Lecture 7:

Introduction to Geometry

Computer Graphics and Imaging

UC Berkeley CS184

Course Roadmap

Rasterization Pipeline

Core Concepts

- Sampling
- Antialiasing
- Transforms

Intro

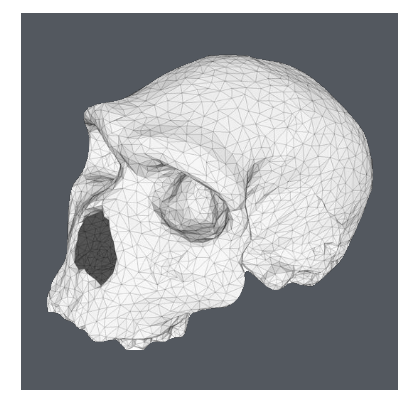
Rasterization

Transforms & Projection

Texture Mapping

Today: Visibility, Shading, Overall Pipeline

Geometric Modeling



Today

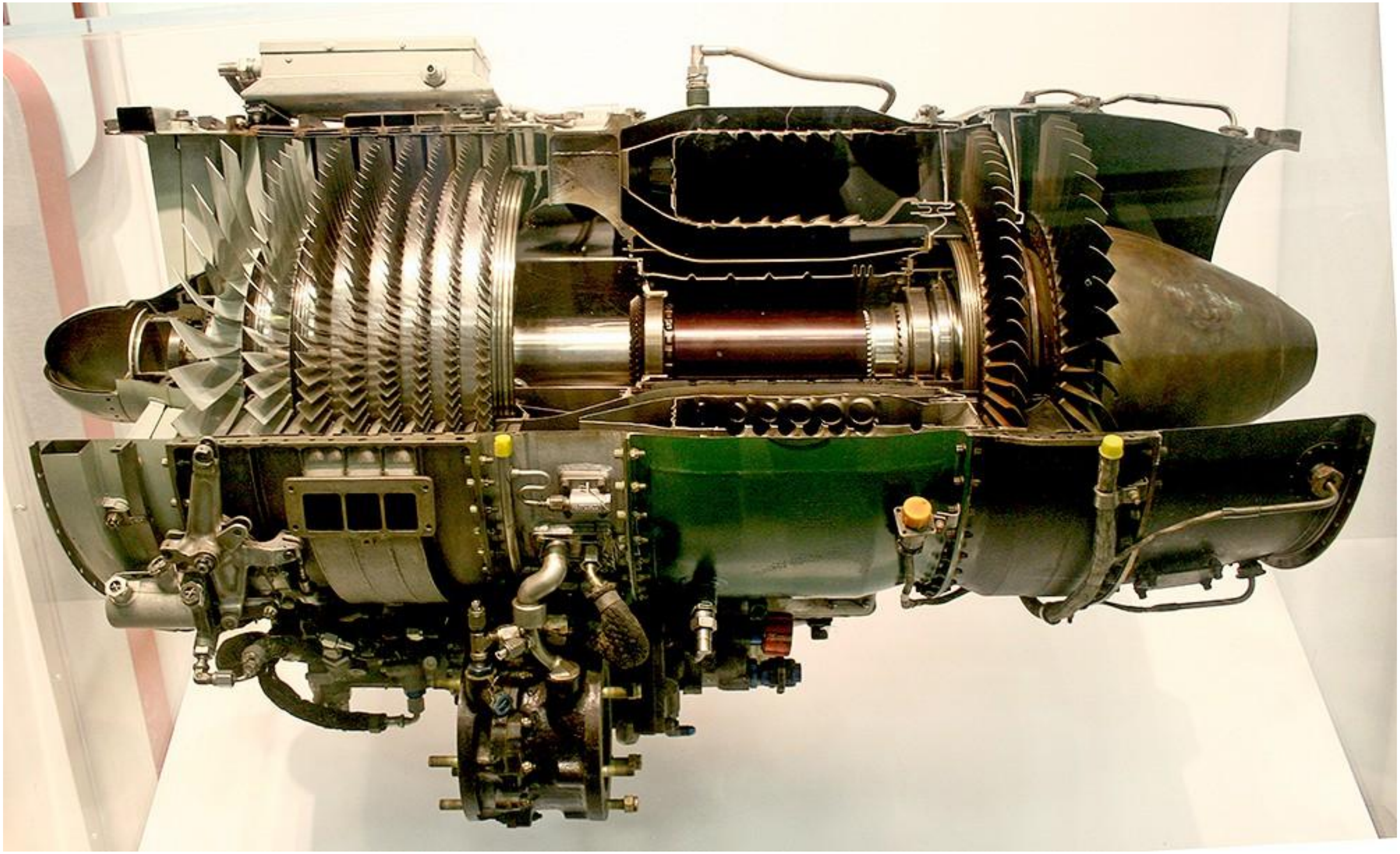
Lighting & Materials

Cameras & Imaging

Examples of Geometry

Examples of Geometry

Examples of Geometry



Examples of Geometry

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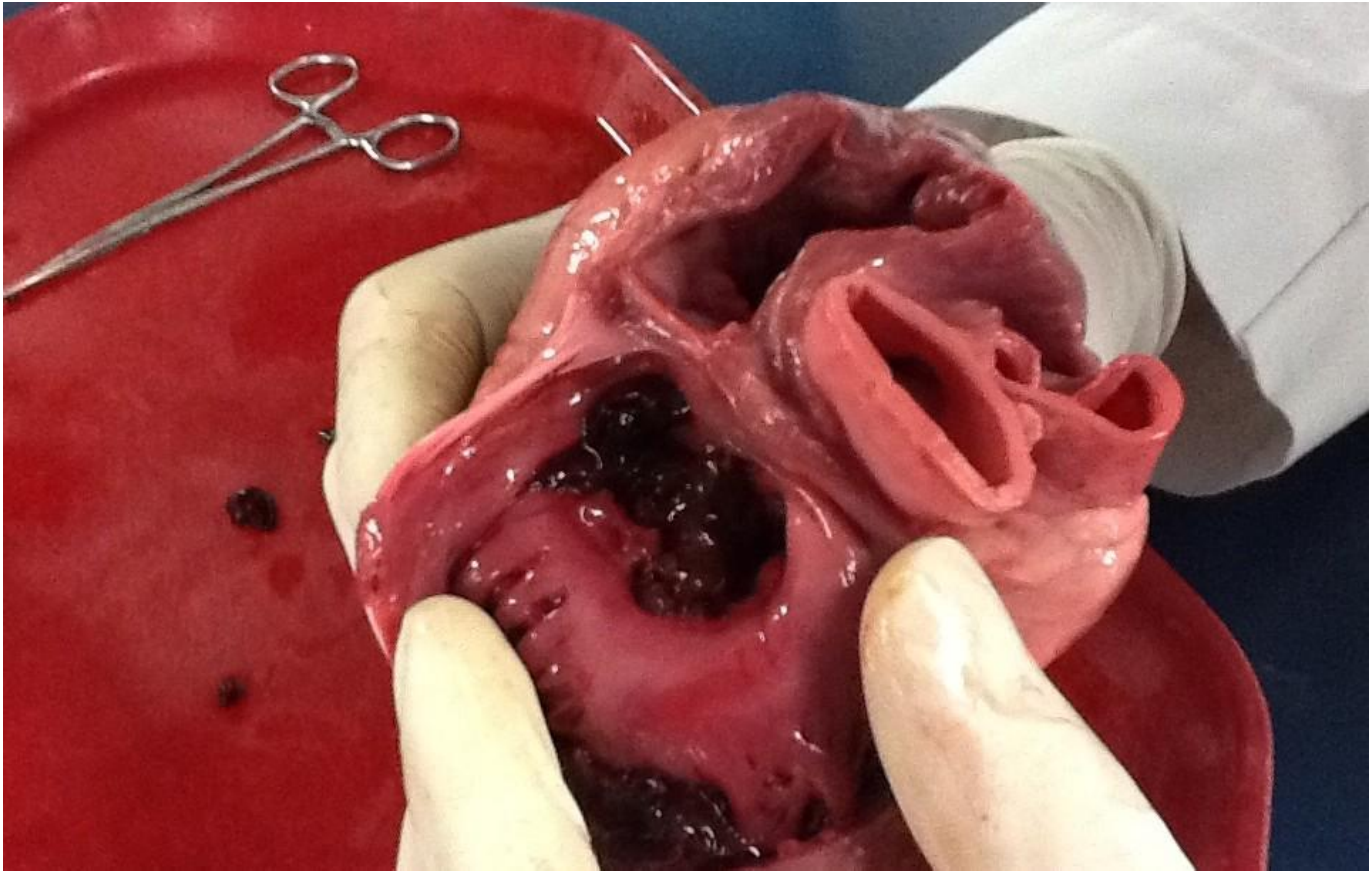
Examples of Geometry



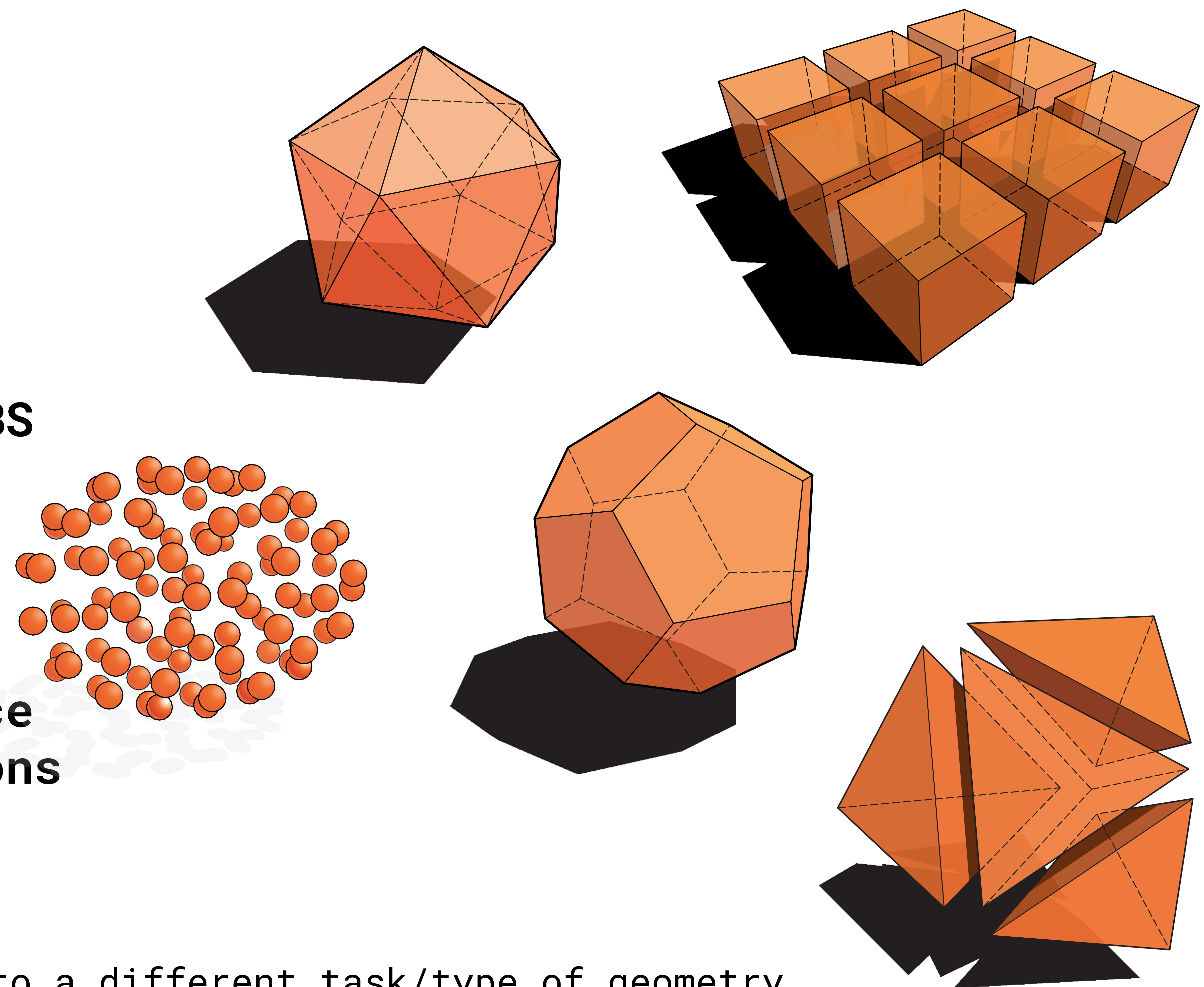
Many Ways to Represent Geometry

Explicit

- **point cloud**
- **polygon mesh**
- **subdivision, NURBS**
- **...**

Implicit

- **level sets**
- **algebraic surface**
- **distance functions**
- **...**



Each choice best suited to a different task/type of geometry

Smooth Curves

Smooth Curves and Surfaces

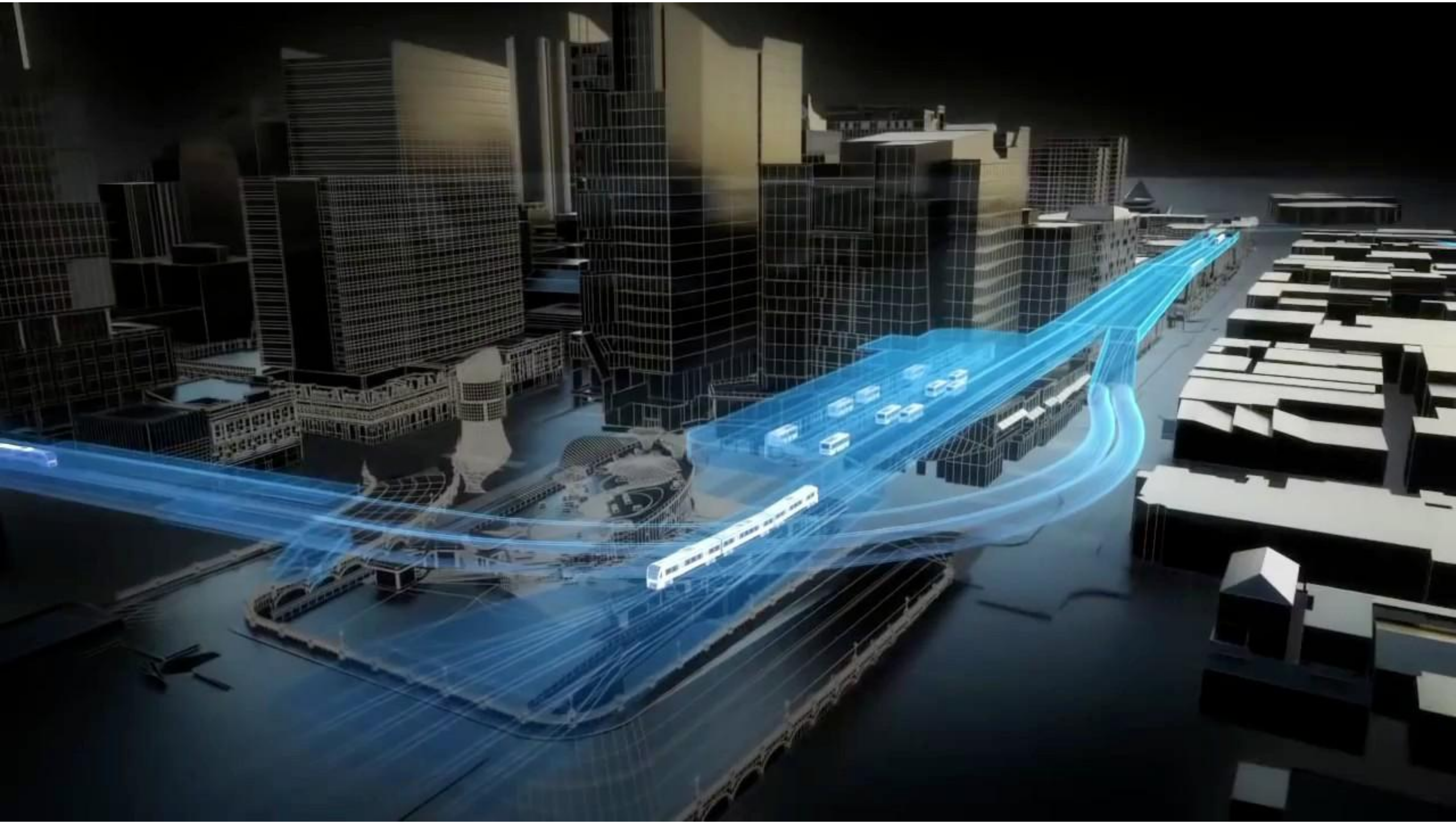
So far we can make:

- Things with corners (line, triangle, Cube, …)
- Piecewise shapes (circle, ellipse, …)

Many applications require smooth shapes

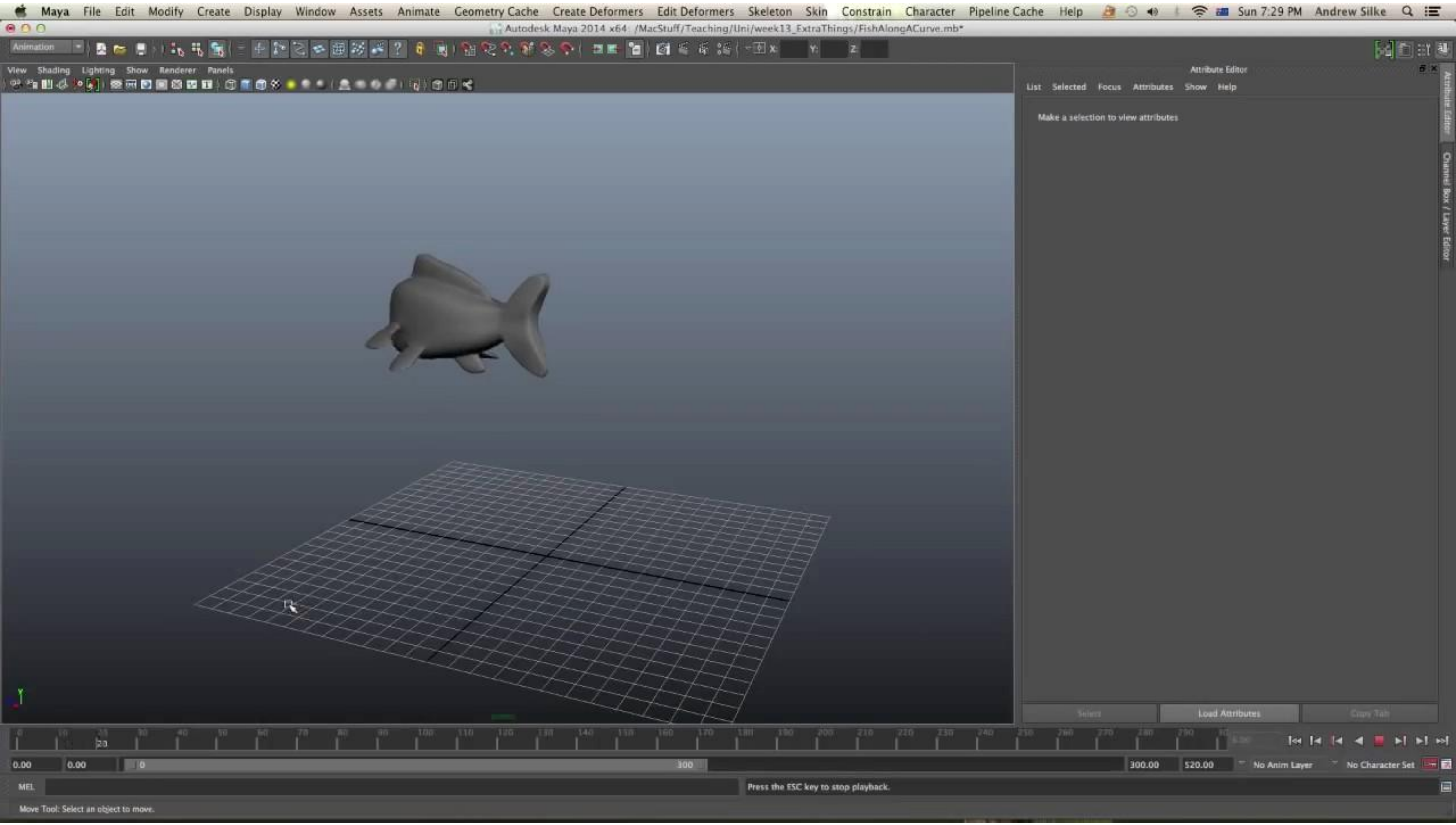
- Camera paths, vector fonts, …
- CAD design, object modeling, …
- Resampling filter functions

