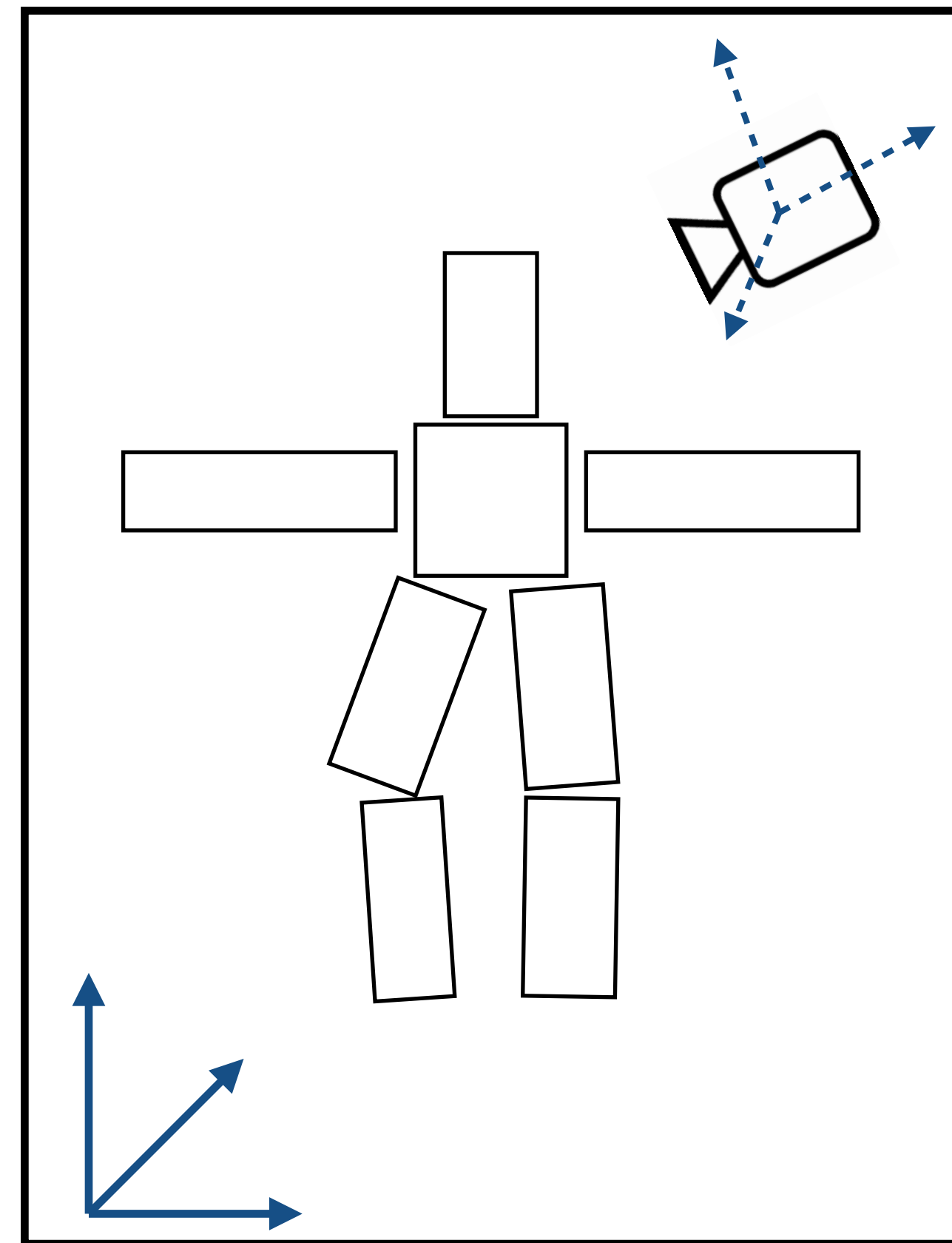
Transforms Recap



Object coords

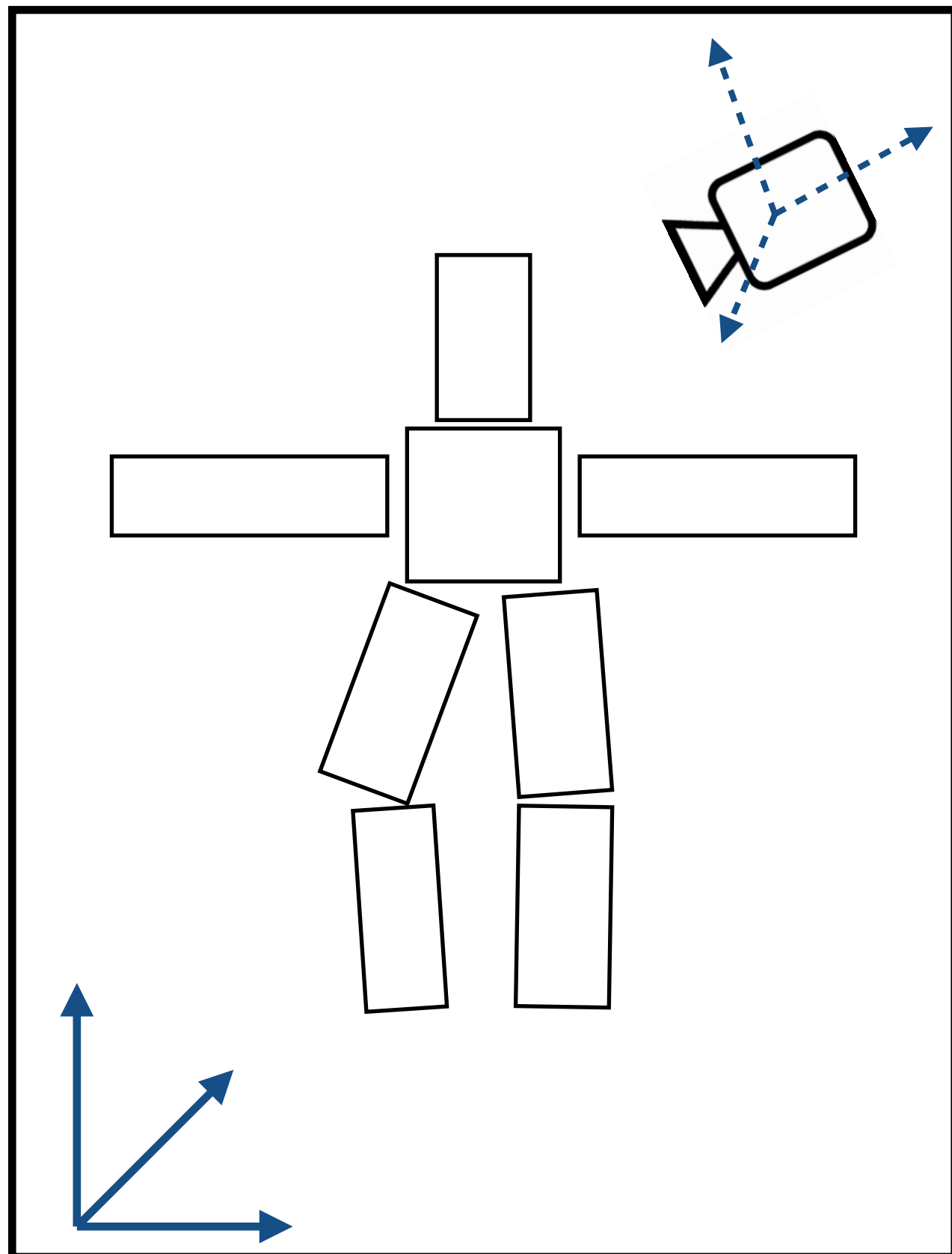


**Modeling
transforms**



