Lecture 9:

Splines, Bezier Curves & Surfaces
Smooth Curves and Surfaces

So far we can make:

• Things with corners (lines, triangles, squares, …)
• Specialty shapes (circles, ellipses, …)

Many applications require designed, smooth shapes

• Camera paths, vector fonts, …
• Resampling filter functions
• CAD design, object modeling, …
Camera Paths

Flythrough of proposed Perth Citylink subway, https://youtu.be/rIJMuQPwr3E
Animation Curves

Maya Animation Tutorial: https://youtu.be/b-o5wtZlJPc
Vector Fonts

The Quick Brown Fox Jumps Over The Lazy Dog

ABCDEFghijklmnopqrstuvwxyz
abcdefghijklmnopqrstuvwxyz

0123456789

Baskerville font - represented as cubic Bézier splines
CAD Design

3D Car Modeling with Rhinoceros
Splines
A Real Draftsman’s Spline

http://www.alatown.com/spline-history-architecture/
Spline Topics

Interpolation
- Cubic Hermite interpolation
- Catmull-Rom interpolation

Bezier curves

Bezier surfaces
Cubic Hermite Interpolation
Goal: Interpolate Values
Nearest Neighbor Interpolation

Problem: values not continuous
Linear Interpolation

Problem: derivatives not continuous
Smooth Interpolation?
Cubic Hermite Interpolation

Inputs: values and derivatives at endpoints
Cubic Polynomial Interpolation

Cubic polynomial

\[ P(t) = a \, t^3 + b \, t^2 + c \, t + d \]

Why cubic?

4 input constraints – need 4 degrees of freedom

\[ P(0) = h_0 \]
\[ P(1) = h_1 \]
\[ P'(0) = h_2 \]
\[ P'(1) = h_3 \]
Cubic Polynomial Interpolation

Cubic polynomial

\[ P(t) = a \, t^3 + b \, t^2 + c \, t + d \]
\[ P'(t) = 3a \, t^2 + 2b \, t + c \]

Set up constraint equations

\[ P(0) = h_0 = d \]
\[ P(1) = h_1 = a + b + c + d \]
\[ P'(0) = h_2 = c \]
\[ P'(1) = h_3 = 3a + 2b + c \]
## Solve for Polynomial Coefficients

\[ h_0 = d \]
\[ h_1 = a + b + c + d \]
\[ h_2 = c \]
\[ h_3 = 3a + 2b + c \]

\[
\begin{bmatrix}
  h_0 \\
  h_1 \\
  h_2 \\
  h_3
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 0 & 1 \\
  1 & 1 & 1 & 1 \\
  0 & 0 & 1 & 0 \\
  3 & 2 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  a \\
  b \\
  c \\
  d
\end{bmatrix}
\]
Solve for Polynomial Coefficients

\[
\begin{bmatrix}
a \\
b \\
c \\
d \\
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0 \\
\end{bmatrix}^{-1}
\begin{bmatrix}
h_0 \\
h_1 \\
h_2 \\
h_3 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
h_0 \\
h_1 \\
h_2 \\
h_3 \\
\end{bmatrix}
\]

(Check that these matrices are inverses)
Matrix Form of Hermite Function

\[ P(t) = a \, t^3 + b \, t^2 + c \, t + d \]

\[
= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}
\]

\[
= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}
\]
Interpretation 1: Matrix Rows = Coefficient Formulas

\[ P(t) = a t^3 + b t^2 + c t + d \]

\[
= \begin{bmatrix}
  t^3 & t^2 & t & 1
\end{bmatrix}
\begin{bmatrix}
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  h_0 \\
  h_1 \\
  h_2 \\
  h_3
\end{bmatrix}
\]
Interpretation 2: Matrix Columns = ?

\[ P(t) = a \ t^3 + b \ t^2 + c \ t + d \]

\[
= \begin{bmatrix}
    t^3 & t^2 & t & 1
\end{bmatrix}
\begin{bmatrix}
    2 & -2 & 1 & 1 \\
    -3 & 3 & -2 & -1 \\
    0 & 0 & 1 & 0 \\
    1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    h_0 \\
    h_1 \\
    h_2 \\
    h_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    2t^3 - 3t^2 + 1 \\
    -2t^3 + 3t^2 \\
    t^3 - 2t^2 + t \\
    t^3 - t^2
\end{bmatrix}^T
\begin{bmatrix}
    h_0 \\
    h_1 \\
    h_2 \\
    h_3
\end{bmatrix}
\]
Hermite Basis Functions

\[ P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} H_0(t) & H_1(t) & H_2(t) & H_3(t) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} \]

\[ \begin{align*}
H_0(t) &= 2t^3 - 3t^2 + 1 \\
H_1(t) &= -2t^3 + 3t^2 \\
H_2(t) &= t^3 - 2t^2 + t \\
H_3(t) &= t^3 - t^2
\end{align*} \]

Basis functions for cubic polynomials  
Hermite basis functions for cubic polynomials

Either basis can represent any cubic polynomial through linear combination
Recap: Cubic Hermite Interpolation