Camera Paths



Flythrough of proposed Perth Citylink subway, https://youtu.be/rIJMuQPwr3E

Animation Curves

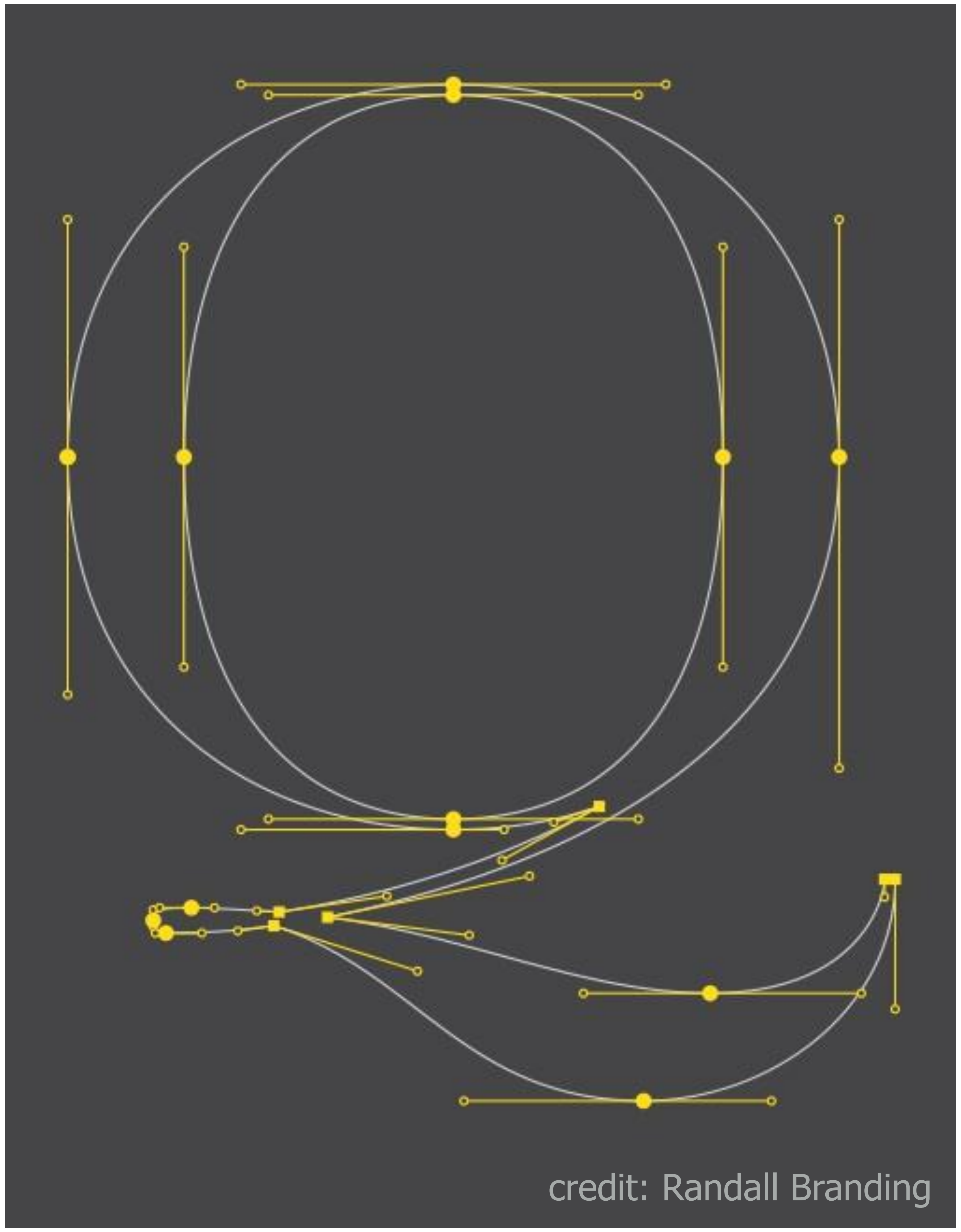


Maya Animation Tutorial: https://youtu.be/b-o5wtZlJPc

Vector Fonts

The Quick Brown Fox Jumps Over The Lazy Dog

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 0123456789



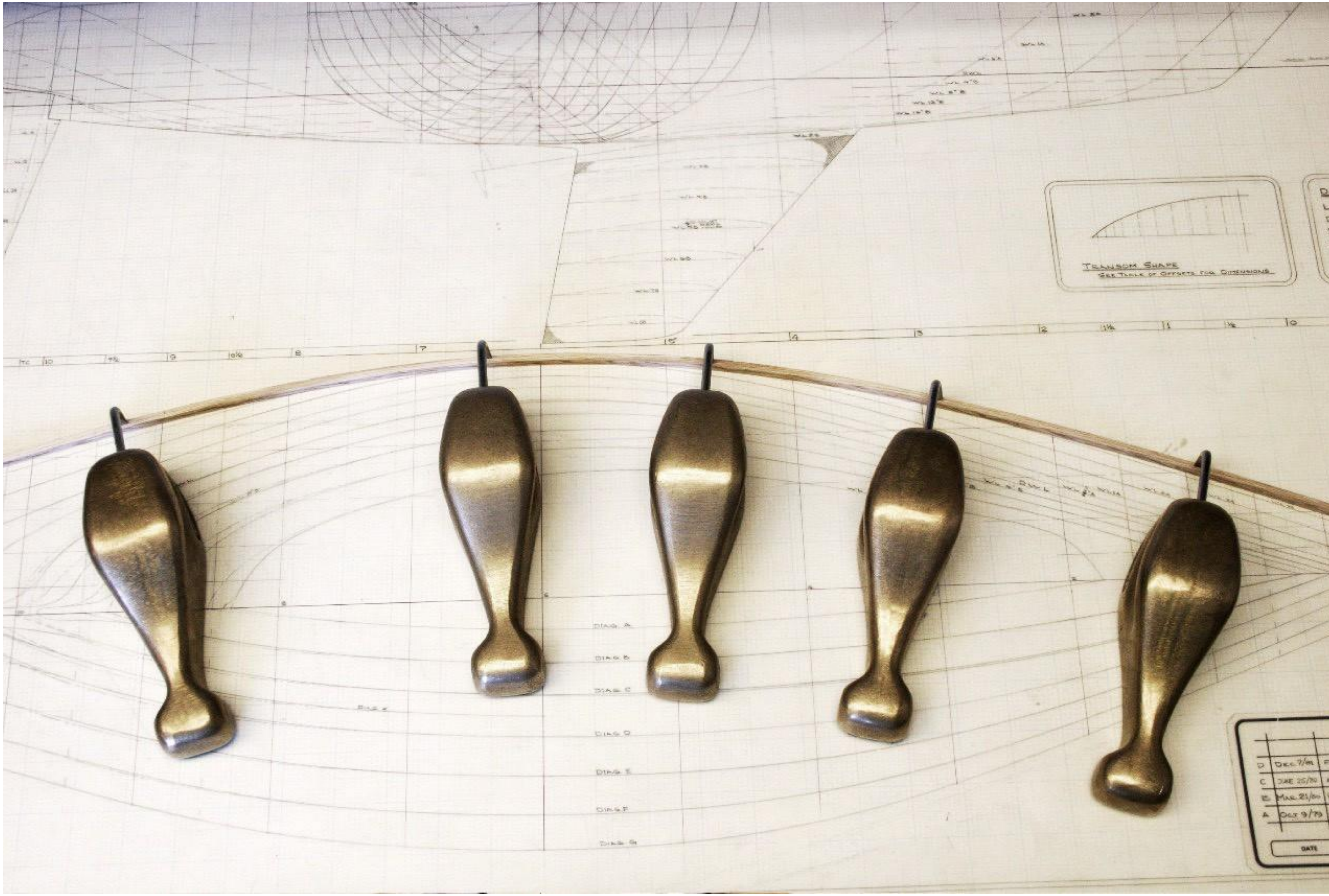
Baskerville font - represented as cubic Bézier splines

CAD Design

3D Car Modeling with Rhinoceros

Splines

Physical Spline for Hand-Drafting



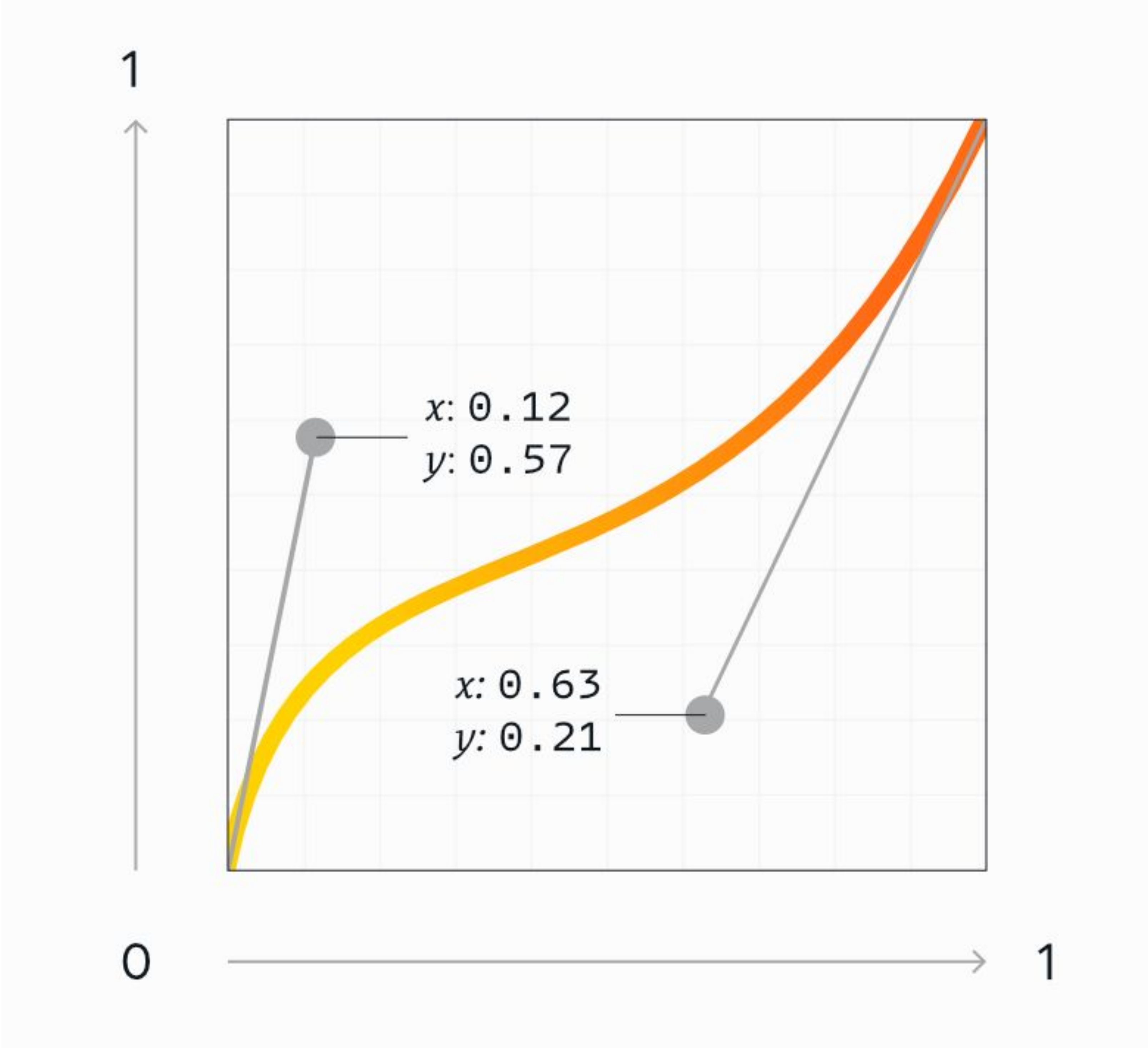
http://www.alatown.com/spline-history-architecture/

Spline Topics

Interpolation

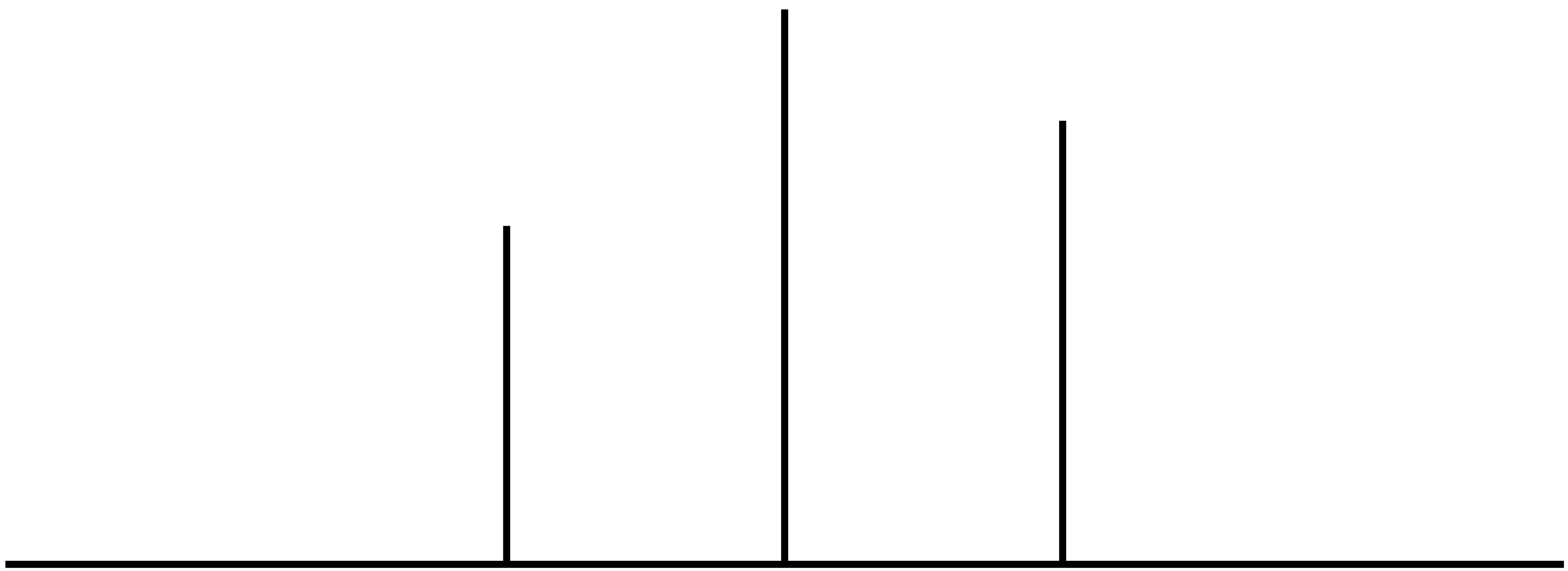
- Cubic Hermite interpolation
- Catmull-Rom interpolation

- Bezier curves
- Bezier surfaces

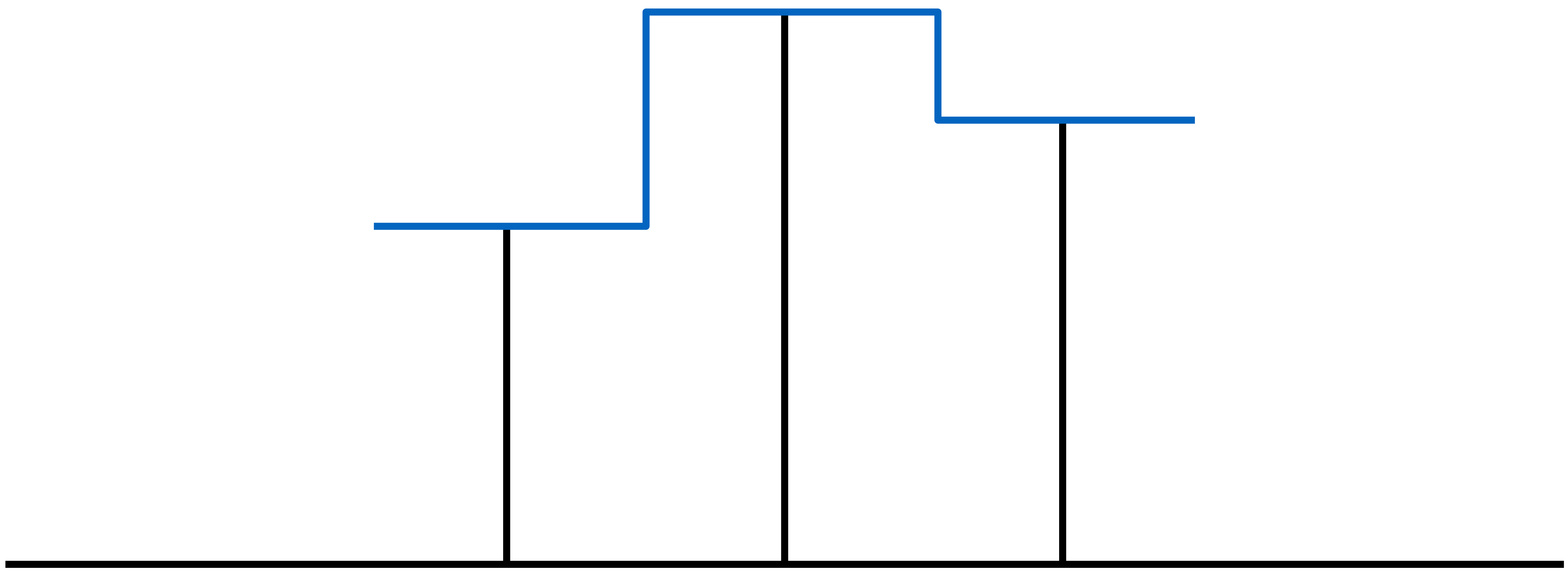


Cubic Hermite Interpolation

Goal: Interpolate Values

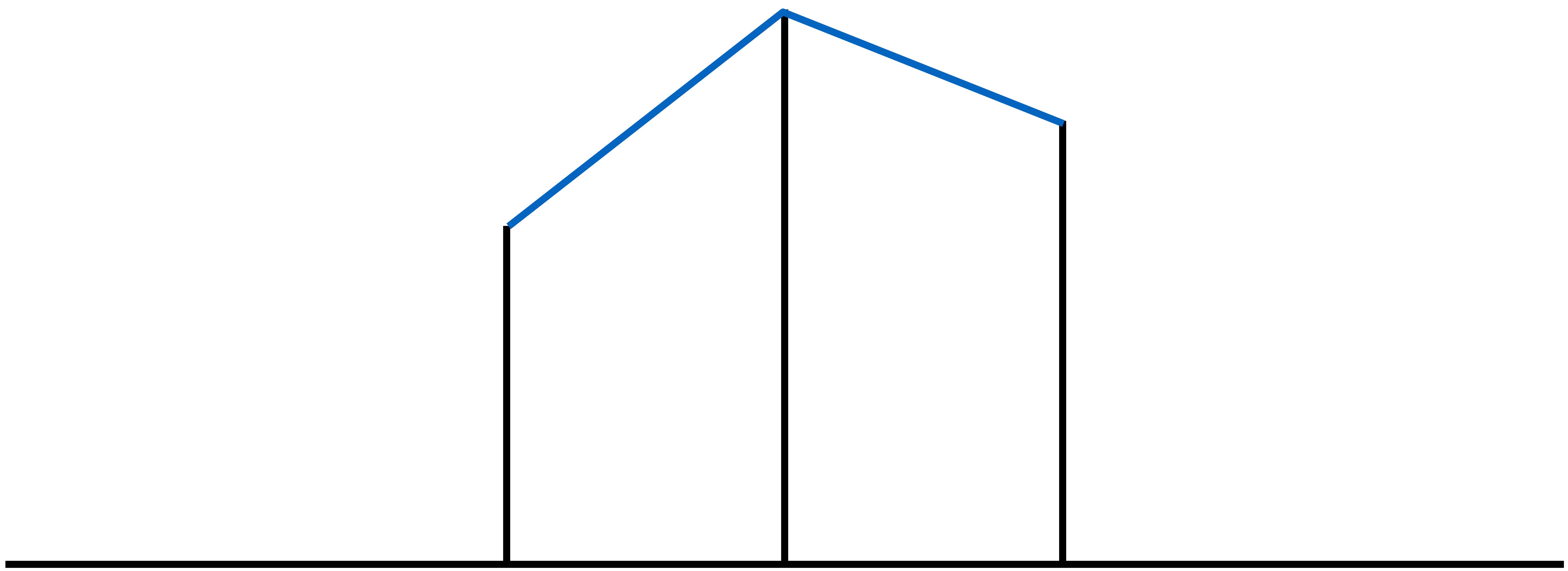


Nearest Neighbor Interpolation



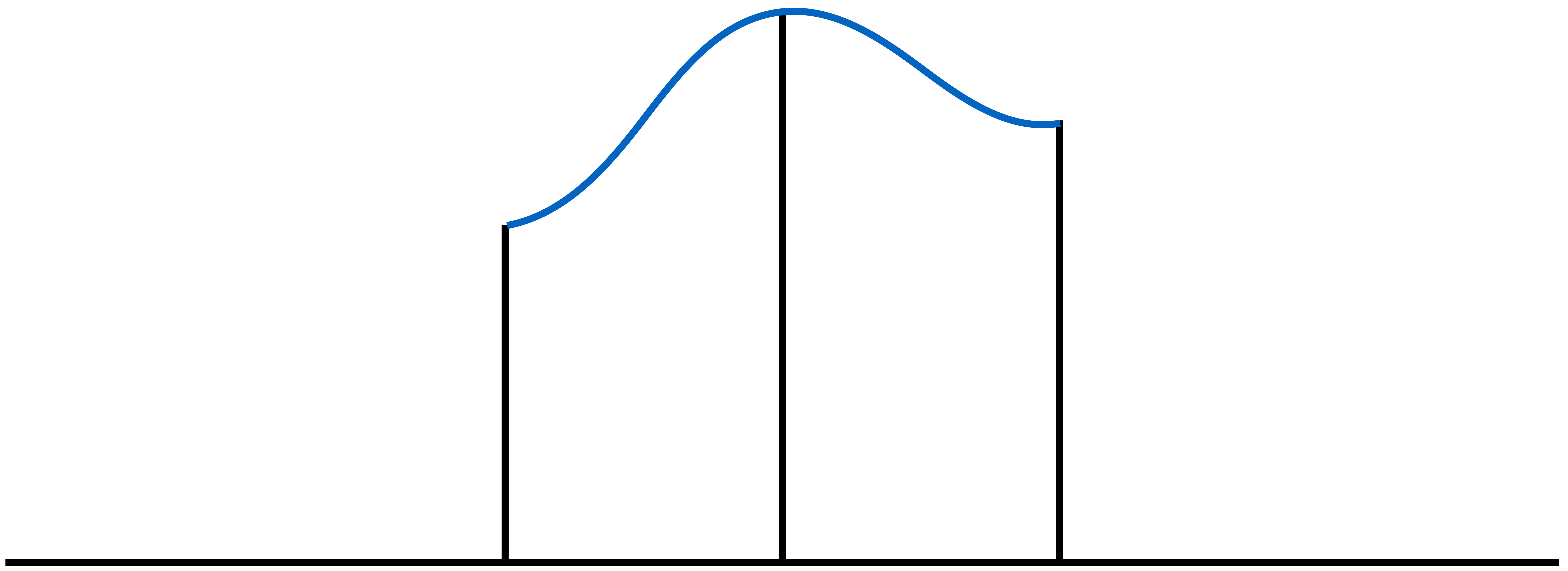
Problem: values not continuous

Linear Interpolation

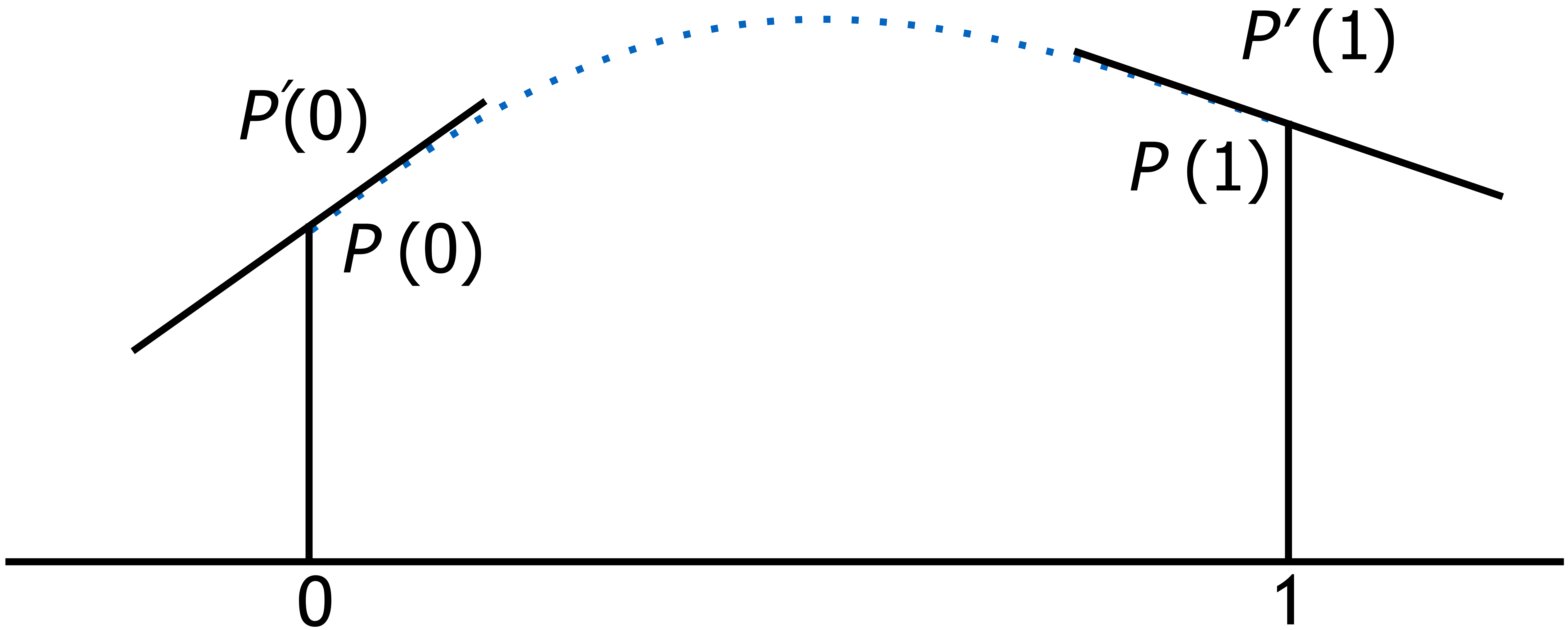


Problem: derivatives not continuous

Smooth Interpolation?