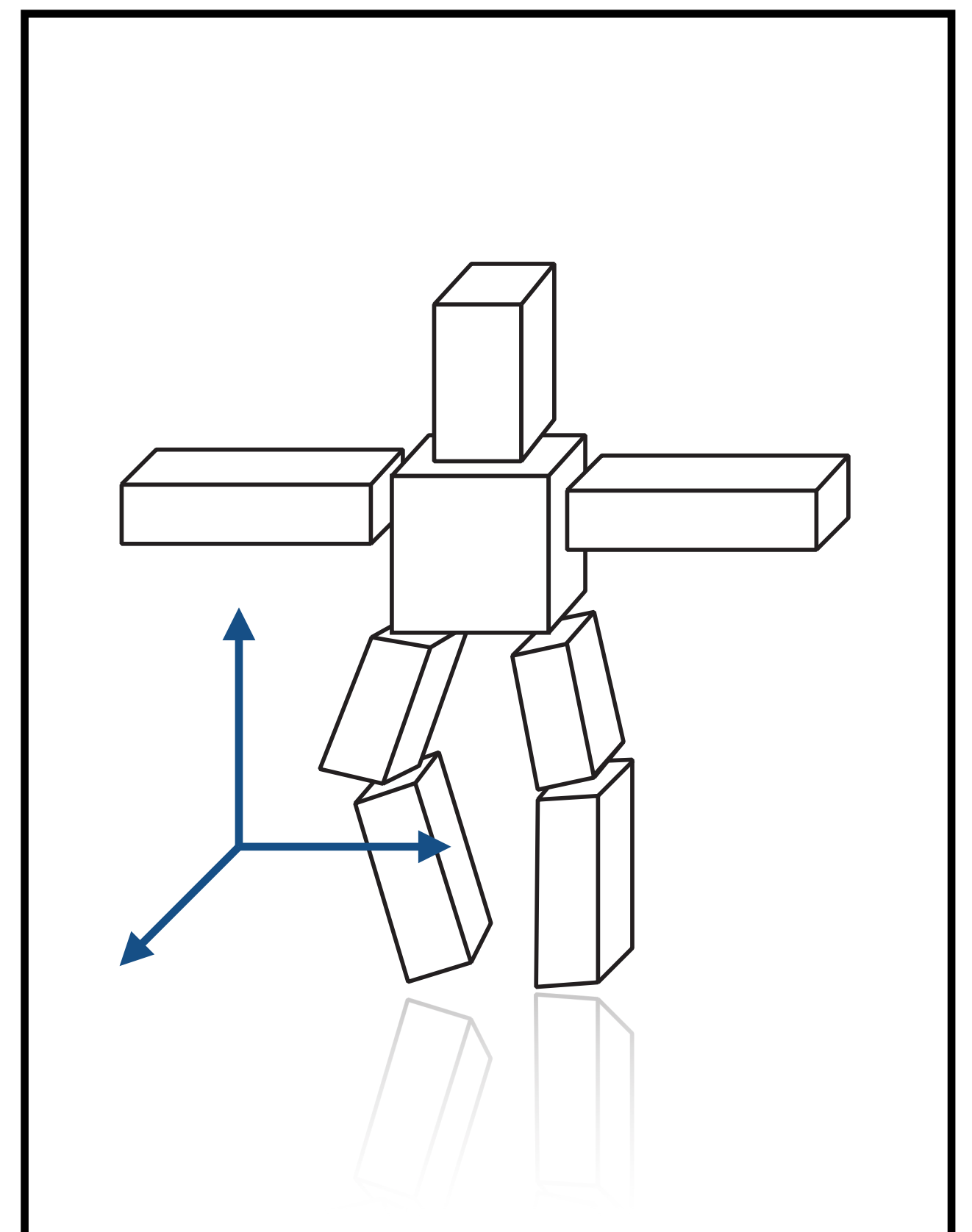
World coords

Transforms Recap



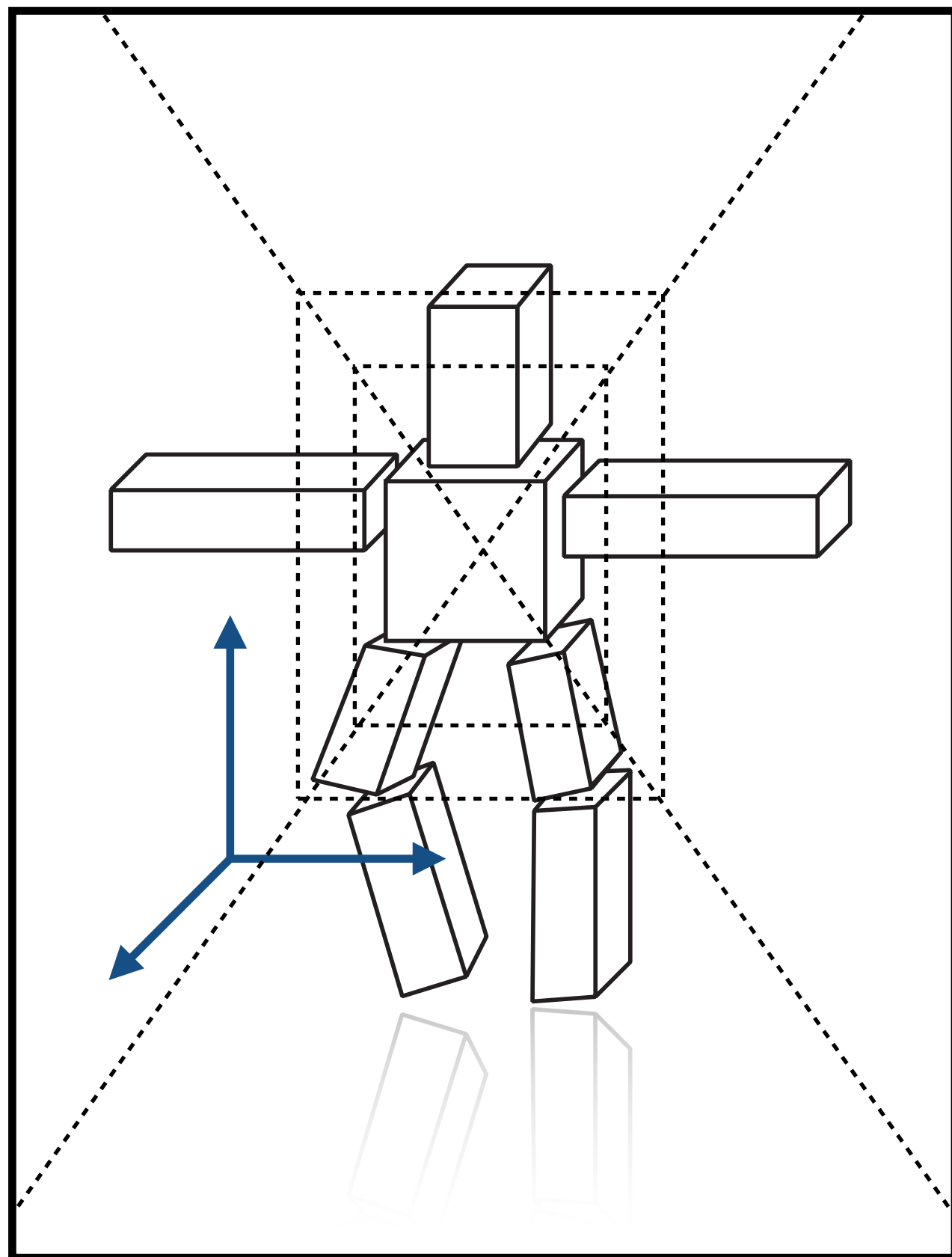
World coords

→
Viewing