Cubic Hermite Interpolation



Inputs: values and derivatives at endpoints

Cubic Polynomial Interpolation

Cubic polynomial

$$P(t) = a\,t^3 + bt^2 + c\,t + d$$

Why cubic?

We have 4 input constraints – need 4 degrees of freedom:

$$P(0) = h_0$$
$$P(1) = h_1$$
$$P'(0) = h_2$$
$$P'(1) = h_3$$

Cubic Polynomial Interpolation

Cubic polynomial

$$P(t) = at^3 + bt^2 + ct + d$$

$$P'(t) = 3at^2 + 2bt + c$$

Set up constraint equations

$$P(0) = h_0 = d$$

$$P(1) = h_1 = a + b + c + d$$

$$P'(0) = h_2 = c$$

$$P'(1) = h_3 = 3a + 2b + c$$

Hermite Basis Functions

$$P(t) = [\ t^3 \quad t^2 \quad t \quad 1\ ] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = [\ H_0(t) \quad H_1(t) \quad H_2(t) \quad H_3(t)\ ] \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

t^3

t^2

t

1

Basis functions for cubic polynomials

$H_0(t) = 2t^3 - 3t^2 + 1$

$H_1(t) = -2t^3 + 3t^2$

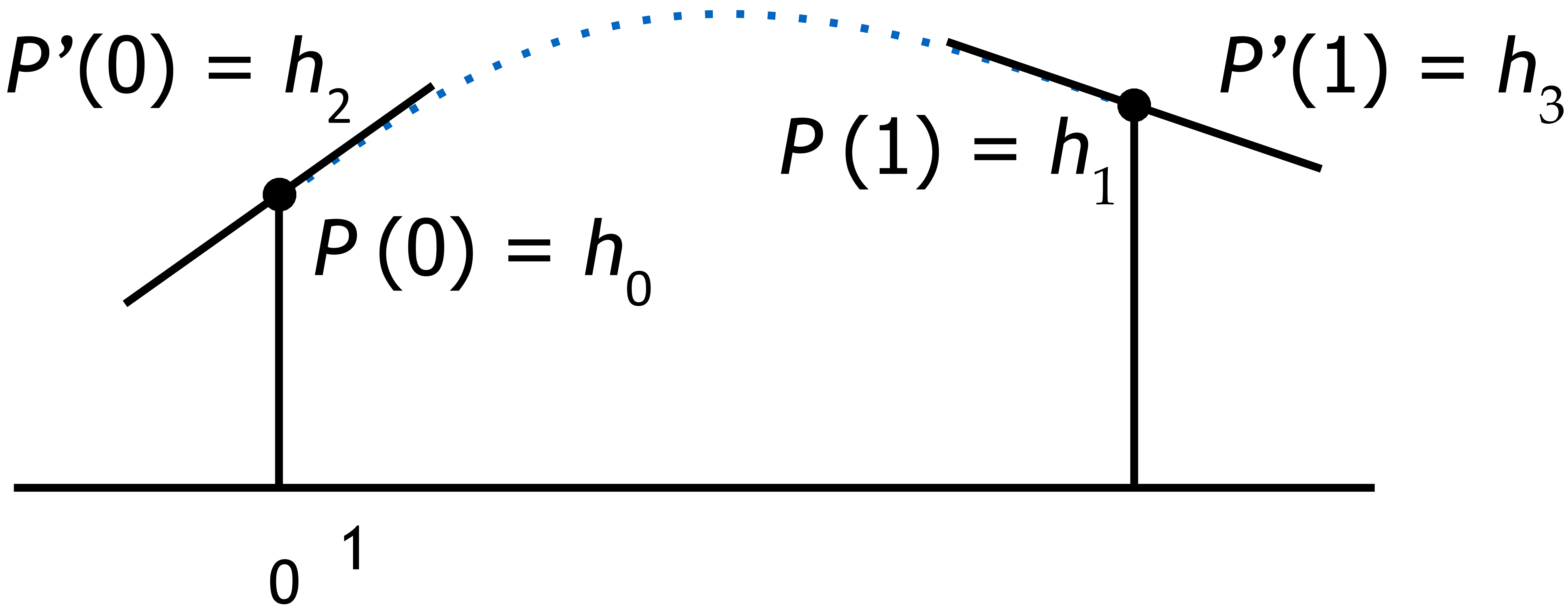
$H_2(t) = t^3 - 2t^2 + t$

$H_3(t) = t^3 - t^2$

Hermite basis functions for cubic polynomials

Either basis can represent any cubic polynomial through linear combination

Recap: Cubic Hermite Interpolation



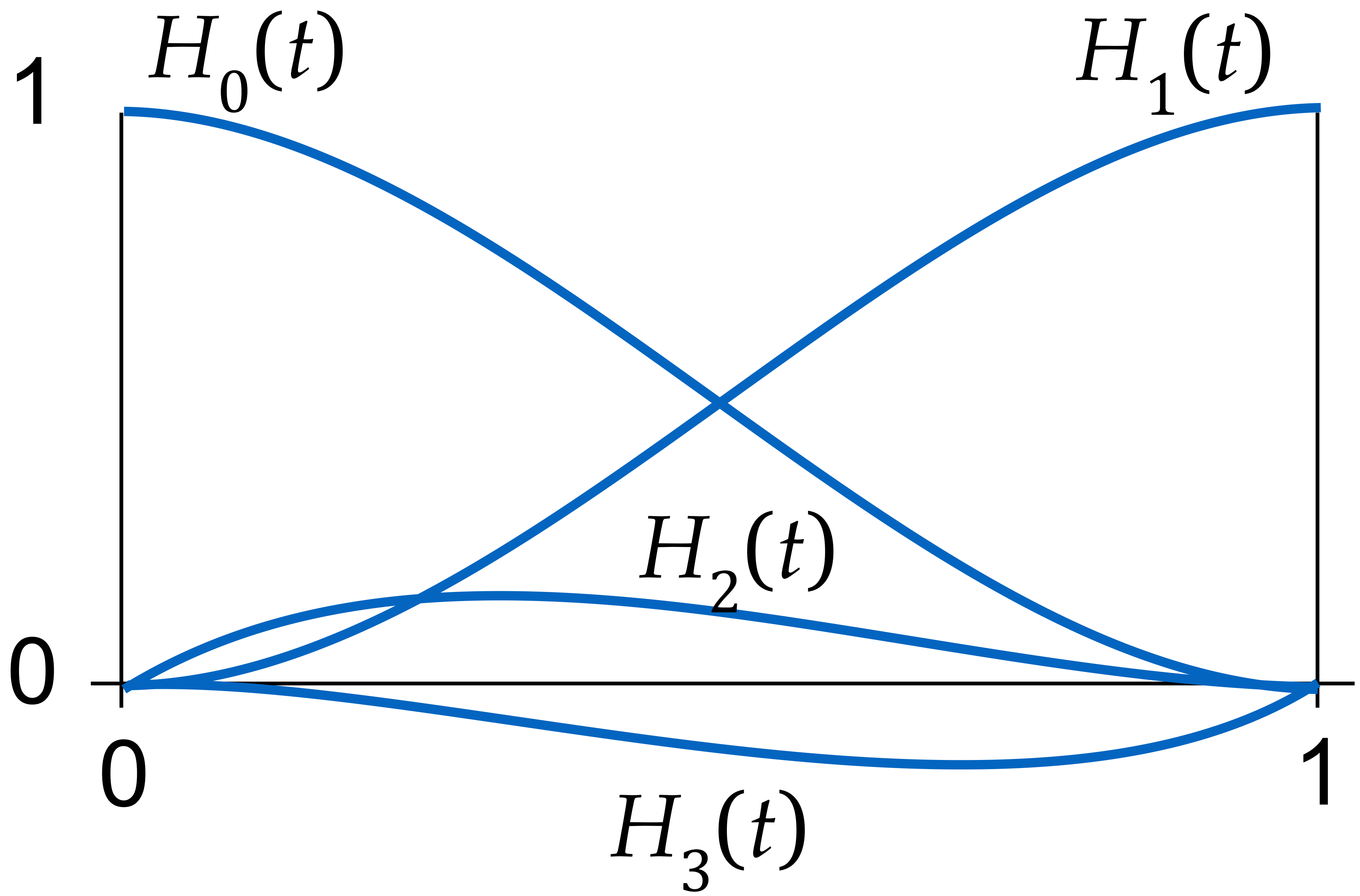
Inputs: values and derivatives at endpoints

Output: cubic polynomial that interpolates

Solution: weighted sum of Hermite basis functions

$$P(t) = h_0 H_0(t) + h_1 H_1(t) + h_2 H_2(t) + h_3 H_3(t)$$

Hermite Basis Functions



$$H_0(t) = 2t^3 - 3t^2 + 1$$

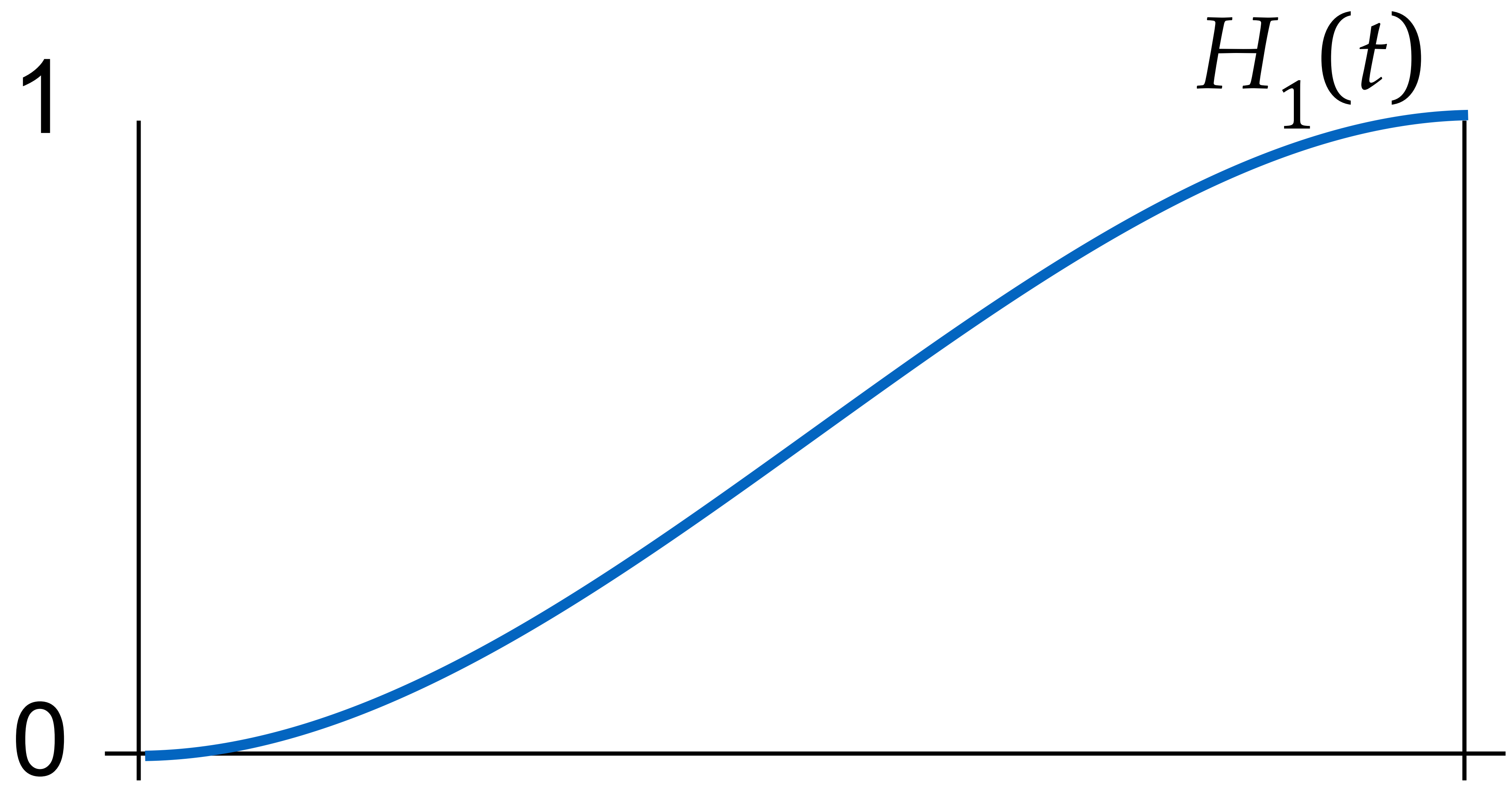
$$H_1(t) = -2t^3 + 3t^2$$

$$H_2(t) = t^3 - 2t^2 + t$$

$$H_3(t) = t^3 - t^2$$

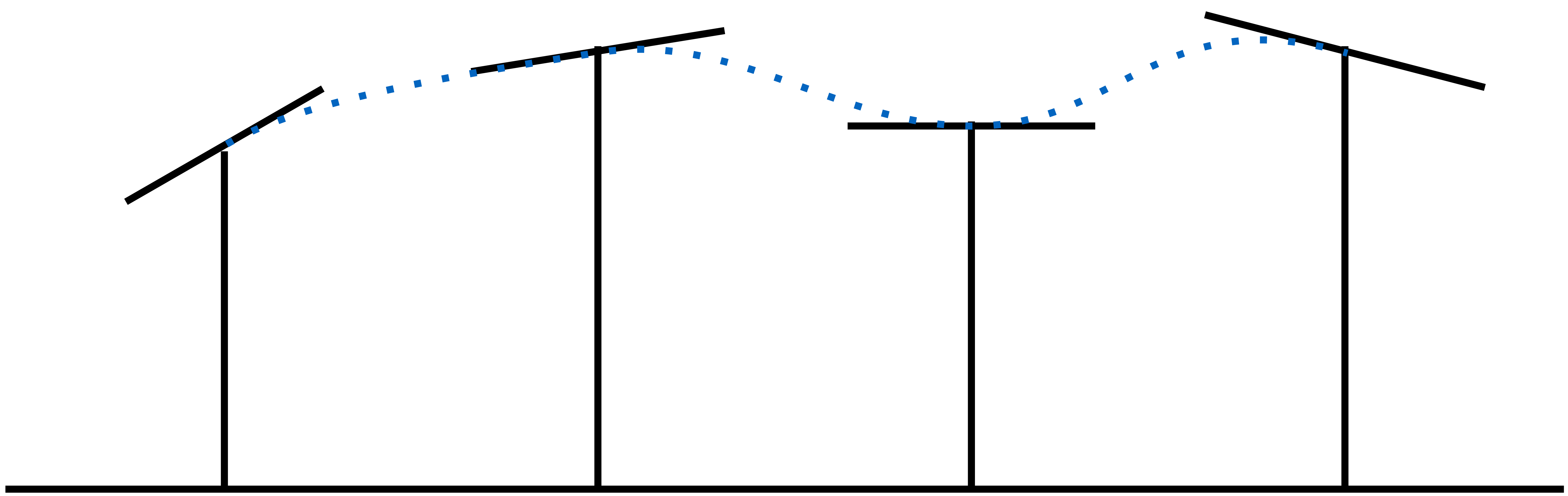
Ease Function

A very useful function in animation, start and stop gently (zero velocity)



$$H_1(t) = -2t^3 + 3t^2 = t^2(3 - 2t)$$

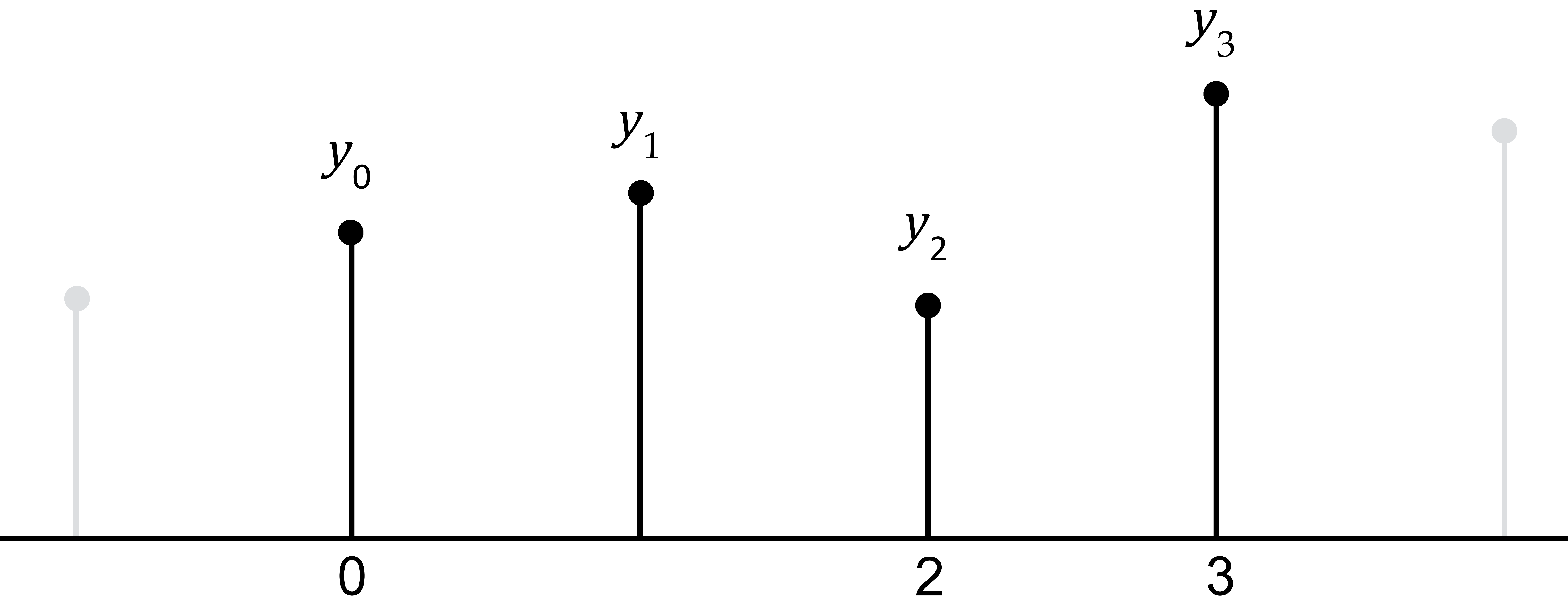
Hermite Spline Interpolation



Inputs: sequence of values and derivatives

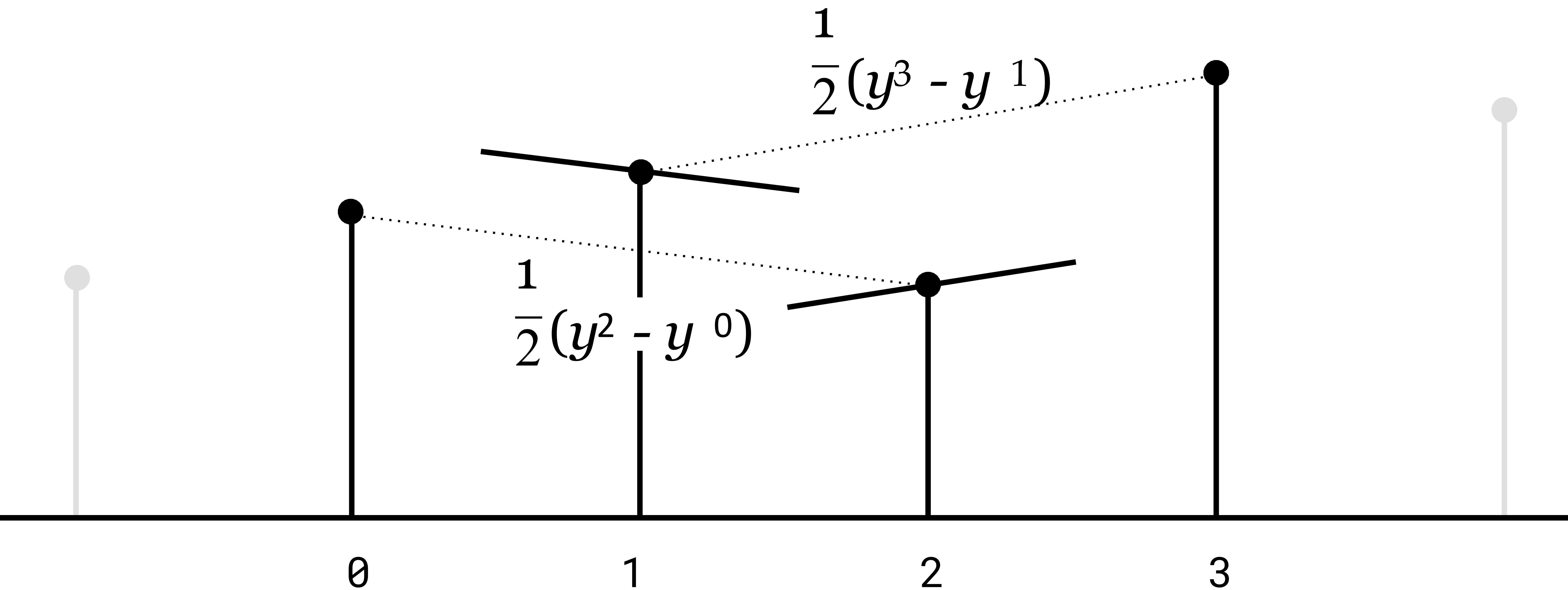
Catmull-Rom Interpolation

Catmull-Rom Interpolation



Inputs: sequence of values

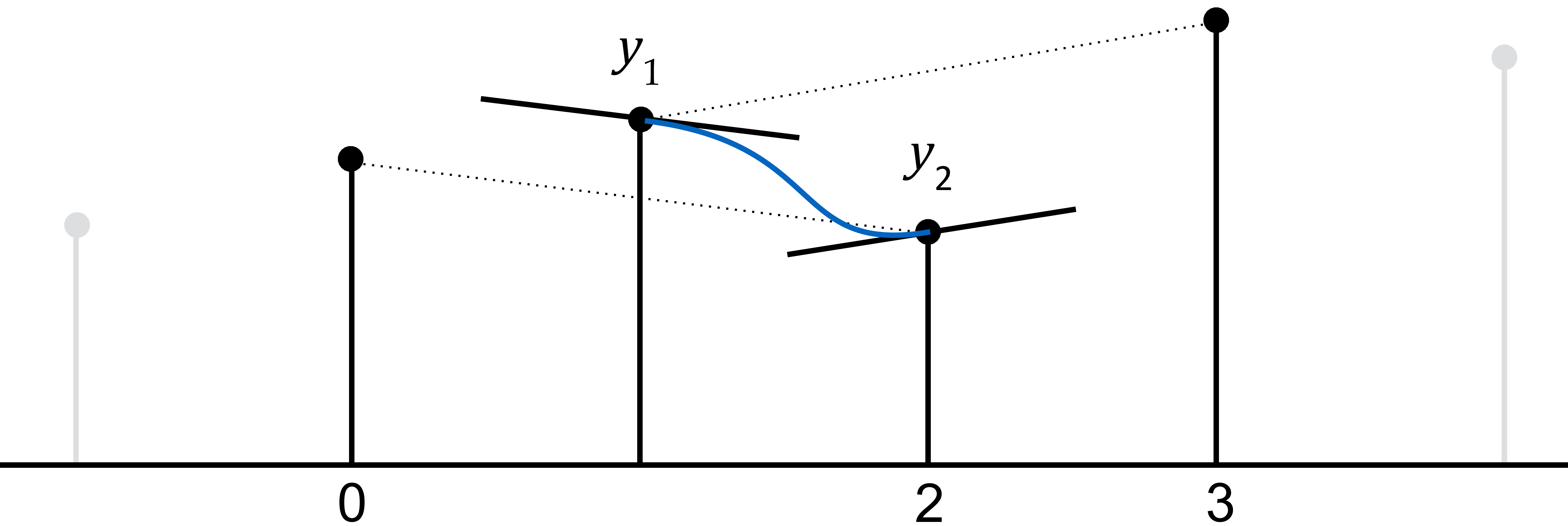
Catmull-Rom Interpolation



Rule for derivatives:

Match slope between previous and next values

Catmull-Rom Interpolation

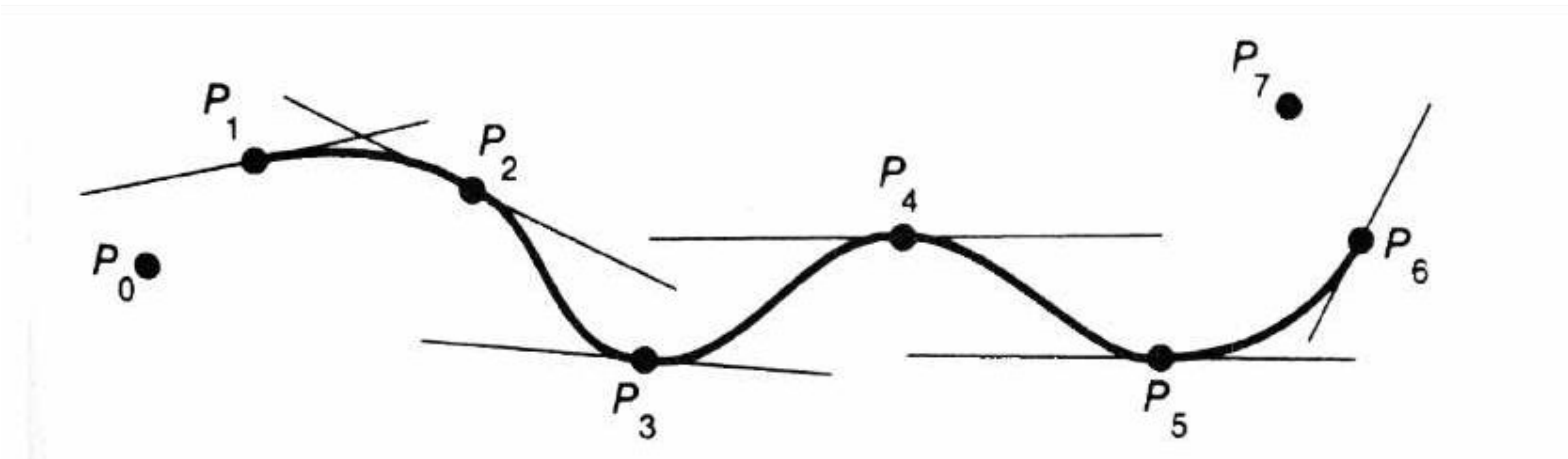


Then use Hermite interpolation

Catmull-Rom Spline

Input: sequence of points

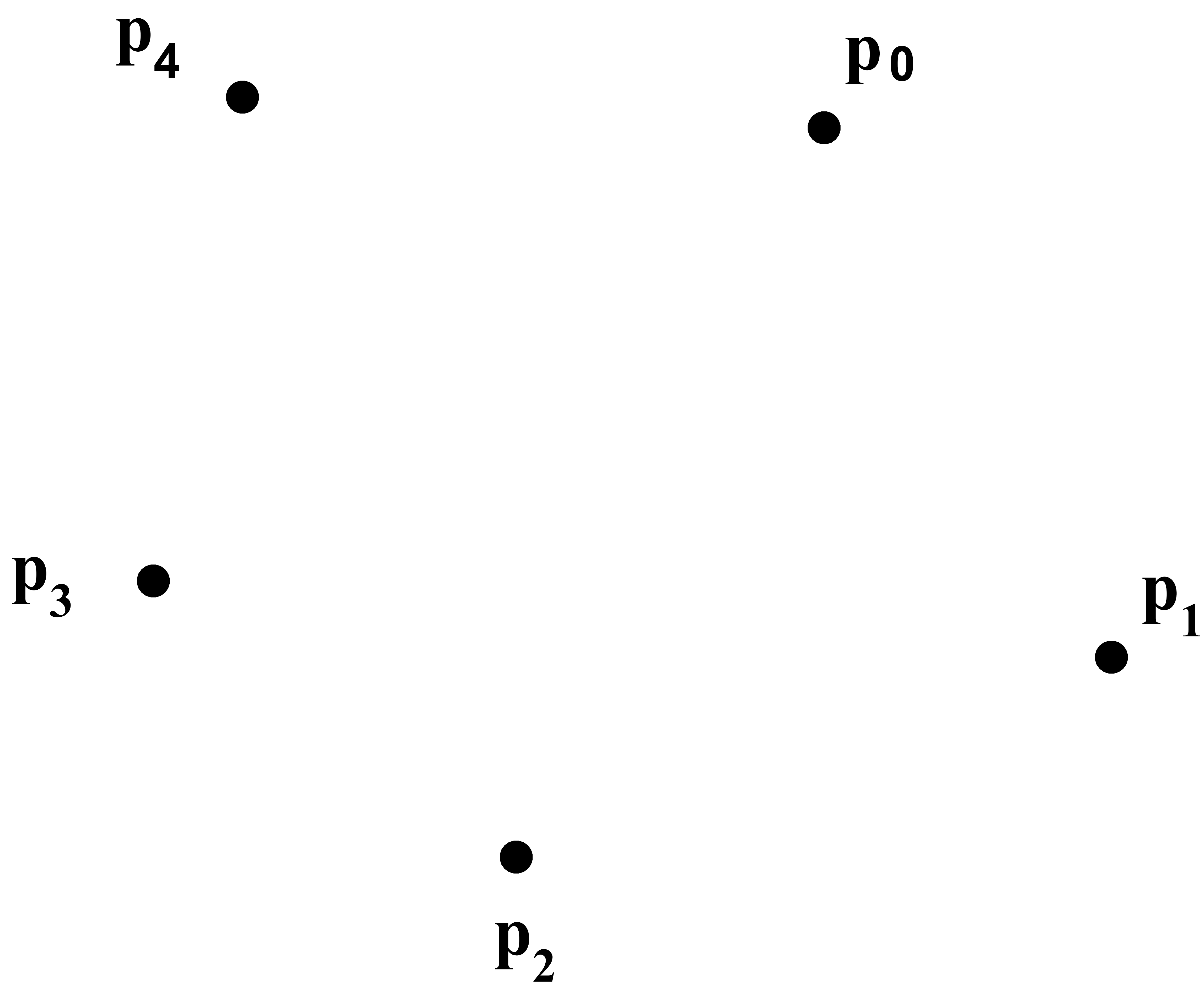
Output: spline that interpolates all points with C1 continuity



Interpolating Points & Vectors

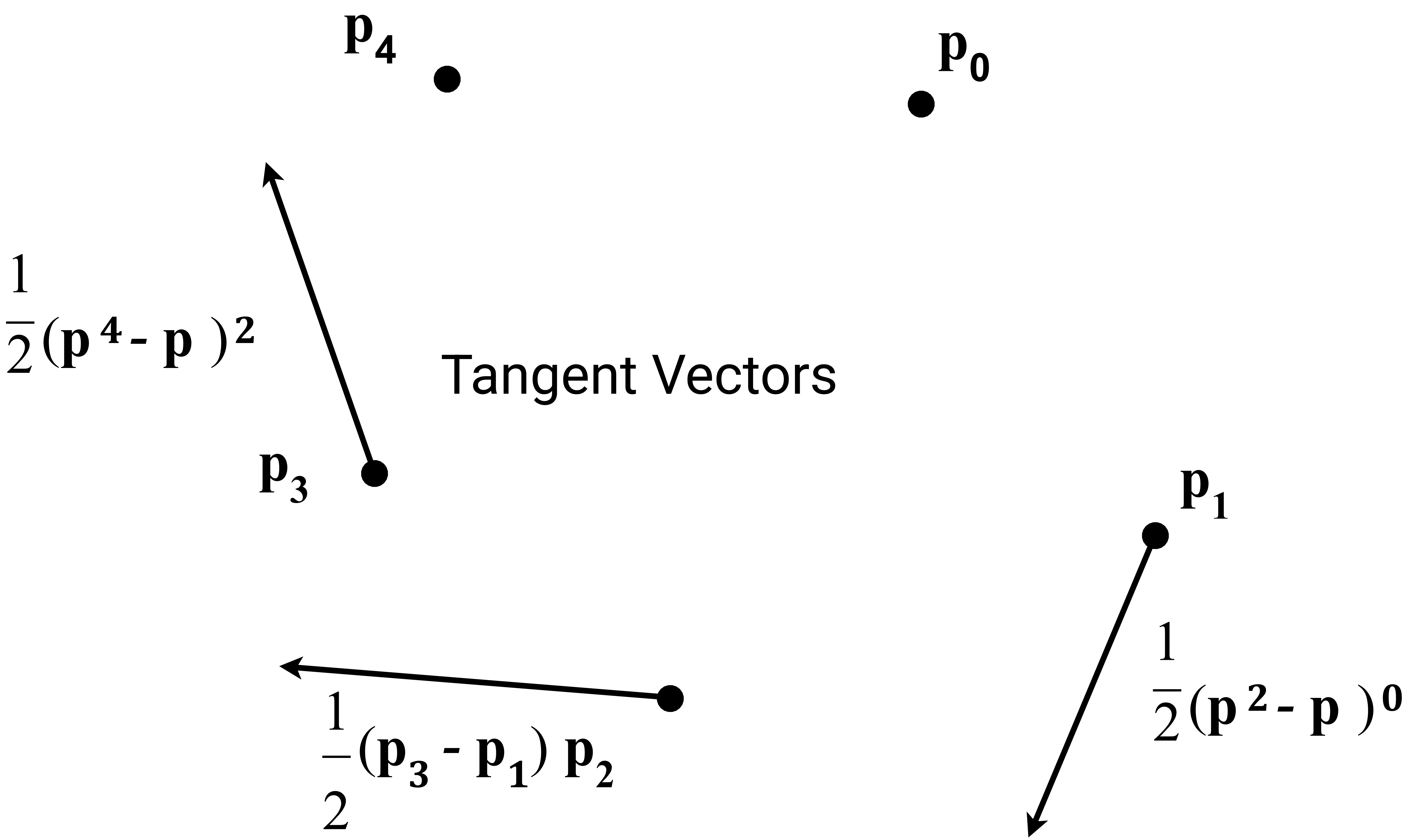
Can Interpolate Points As Easily As Values

E.g. point (0,1,3) in 3D space, or even a general vector in N dimensions



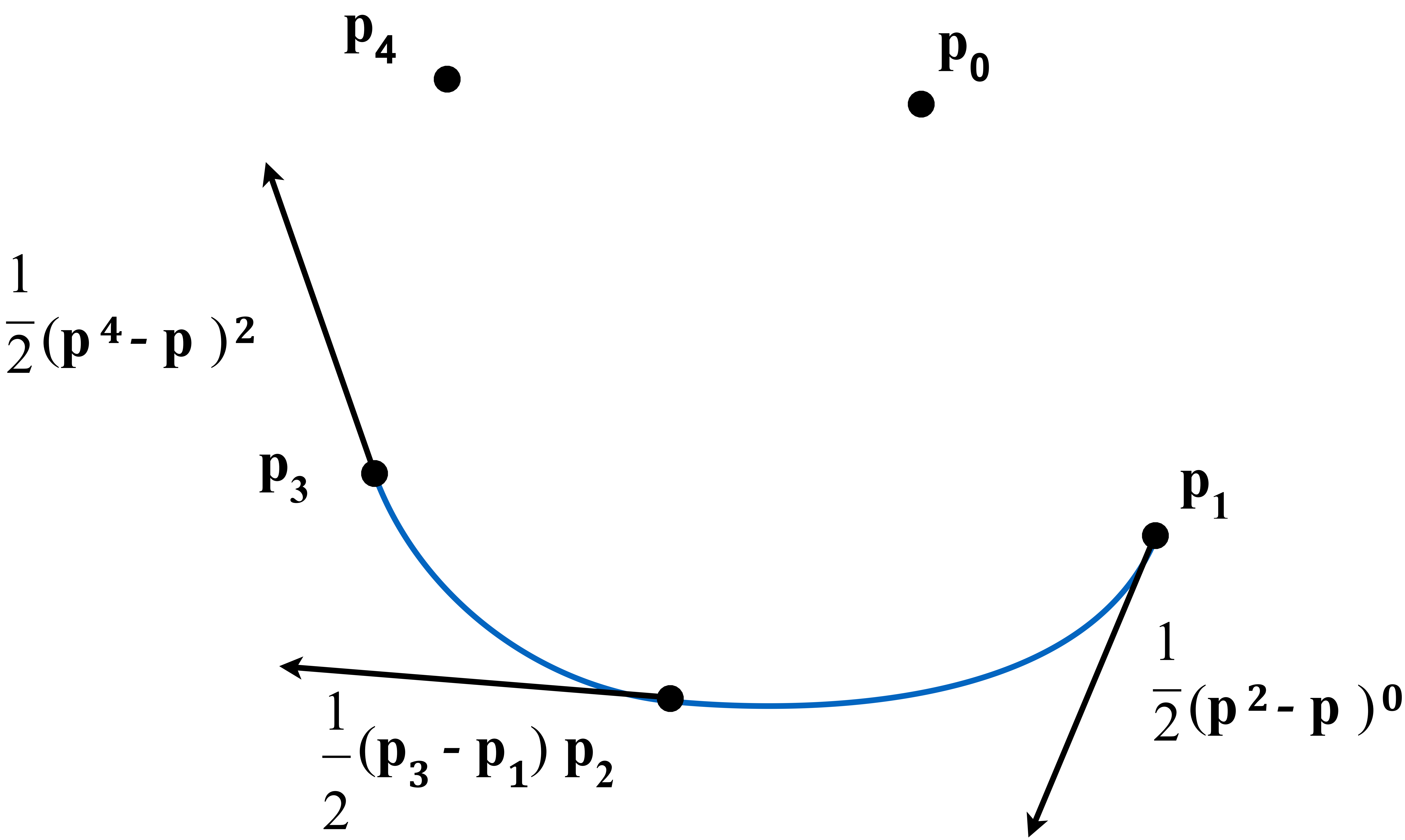
Catmull-Rom 3D spline control points

Can Interpolate Points As Easily As Values



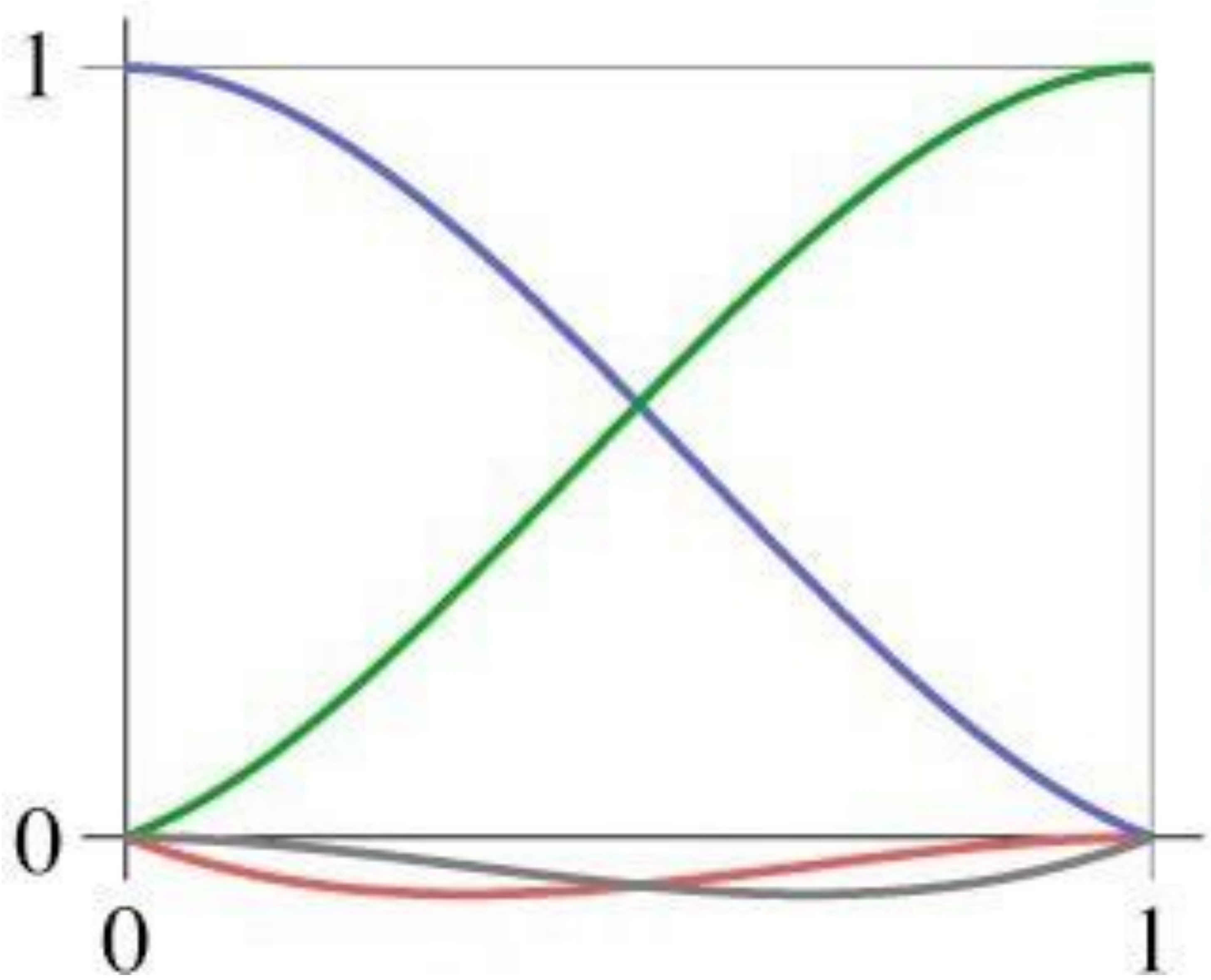
Catmull-Rom 3D tangent vectors

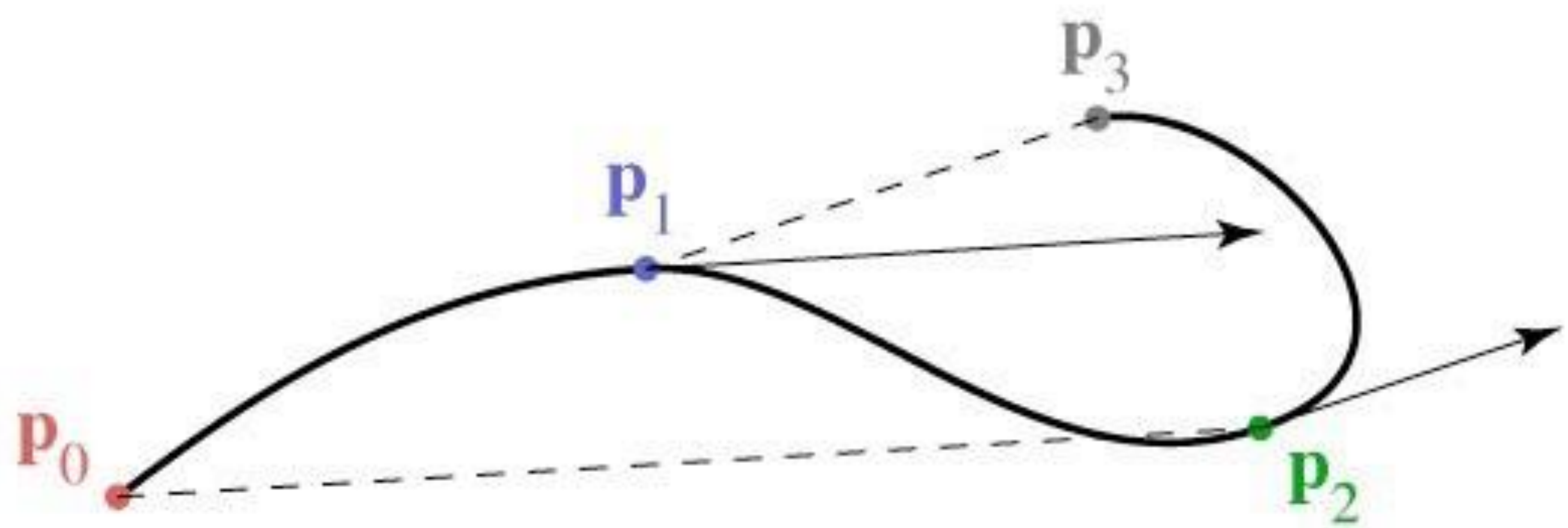
Can Interpolate Points As Easily As Values



Catmull-Rom 3D space curve

Catmull-Rom Basis Functions

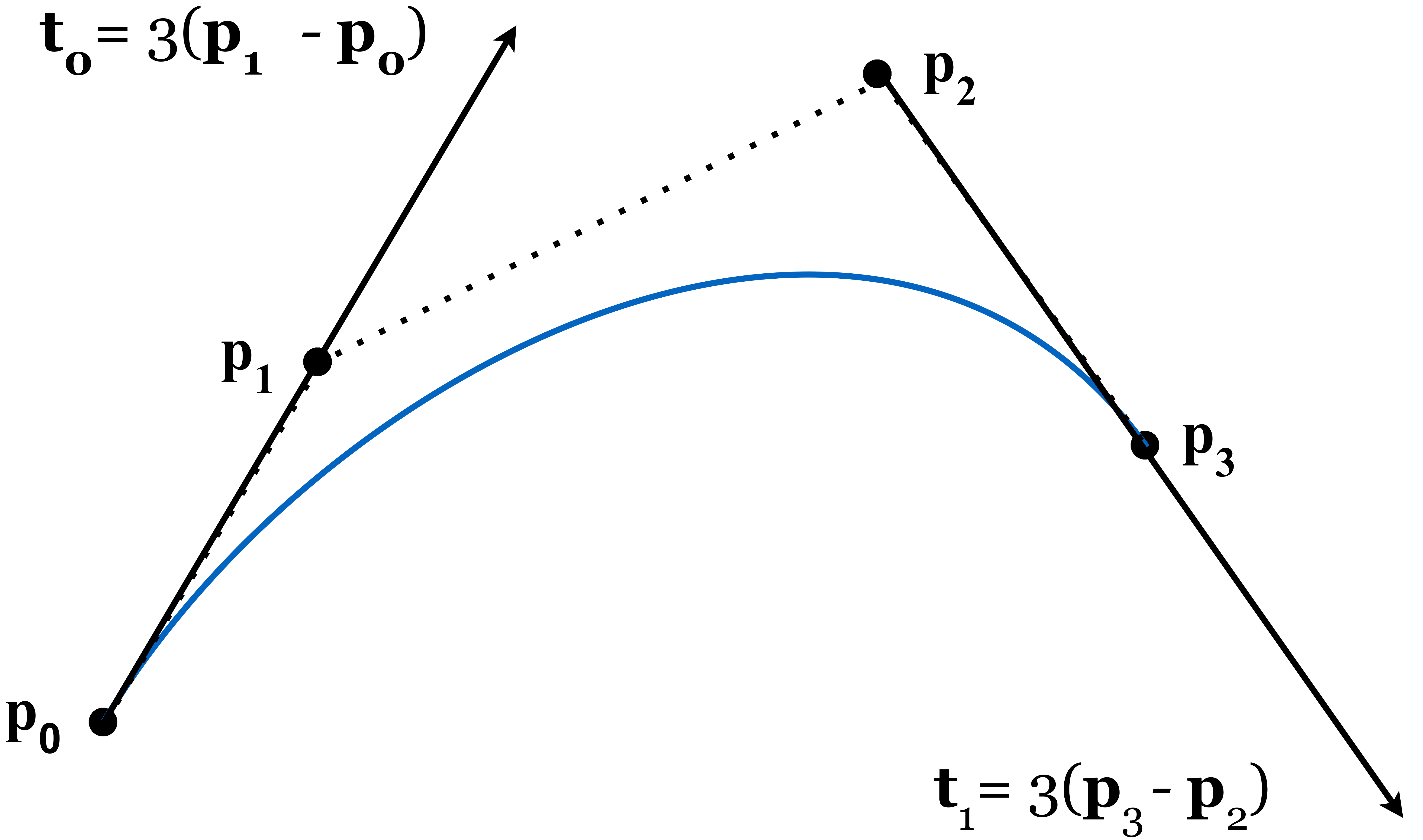




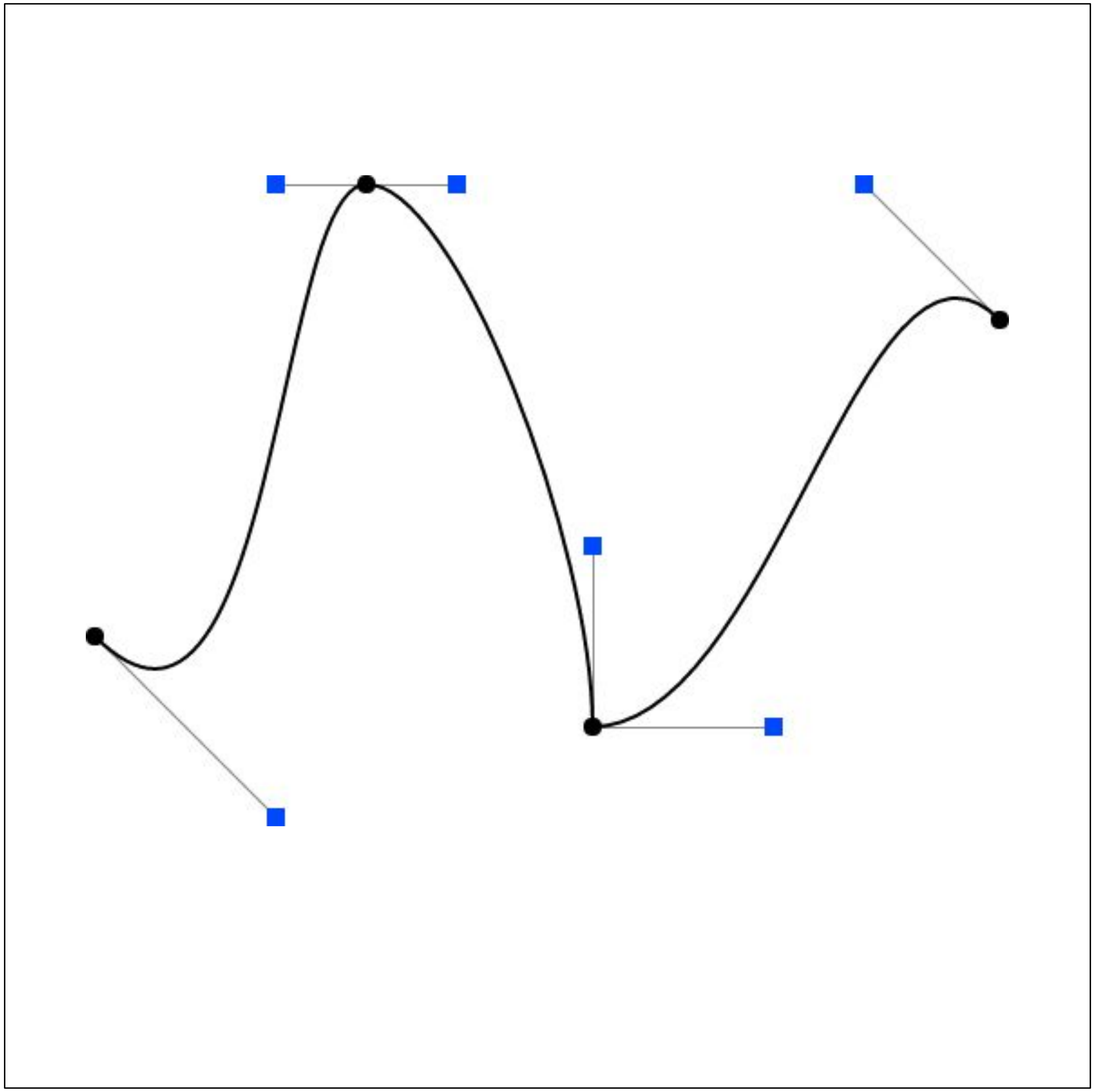
Bézier Curves

Examples of Geometry

Defining Cubic Bézier Curve With Tangents



Piecewise Cubic Bézier Curve



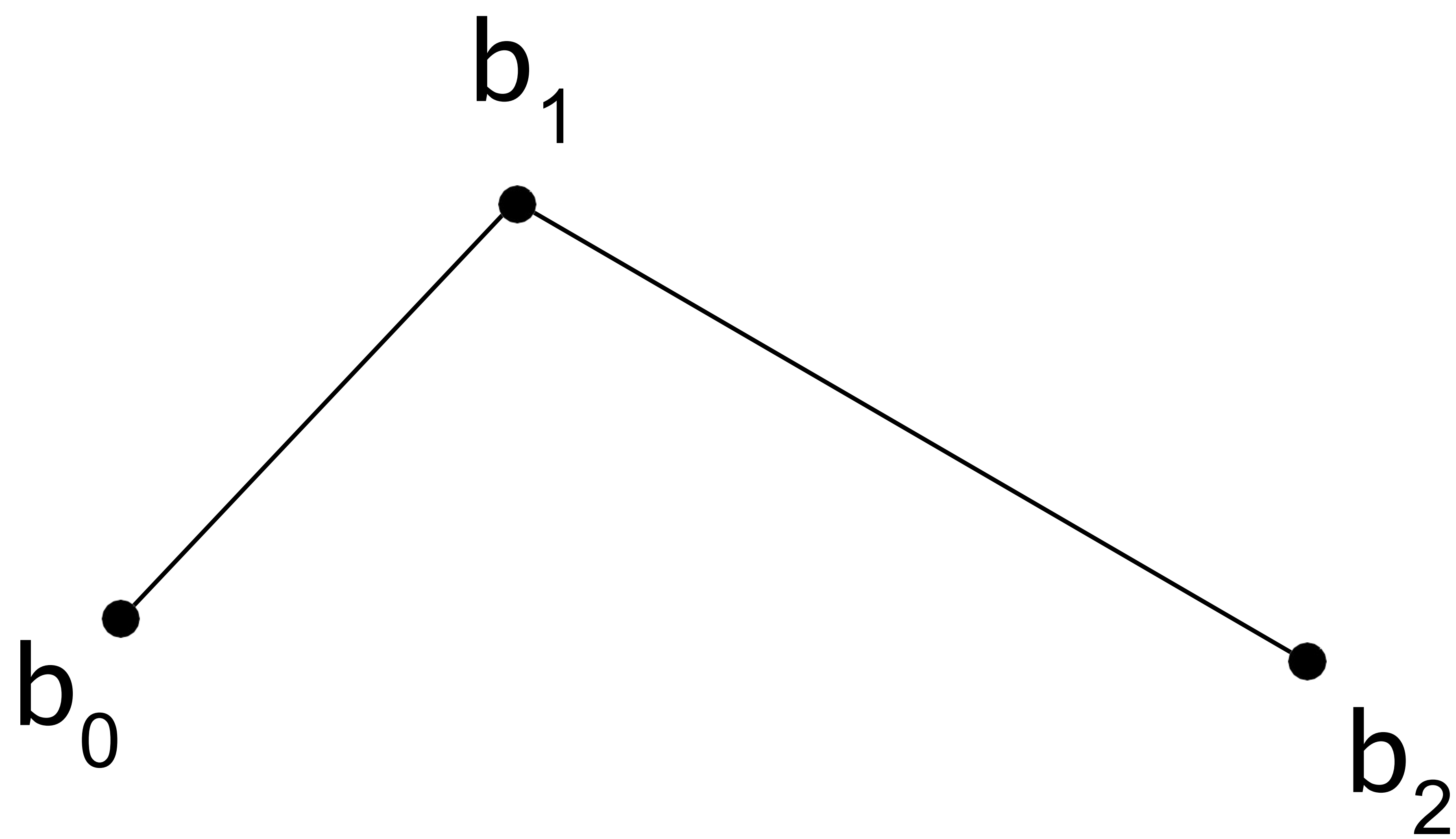
David Eck, http://math.hws.edu/eck/cs424/notes2013/canvas/bezier.html

Evaluating Bézier Curves:

De Casteljau Algorithm

Bézier Curves — de Casteljau Algorithm

Consider three points (quadratic Bezier)

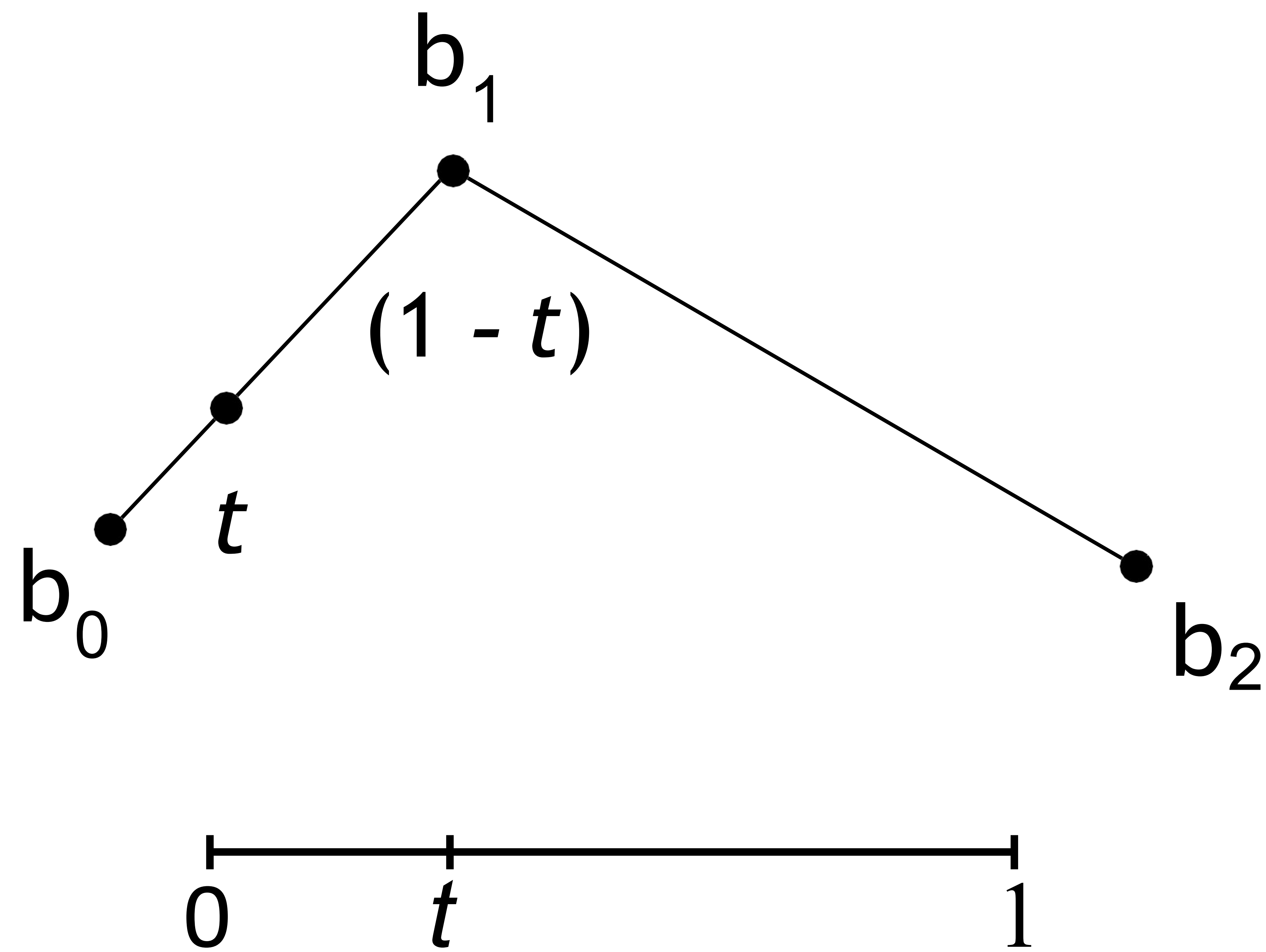


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves — de Casteljau Algorithm

Insert a point using linear interpolation

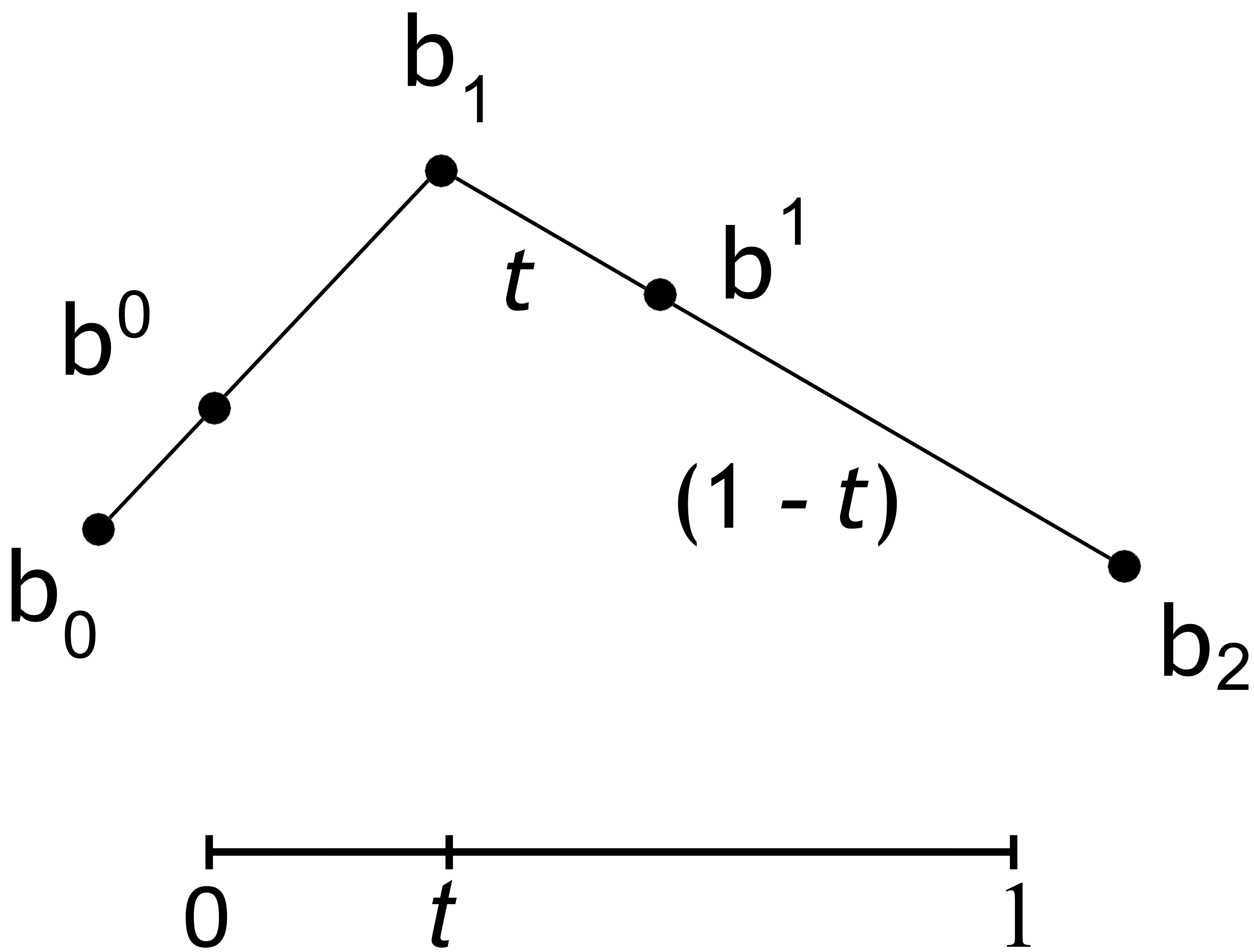


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves — de Casteljau Algorithm