transform



Camera coords

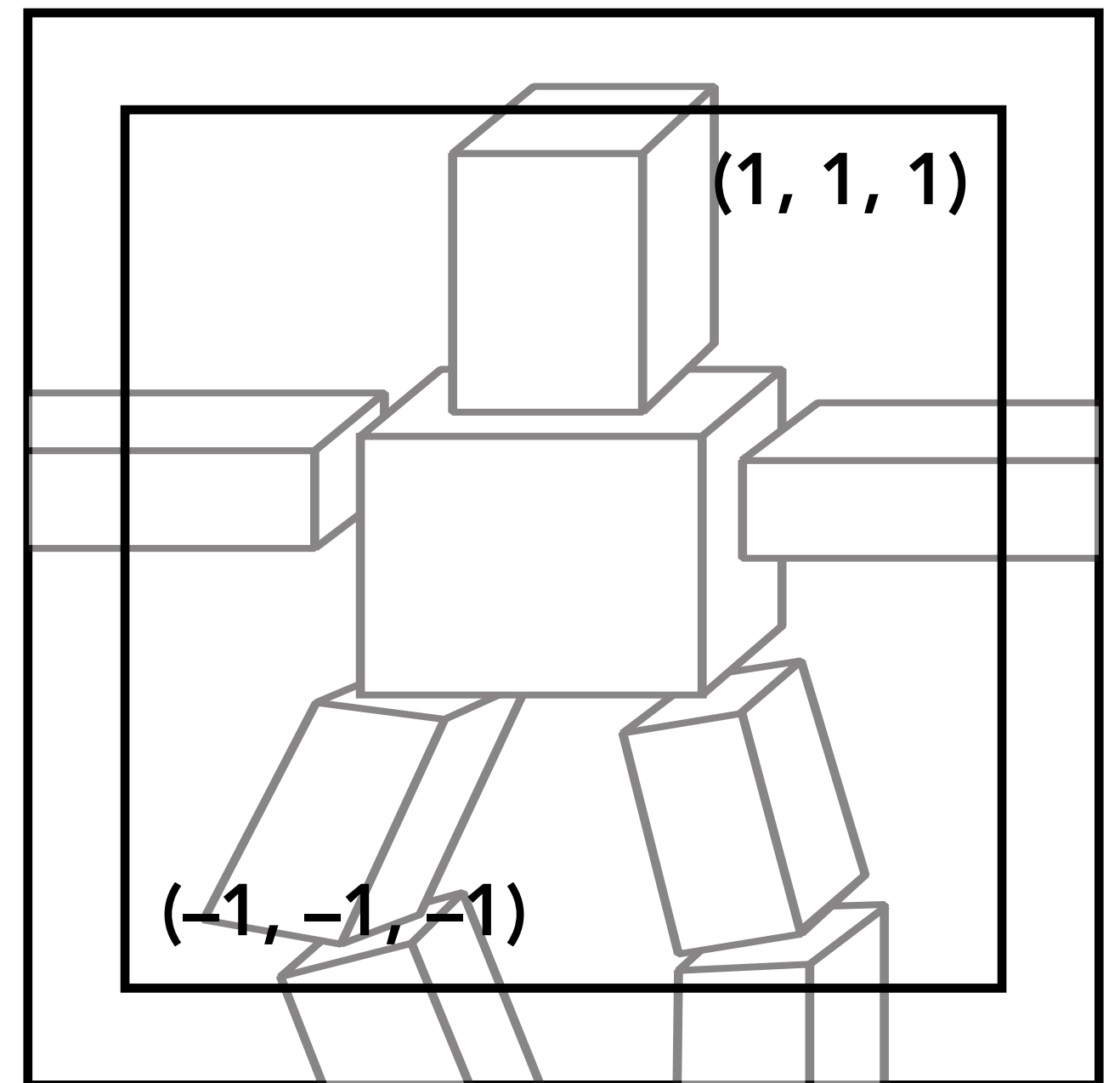
Transforms Recap



Camera coords

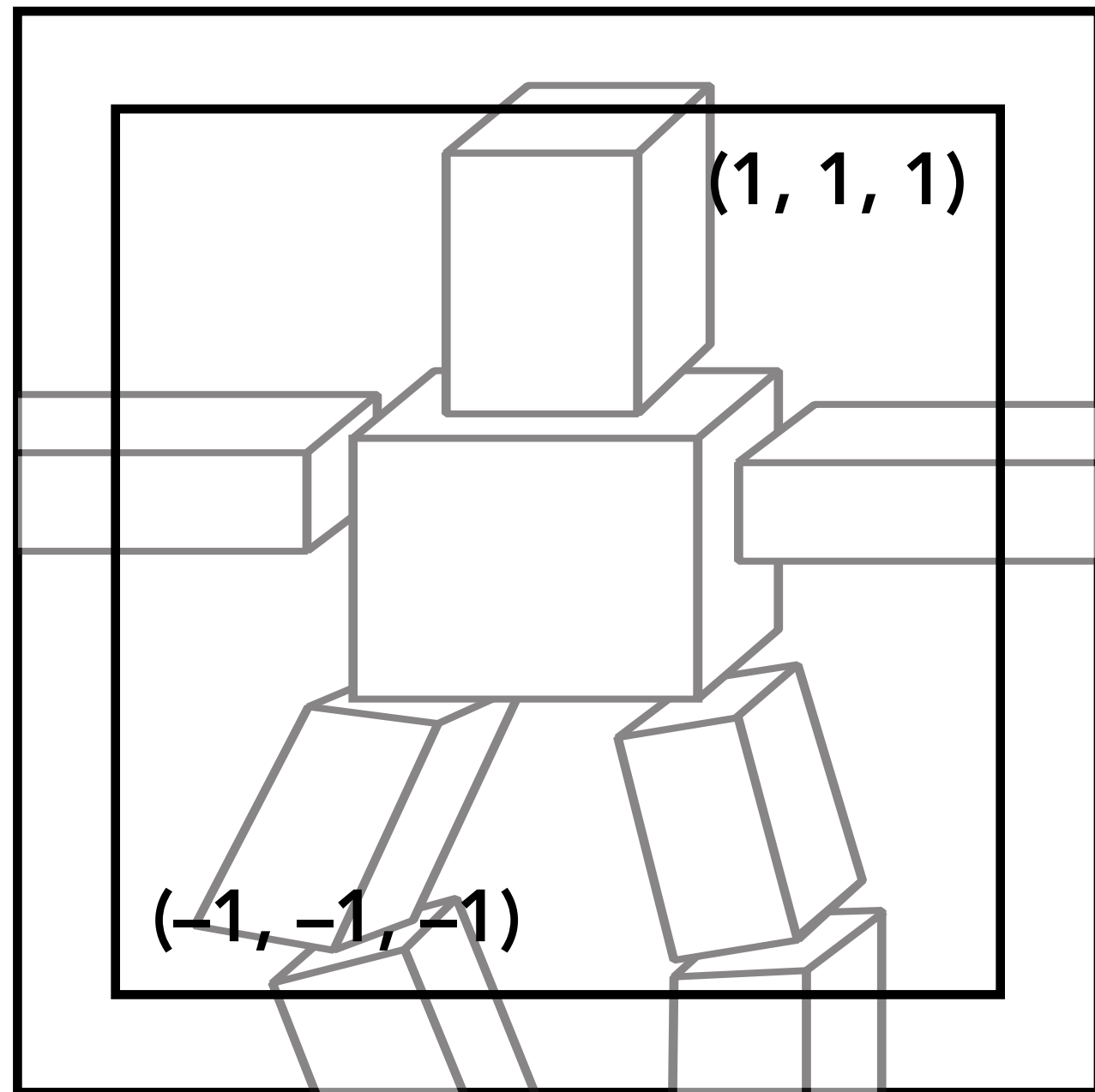


**Perspective
projection
and
homogeneous
divide**



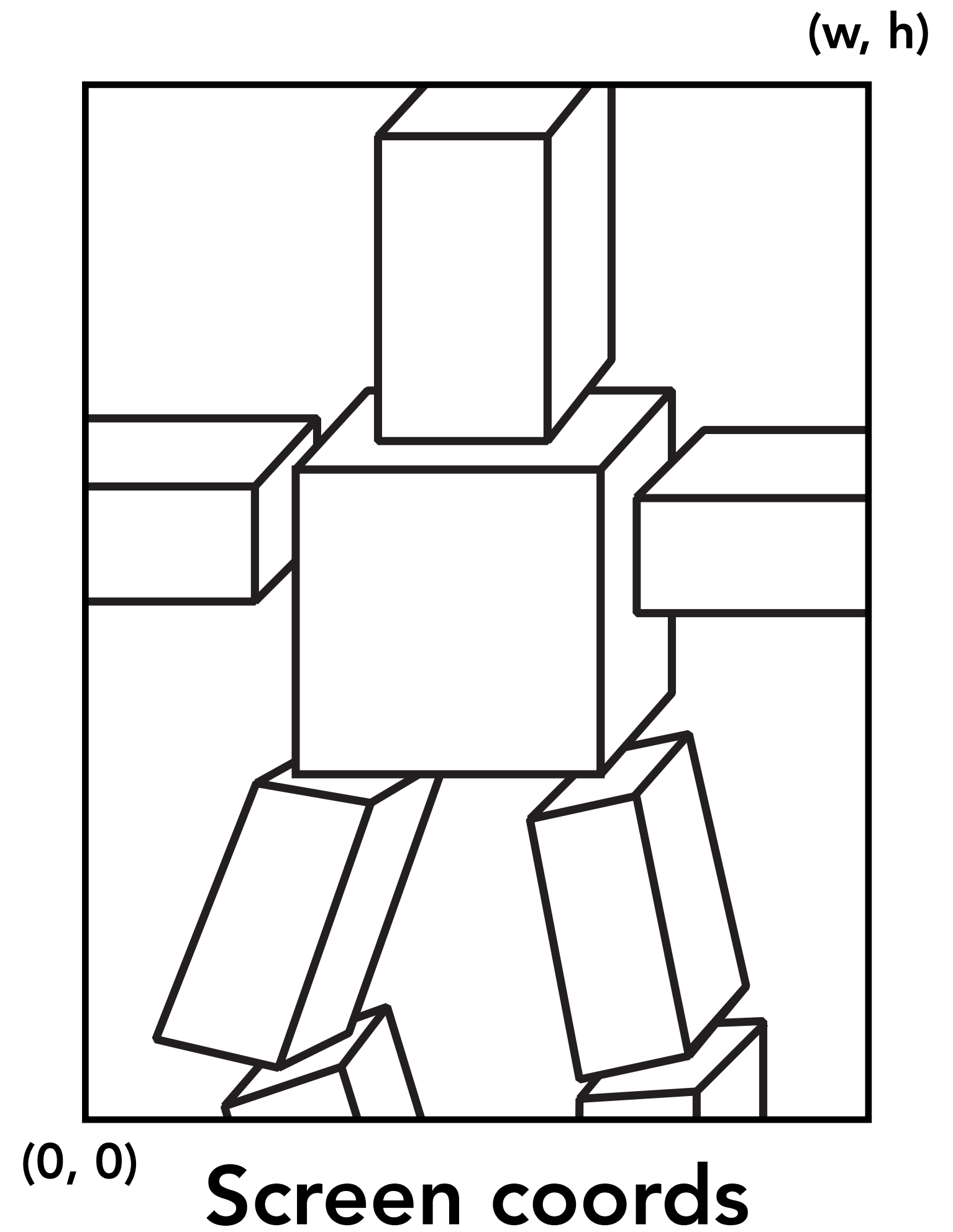
NDC

Transforms Recap