Insert on both edges

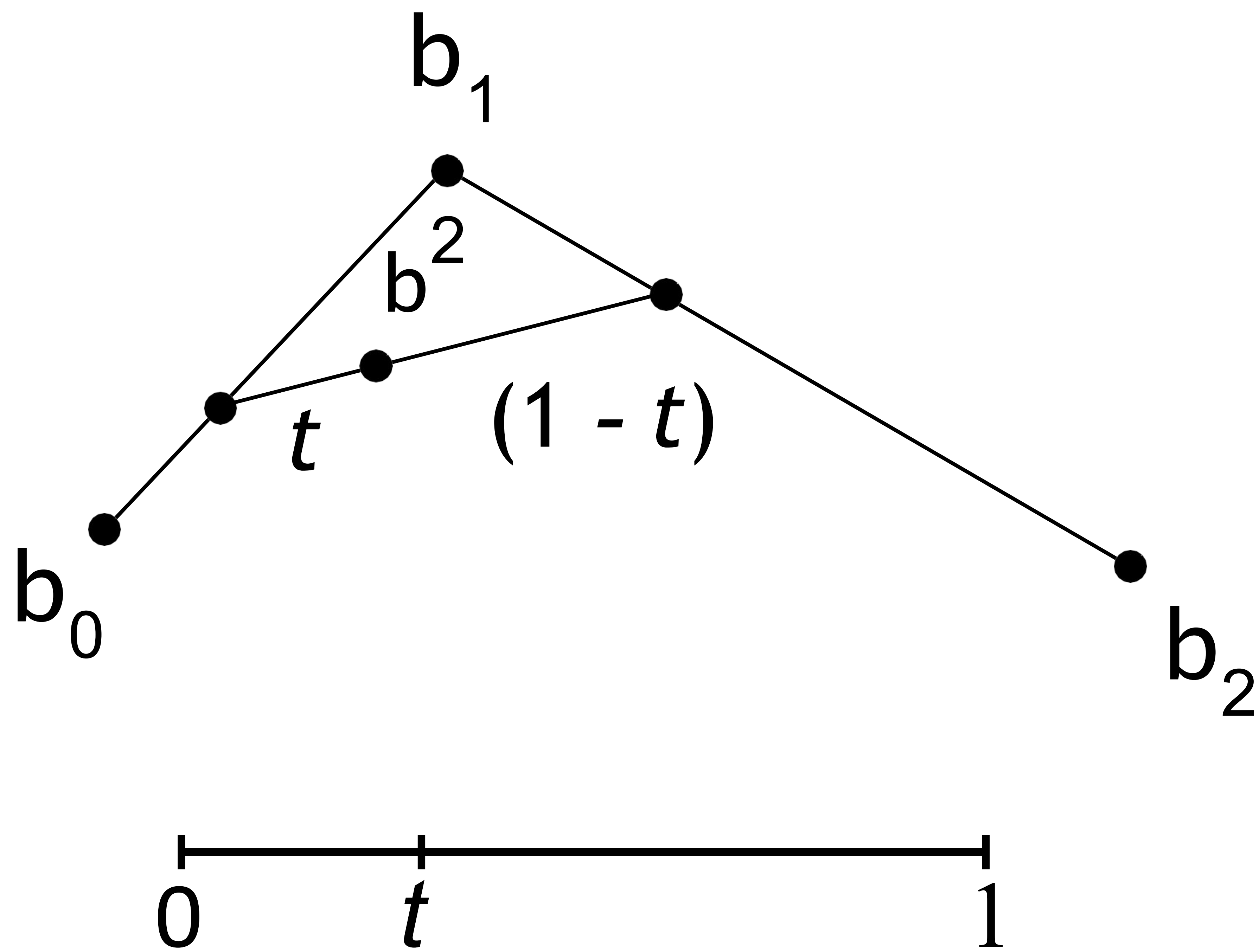


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Repeat recursively

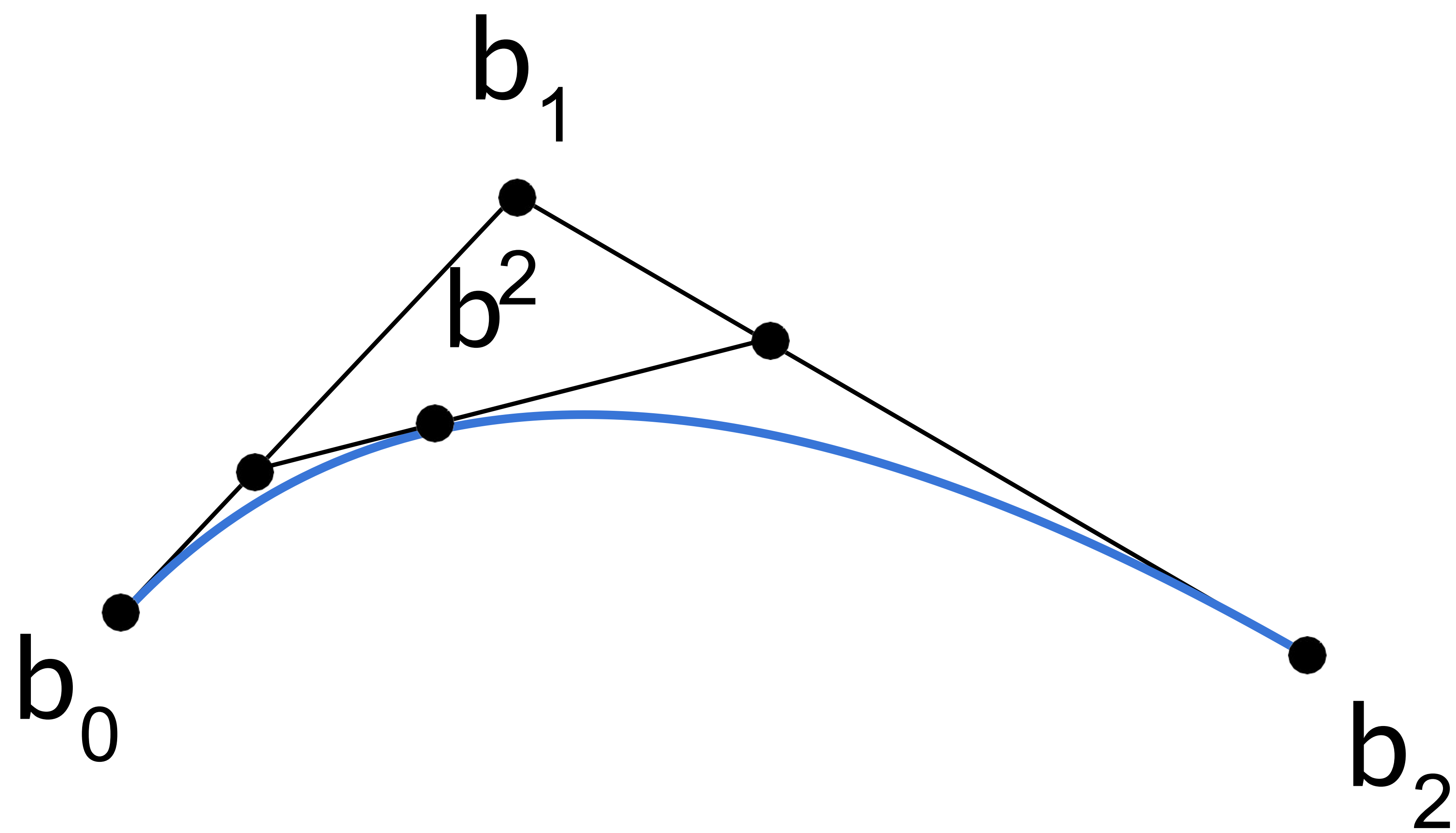


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves — de Casteljau Algorithm

Algorithm defines the curve



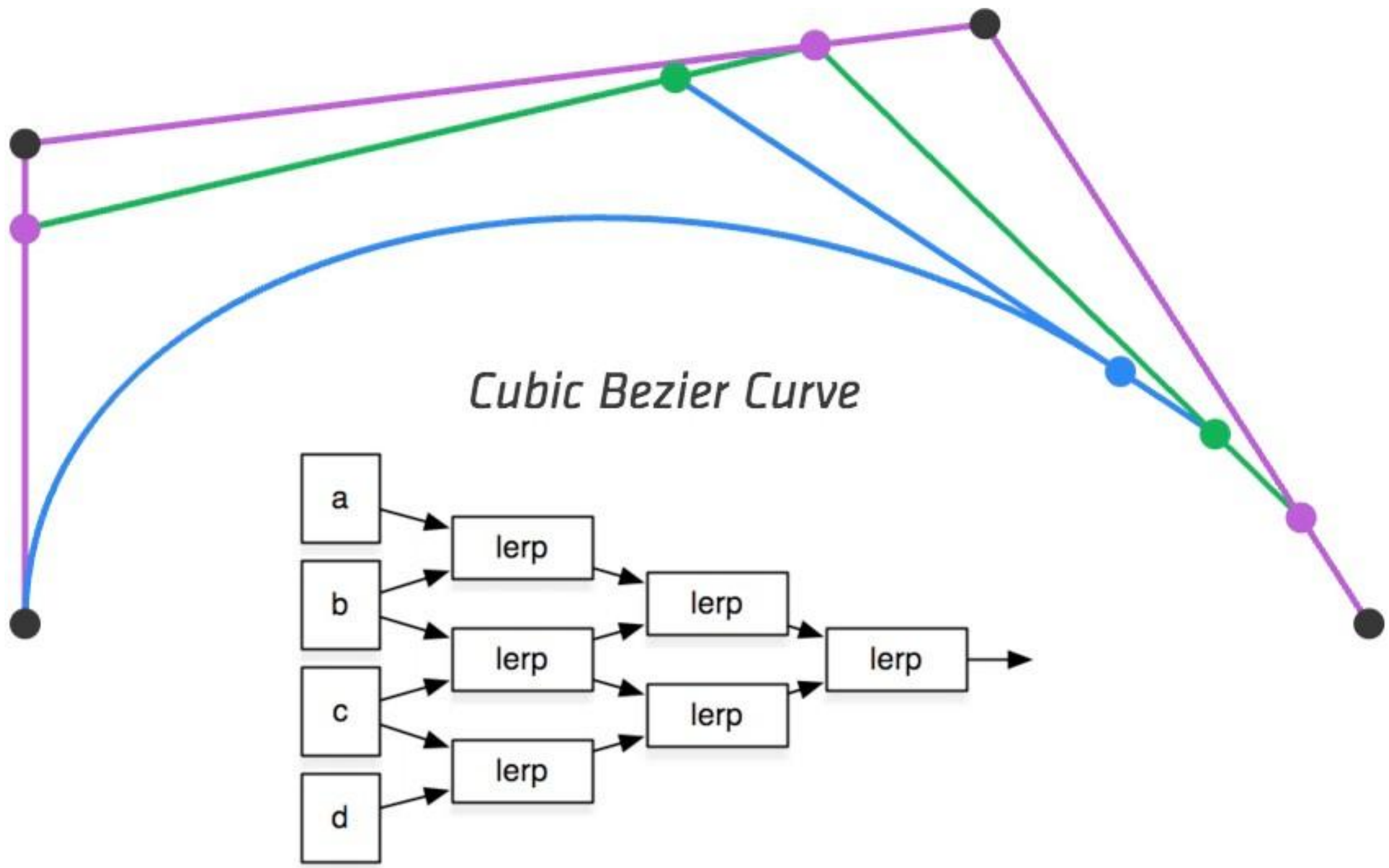
"Corner cutting" recursive subdivision

Successive linear interpolation

Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Visualizing de Casteljau Algorithm

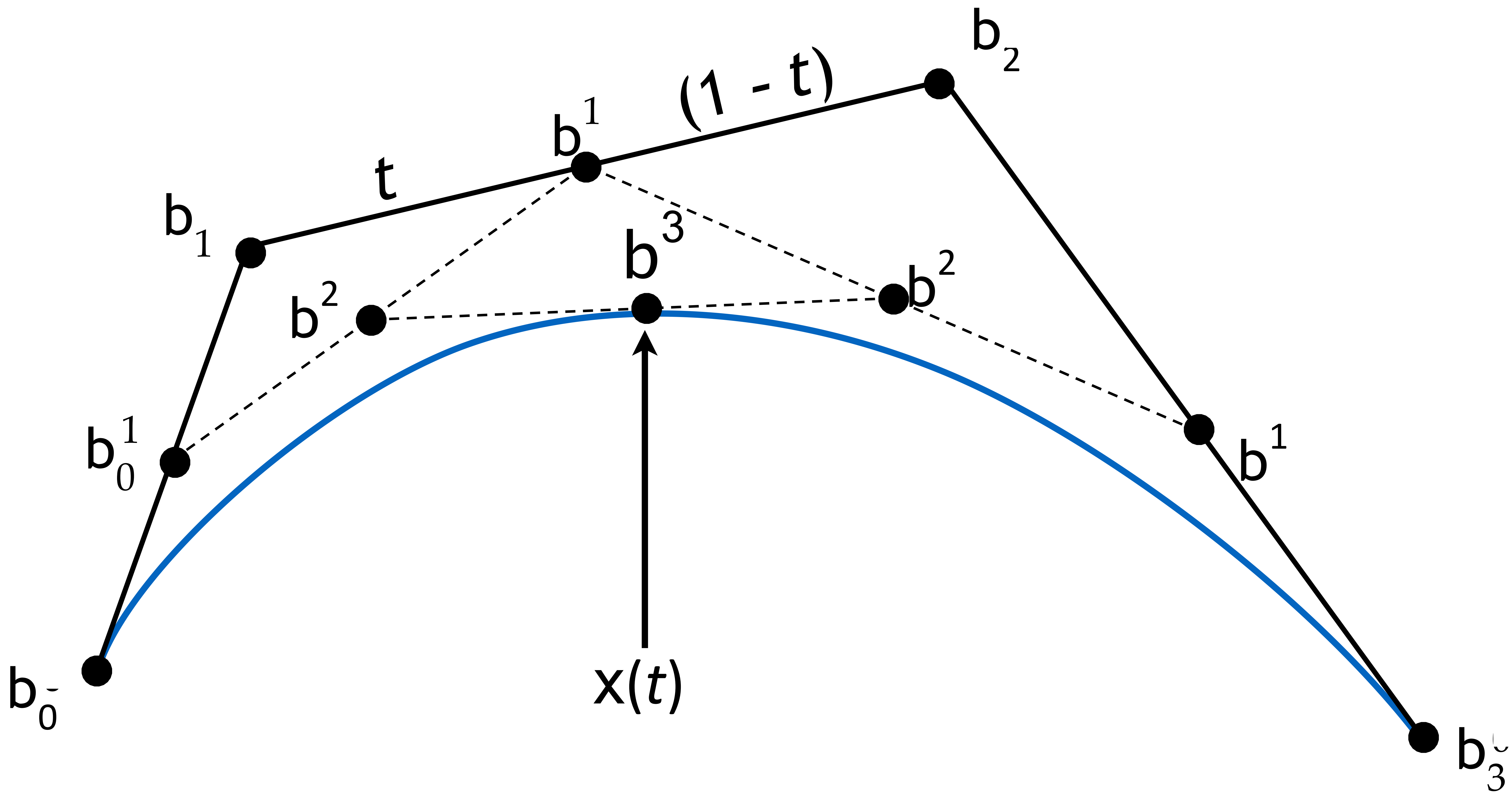


Animation: Steven Wittens, Making Things with Maths, http://acko.net

Cubic Bézier Curve – de Casteljau

Consider four points

Same recursive linear interpolations

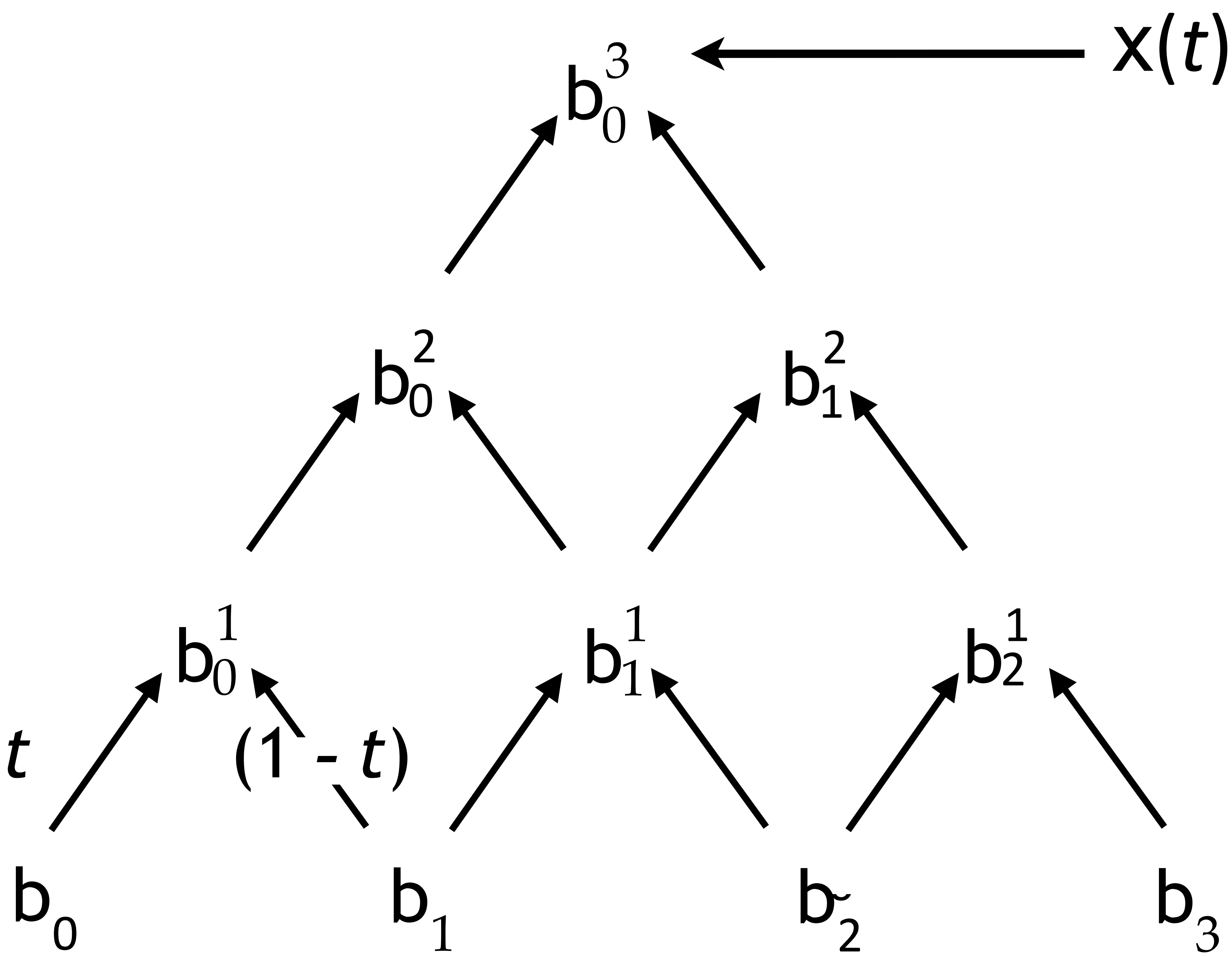


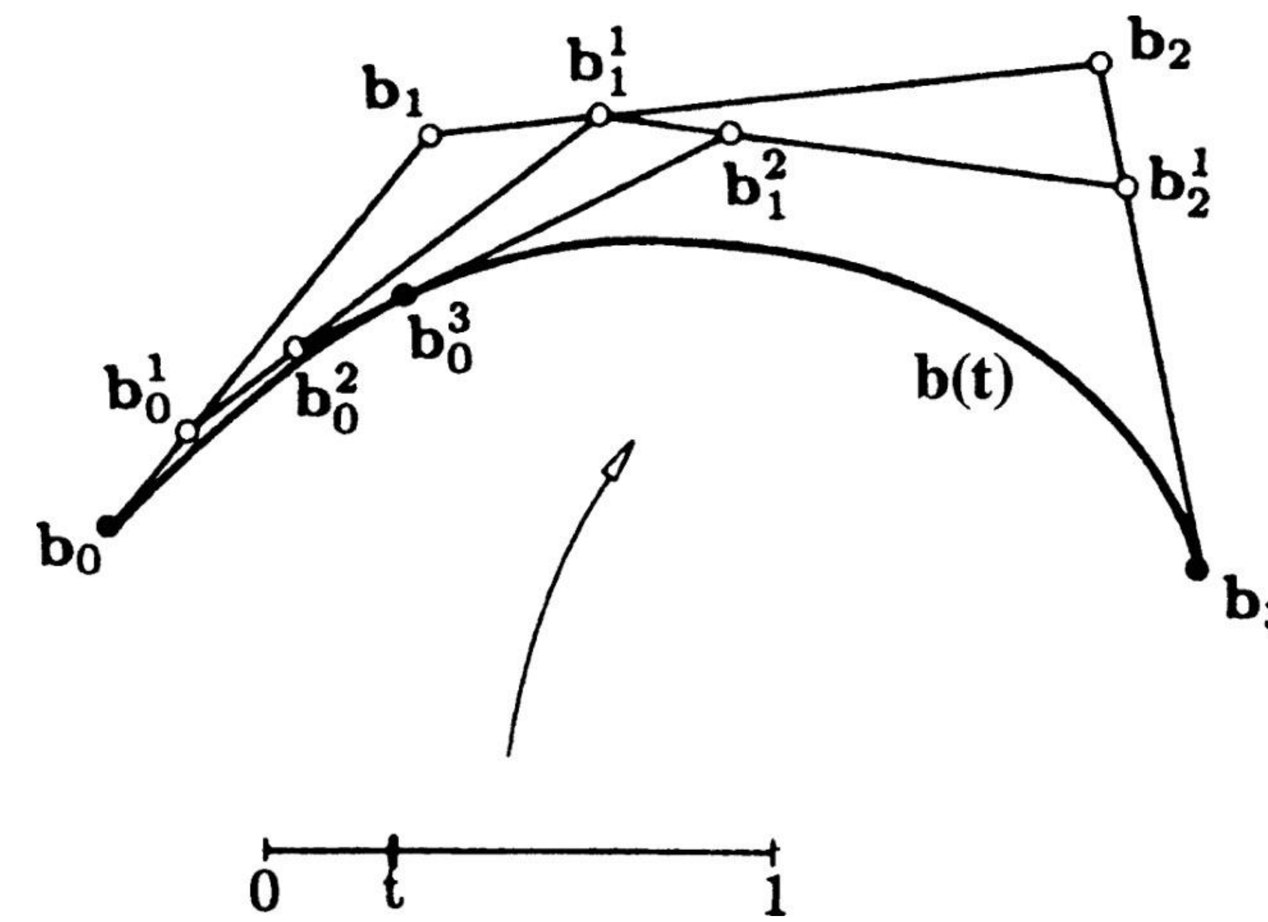
Evaluating Bézier Curves:

Algebraic Formula

Bézier Curve – Algebraic Formula

de Casteljau algorithm gives a pyramid of coefficients





Every rightward arrow is multiplication by t

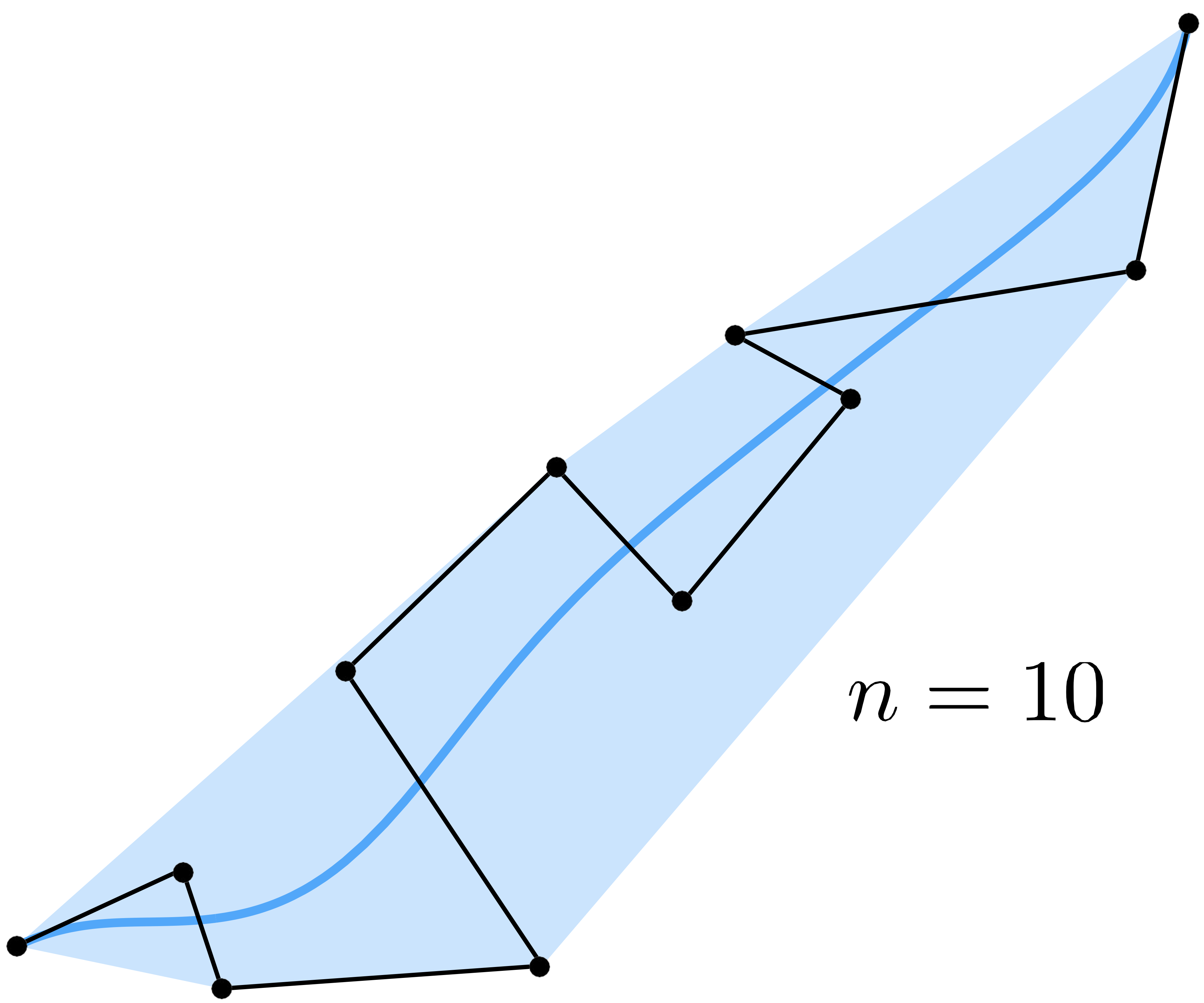
Every leftward arrow by (1−t)

Piecewise Bézier Curves:

(Bézier Spline)

Higher-Order Bézier Curves?

High-degree polynomials don’t interpolate well



$n = 10$

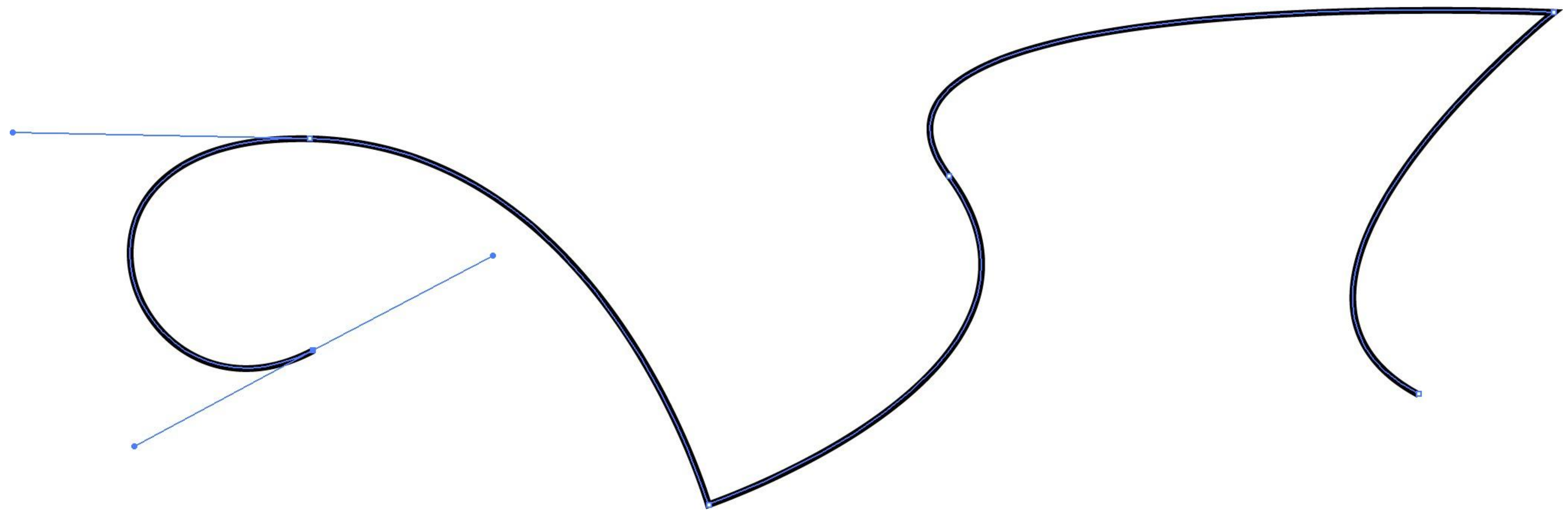
Very hard to control!

Uncommon

Piecewise Bézier Curves

Instead, chain many low-order Bézier curve

Piecewise cubic Bézier the most common technique



Widely used (fonts, paths, Illustrator, Keynote, …)

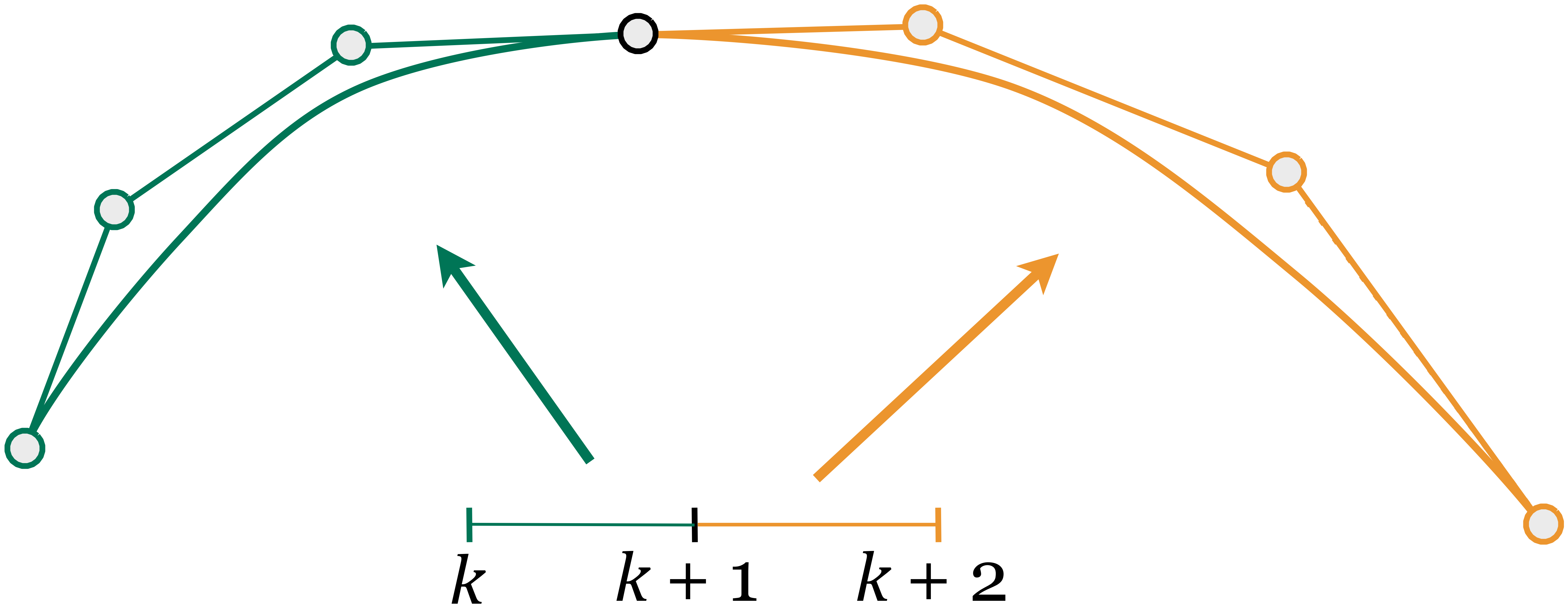
Piecewise Bézier Curve – Continuity

Two Bézier curves

$$\mathbf{a} : [k, k+1] \rightarrow \mathrm{R}^N$$

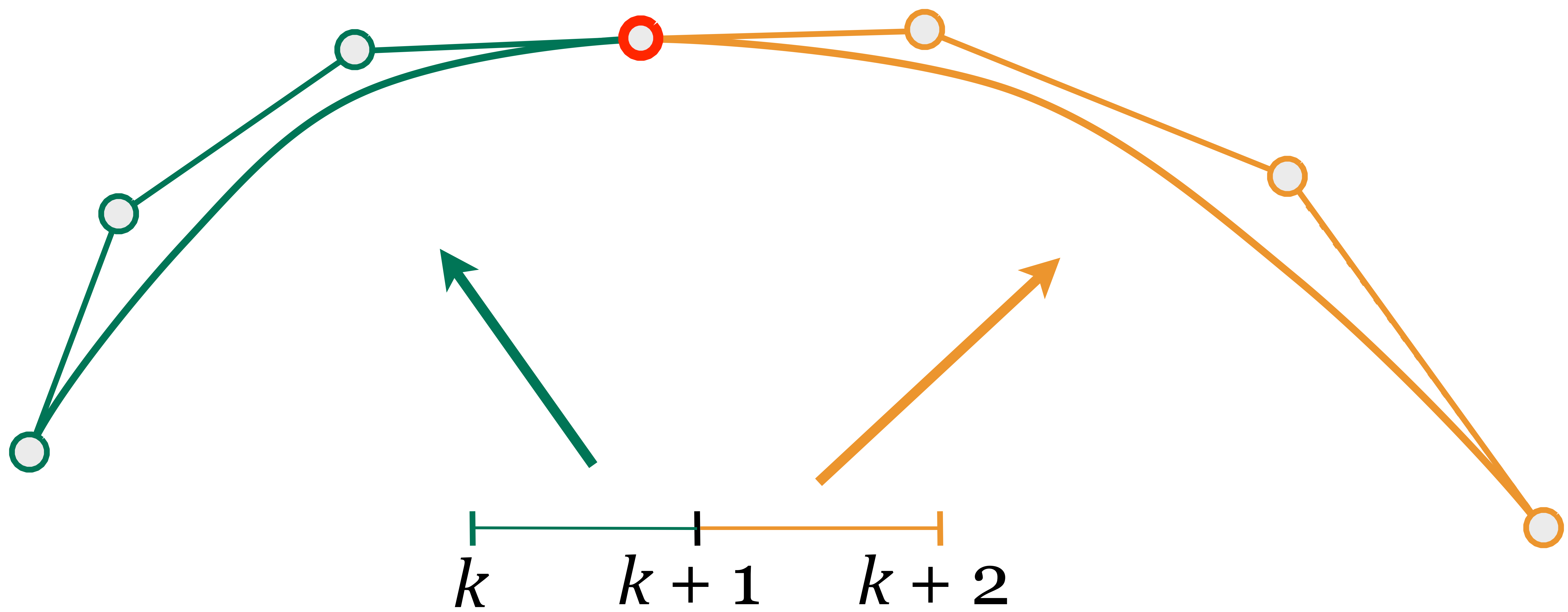
$$\mathbf{b} : [k+1, k+2] \rightarrow \mathrm{R}^N$$

Assuming integer partitions here



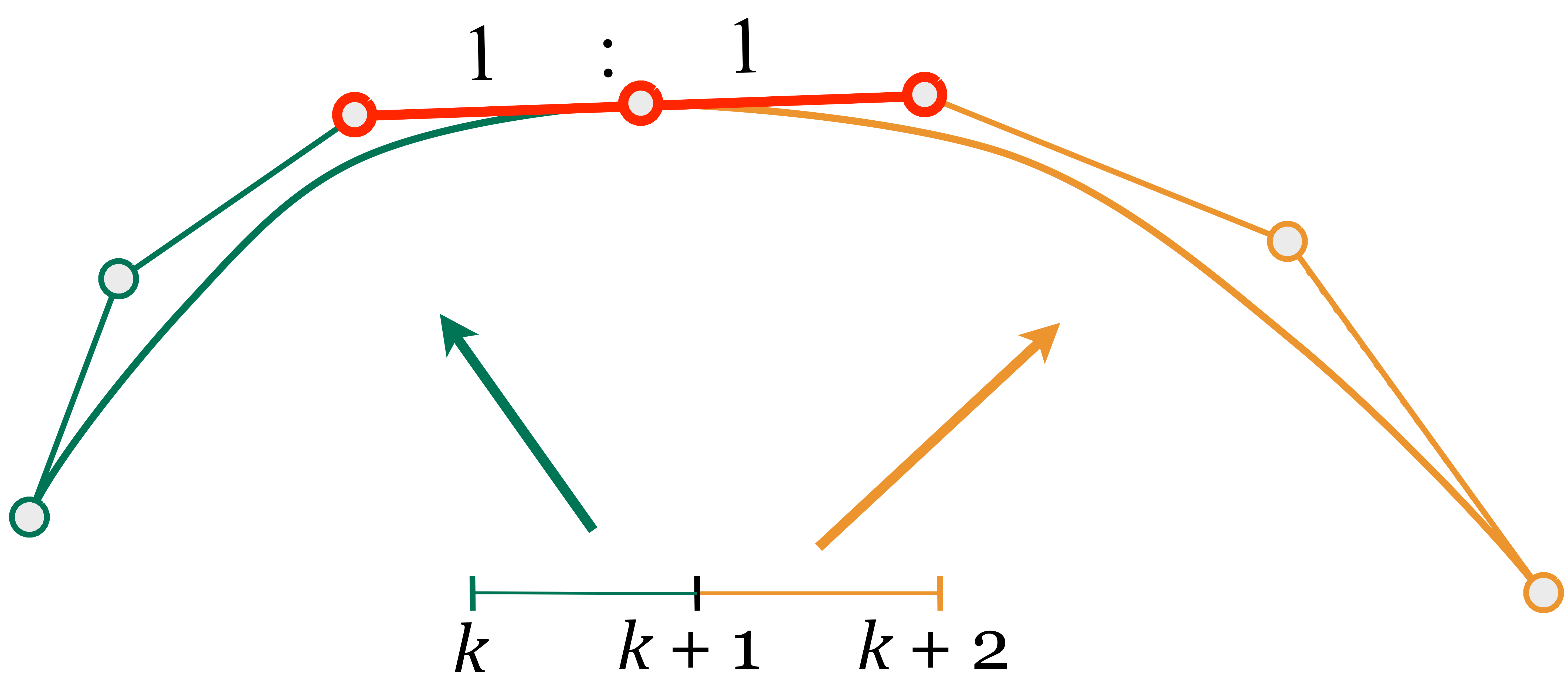
Piecewise Bézier Curve – Continuity

C^0 continuity: $a_n = b_0$



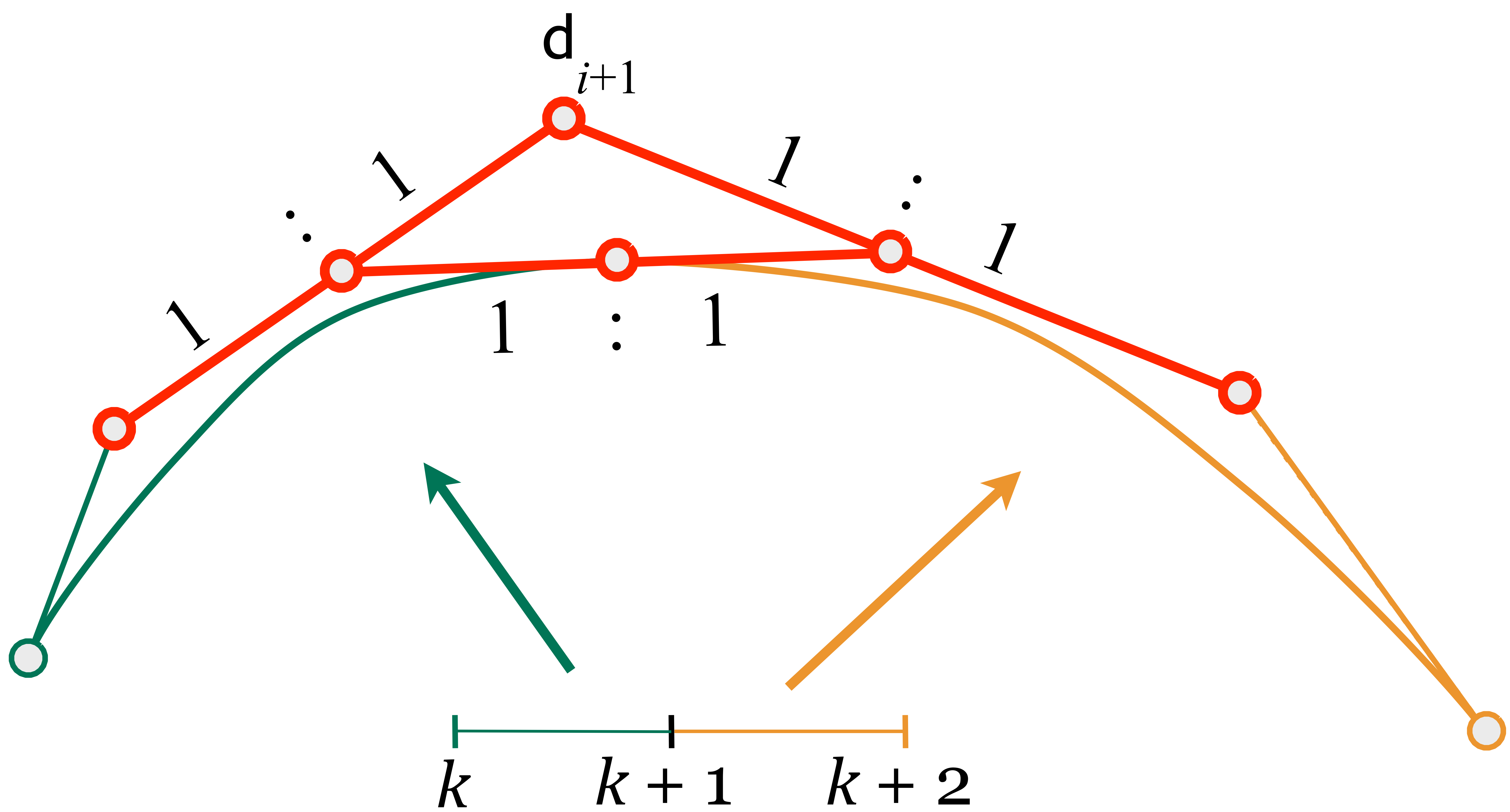
Piecewise Bézier Curve – Continuity

C¹ continuity: $a_n = b_0 = \frac{1}{2}(a_{n-1} + b_1)$



Piecewise Bézier Curve – Continuity

C^2 continuity: "A-frame" construction



Properties of Bézier Curves

Interpolates endpoints

- For cubic Bézier: $b(0) = b_0; \quad b(1) = b_3$

Affine transformation property

- Transform curve by transforming control points
- Convex hull property

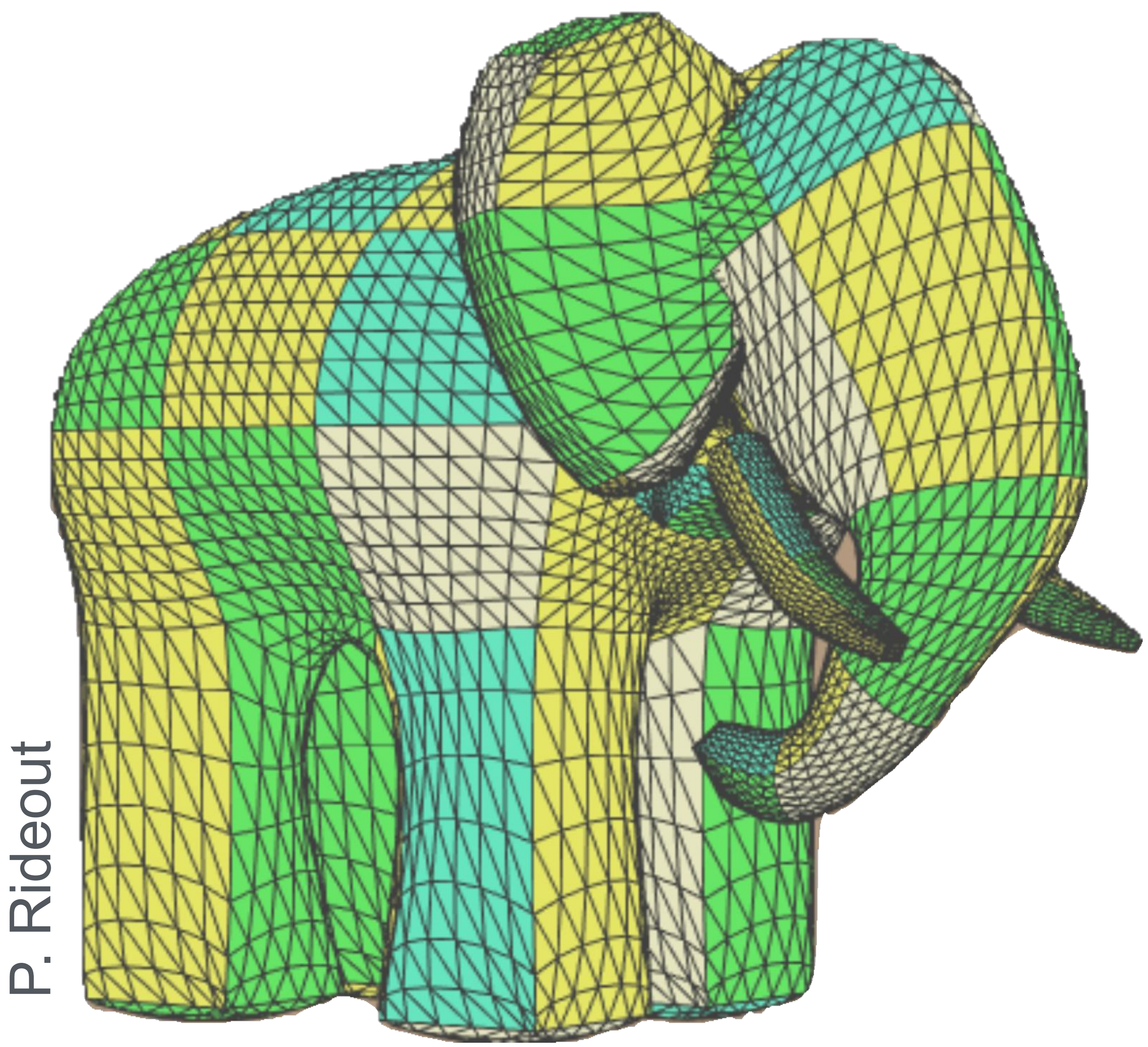
  Curve is within convex hull of control points