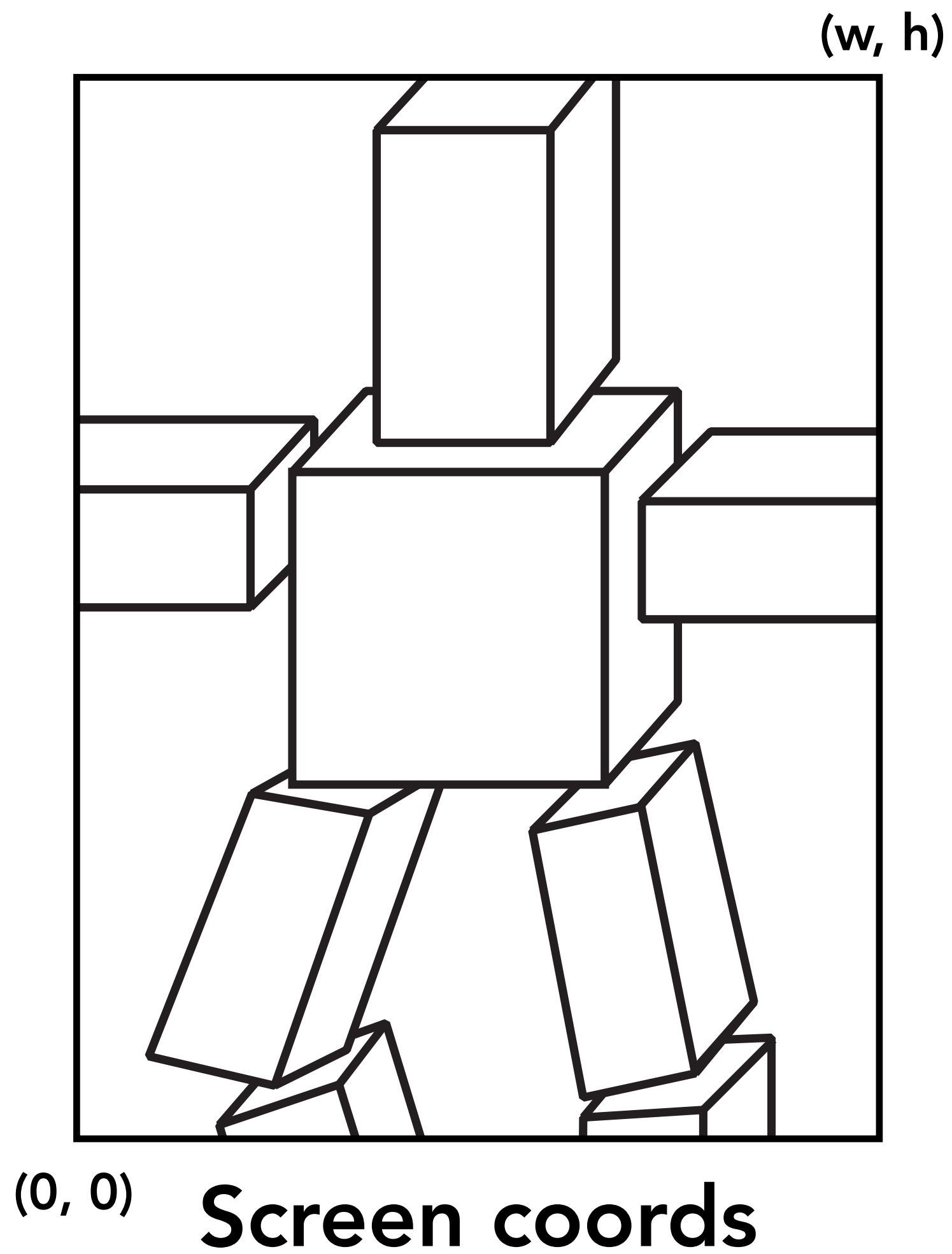


NDC

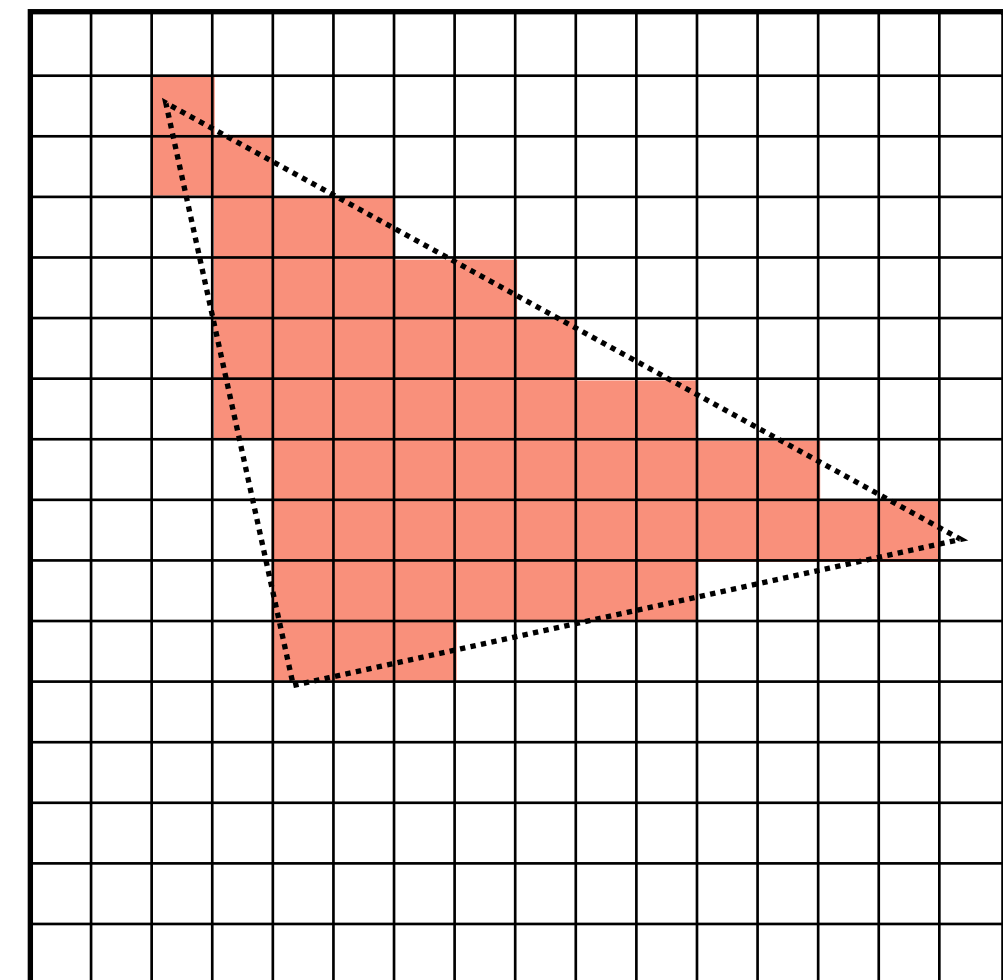
→
**Screen
transform**



Transforms Recap



Rasterization



Things to Remember

Transform uses

- Basic transforms: rotate, scale, translate, ...
- Modeling, viewing, projection, perspective
- Change in coordinate system
- Hierarchical scene descriptions by push/pop

Implementing transforms

- Linear transforms = matrices
- Transform composition = matrix multiplication
- Homogeneous coordinates for translation, projection

Acknowledgments

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