Madrid

Bézier Surfaces

Bézier Surfaces

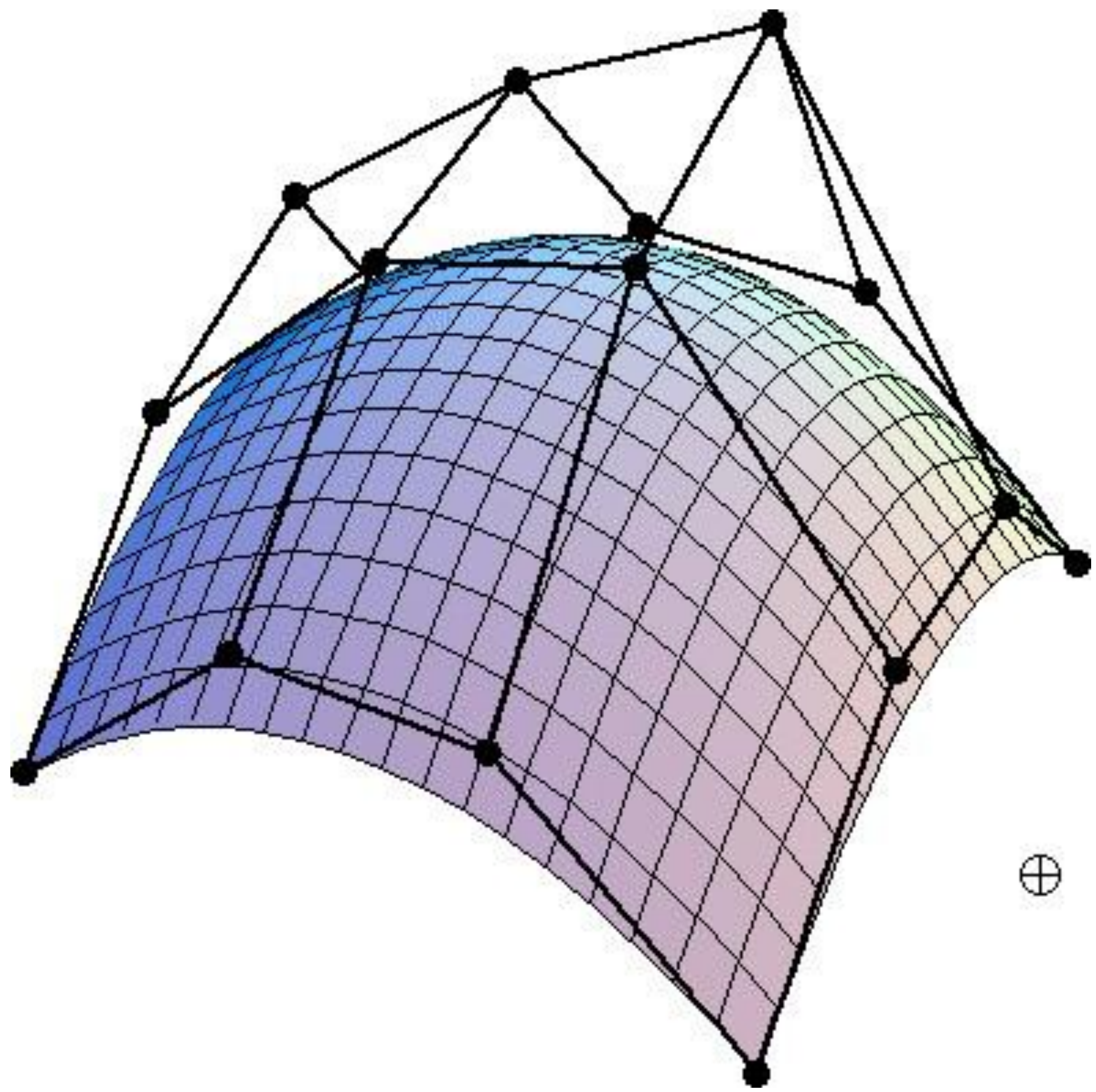
Extend Bézier curves to surfaces



Ed Catmull's "Gumbo" model

Utah Teapot

Bicubic Bézier Surface Patch



Bezier surface and 4 x 4 array of control points

Visualizing Bicubic Bézier Surface Patch

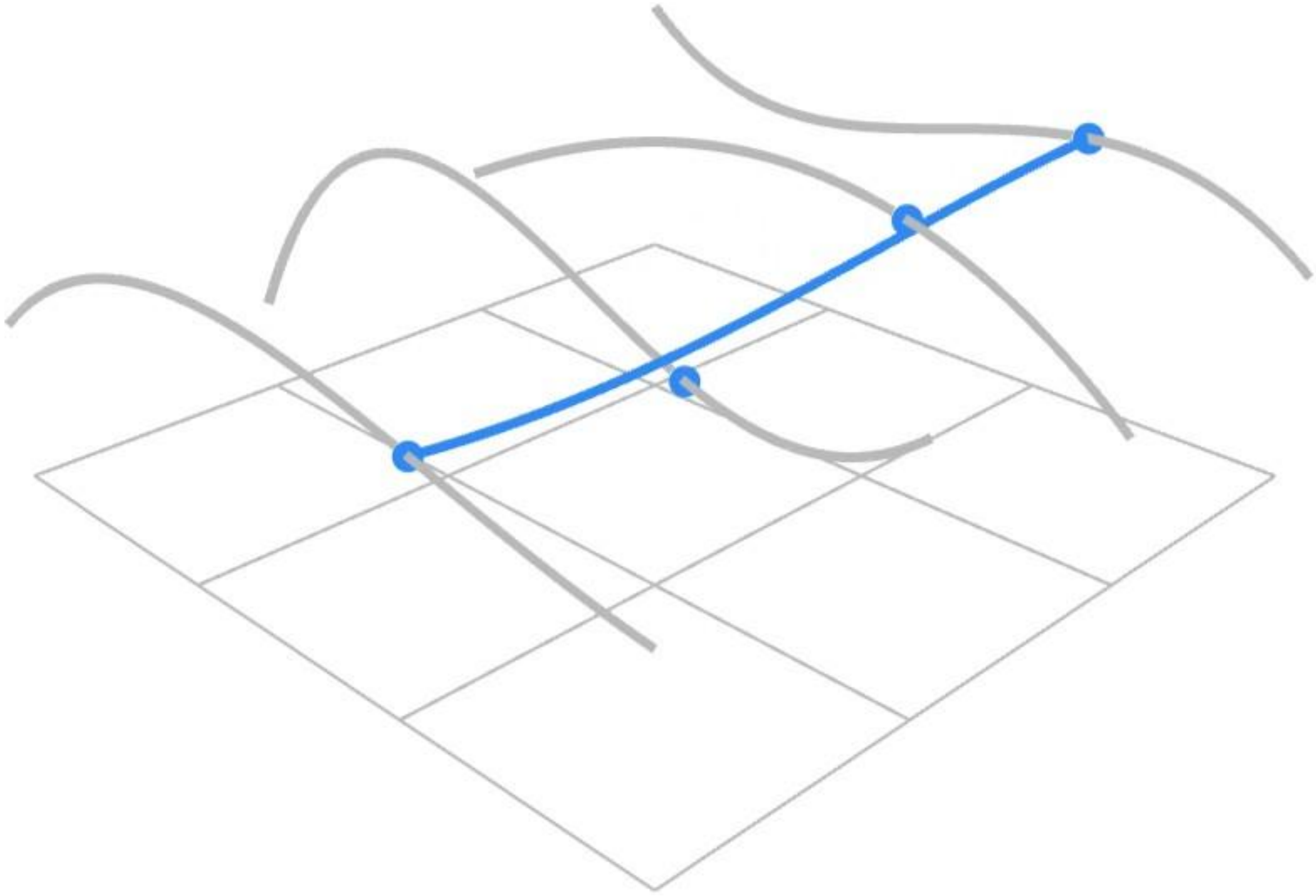
4x4 control points

- Each 4x1 control points in u define a Bezier curve

  (4 Bezier curves in u)

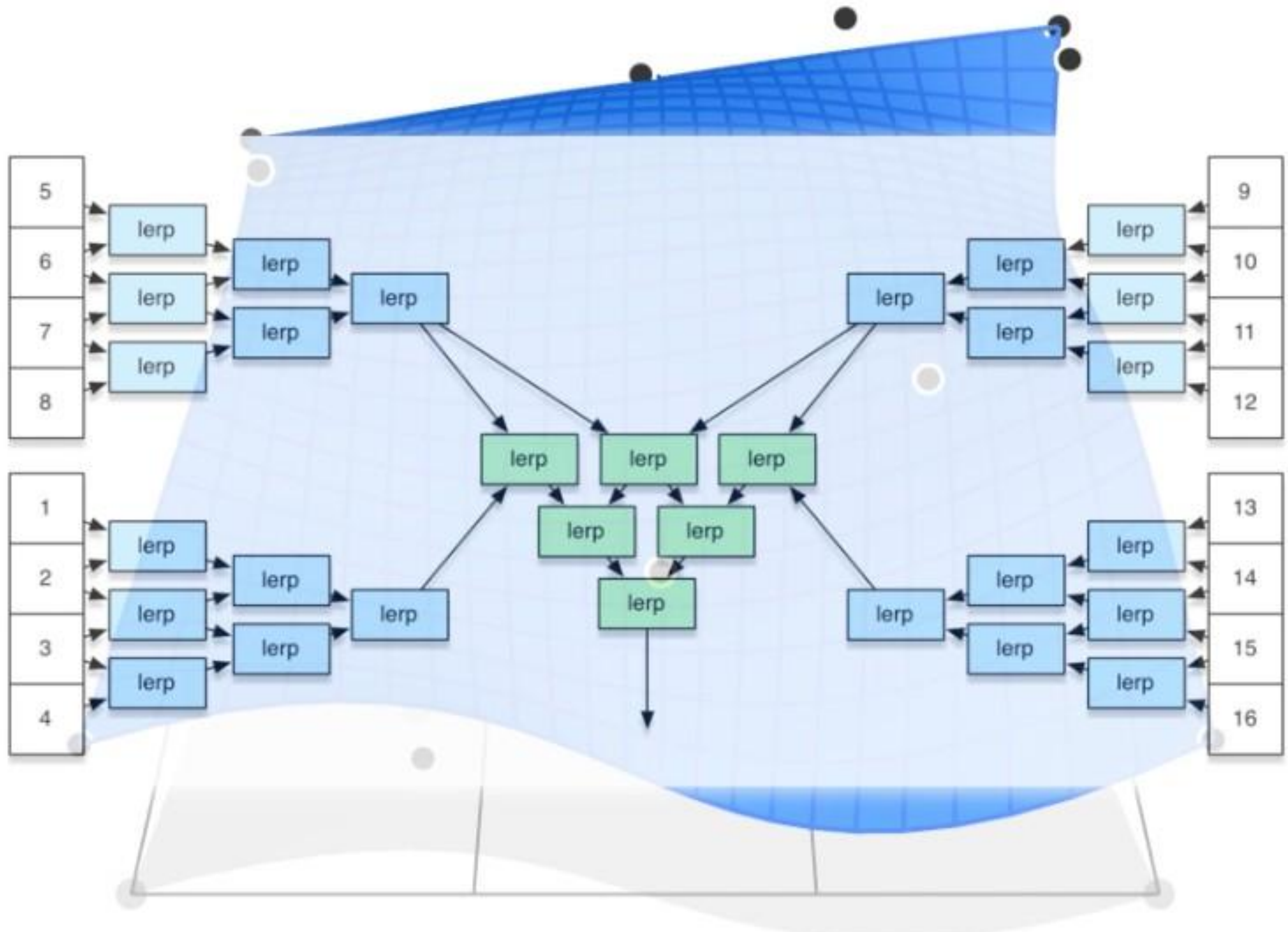
- Corresponding points on these 4 Bezier curves define 4 control points for a “moving curve” in v
- This “moving” curve sweeps out the 2D surface

Visualizing Bicubic Bézier Surface Patch



Animation: Steven Wittens, Making Things with Maths, http://acko.net

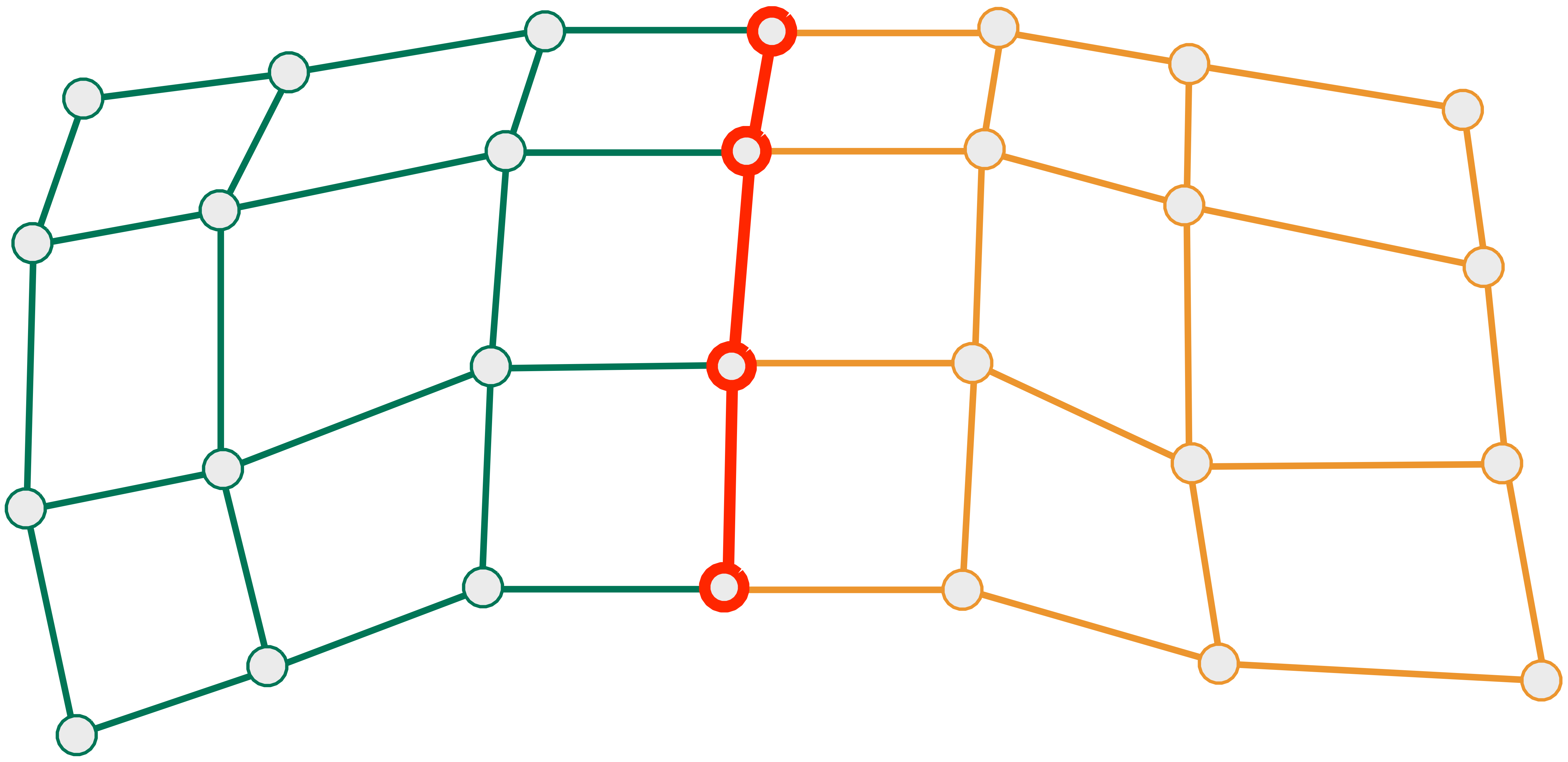
Separable 1D de Casteljau Algorithm



Bézier Surface Continuity

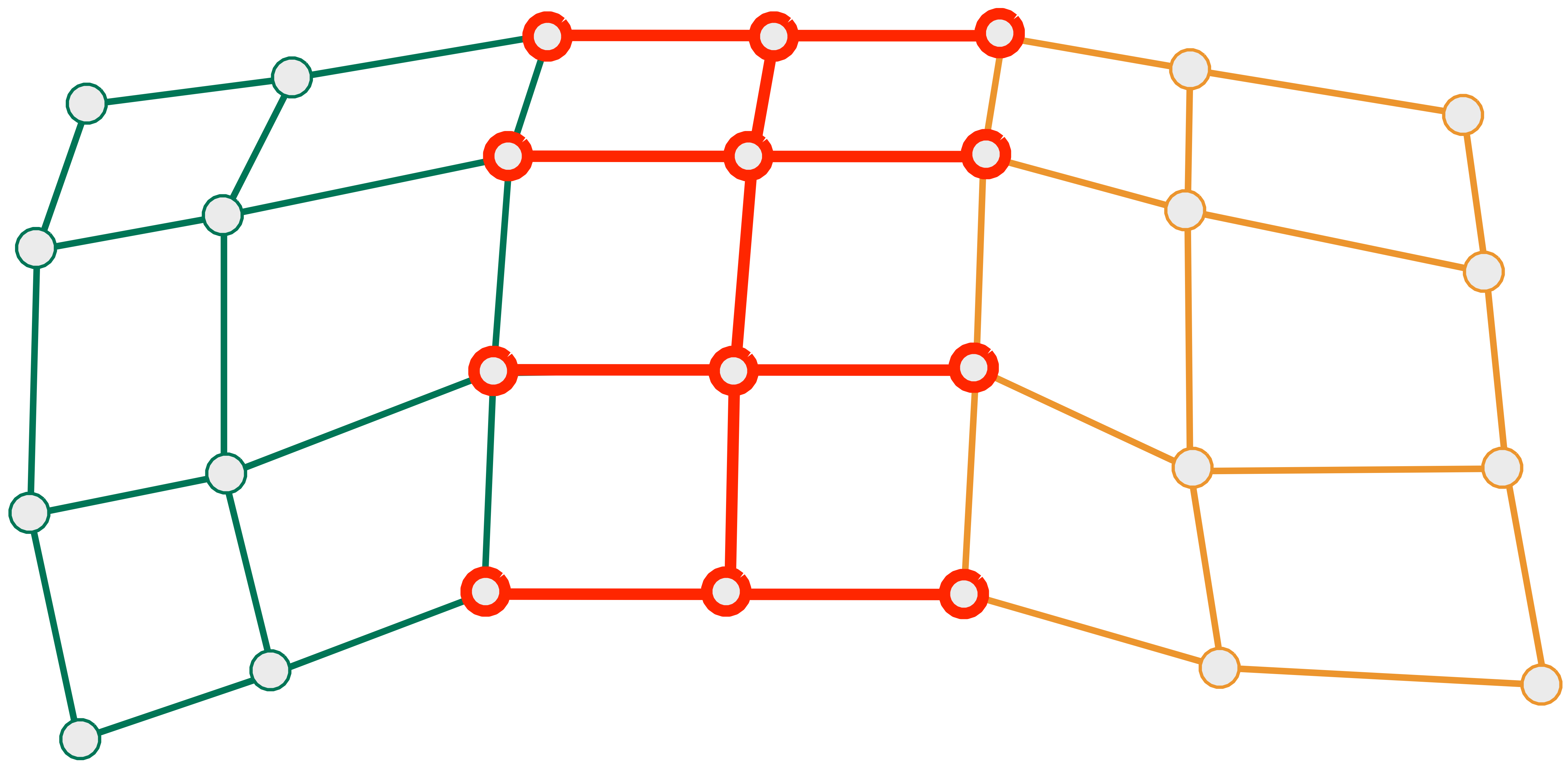
Piecewise Bézier Surfaces

C^0 continuity: Boundary curves



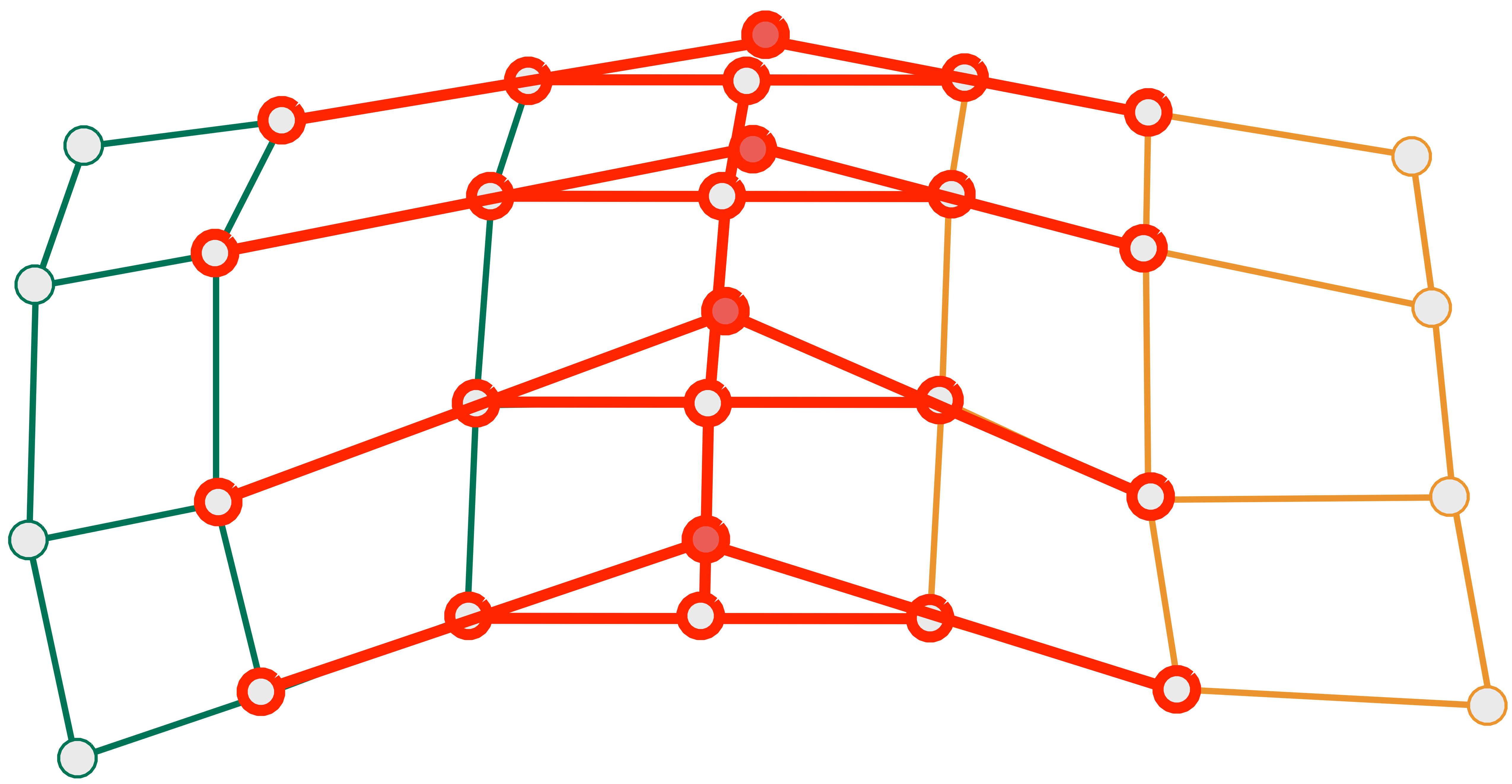
Piecewise Bézier Surfaces

C^1 continuity: Collinearity



Piecewise Bézier Surfaces

C^2 continuity: A-frames



Things to Remember

Splines

- Cubic Hermite and Catmull-Rom interpolation
- Matrix representation of cubic polynomials

Bézier curves

- Easy-to-control spline
- Recursive linear interpolation – de Casteljau algorithm
- Properties of Bézier curves
- Piecewise Bézier curve – continuity types and how to achieve Bézier

surfaces

- Bicubic Bézier patches – tensor product surface
- 2D de Casteljau algorithm

Acknowledgments

Thanks to Pat Hanrahan, Mark Pauly and Steve Marschner for presentation resources.