- Inputs: values and derivatives at endpoints
- Output: cubic polynomial that interpolates
- Solution: weighted sum of Hermite basis functions

\[ P(t) = h_0 \, H_0(t) + h_1 \, H_1(t) + h_2 \, H_2(t) + h_3 \, H_3(t) \quad h_3 \]
Hermite Basis Functions

\[ H_0(t) = 2t^3 - 3t^2 + 1 \]
\[ H_1(t) = -2t^3 + 3t^2 \]
\[ H_2(t) = t^3 - 2t^2 + t \]
\[ H_3(t) = t^3 - t^2 \]
Ease Function

A very useful function

In animation, start and stop gently (zero velocity)

\[ H_1(t) = -2t^3 + 3t^2 = t^2(3 - 2t) \]
Hermite Spline Interpolation

Inputs: sequence of values and derivatives
Catmull-Rom Interpolation
Catmull-Rom Interpolation

Inputs: sequence of values
**Catmull-Rom Interpolation**

**Rule for derivatives:**
Match slope between previous and next values

\[
\frac{1}{2}(y_3 - y_1)
\]

\[
\frac{1}{2}(y_2 - y_0)
\]
Catmull-Rom Interpolation

Then use Hermite interpolation

\[ \frac{1}{2}(y_2 - y_0) \quad \frac{1}{2}(y_3 - y_1) \]
Catmull-Rom Spline

Input: sequence of points
Output: spline that interpolates all points with C1 continuity
Interpolating Points & Vectors
Can Interpolate Points As Easily As Values

E.g. point (0,1,3) in 3D space, or even a general vector in N dimensions.

Catmull-Rom 3D spline control points.
Can Interpolate Points As Easily As Values

\[ \frac{1}{2}(p_4 - p_2) \]

\[ \frac{1}{2}(p_3 - p_1) \]

\[ \frac{1}{2}(p_2 - p_0) \]

Catmull-Rom 3D tangent vectors
Can Interpolate Points As Easily As Values

\[ \frac{1}{2} (p_4 - p_2) \rightarrow p_3 \]
\[ \frac{1}{2} (p_3 - p_1) \rightarrow p_2 \]
\[ \frac{1}{2} (p_2 - p_0) \rightarrow p_1 \]

Catmull-Rom 3D space curve
Use Basis Functions to Define Curves

General formula for a particular interpolation scheme: \( p(t) = \sum_{i=0}^{n} p_i F_i(t) \)

\[
\begin{align*}
x(t) &= \sum_{i=0}^{n} x_i F_i(t) \\
y(t) &= \sum_{i=0}^{n} y_i F_i(t) \\
z(t) &= \sum_{i=0}^{n} z_i F_i(t)
\end{align*}
\]

Coefficients \( p_i \) can be points & vectors, not just values. \( F_i(t) \) are basis functions for the interpolation scheme.

Saw \( H_i(t) \) for Hermite interpolation earlier. Will see \( C_i(t) \) for Catmull-Rom shortly, and \( B_i(t) \) for Bézier scheme later. The basis functions are properties of the interpolation scheme.
Matrix Form of Catmull-Rom Space Curve?

Use Hermite matrix form

- Points & tangents given by Catmull-Rom rules

Hermite points

\[ h_0 = p_1 \]
\[ h_1 = p_2 \]
\[ h_2 = \frac{1}{2}(p_2 - p_0) \]
\[ h_3 = \frac{1}{2}(p_3 - p_1) \]

Hermite tangents

\[
P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}
\]
Matrix Form of Catmull-Rom Space Curve

\[
P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & \frac{3}{1} & -\frac{1}{2} \\ 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = C_0(t) \ p_0 + C_1(t) \ p_1 + C_2(t) \ p_2 + C_3(t) \ p_3
\]

Matrix columns = Catmull-Rom basis functions
Catmull-Rom Basis Functions
Bézier Curves
Examples of Geometry
Defining Cubic Bézier Curve With Tangents

\[ t_0 = 3(p_1 - p_0) \]

\[ t_1 = 3(p_3 - p_2) \]
Matrix Form of Cubic Bézier Curve?

\[
P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = B_0^3(t) \ p_0 + B_1^3(t) \ p_1 + B_2^3(t) \ p_2 + B_3^3(t) \ p_3
\]

Good exercise to derive this matrix yourself. One way: use Hermite matrix equation again. What are the points and tangents?
Demo – Piecewise Cubic Bézier Curve

Evaluating Bézier Curves
De Casteljau Algorithm
Bézier Curves – de Casteljau Algorithm

Consider three points (quadratic Bezier)
Bézier Curves – de Casteljau Algorithm

Insert a point using linear interpolation

\[ b_0, b_1, b_2 \]

\[ b_0 = (1-t)b_0 + tb_1 \]

\[ b_1 = (1-t)b_1 + tb_2 \]

Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930
Bézier Curves – de Casteljau Algorithm

Insert on both edges

\[ b_0, b_1, b_2 \]

\[ b_0, (1-t)b_1 + tb_2, b_1 \]

Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930
Bézier Curves – de Casteljau Algorithm

Repeat recursively

Repeat recursively $(1 - t) (1 - t)$
Bézier Curves – de Casteljau Algorithm

Algorithm defines the curve

“Corner cutting” recursive subdivision
Successive linear interpolation
Visualizing de Casteljau Algorithm

Cubic Bezier Curve

Consider four points
Same recursive linear interpolations
Evaluating Bézier Curves
Algebraic Formula
Bézier Curve – Algebraic Formula

de Casteljau algorithm gives a pyramid of coefficients

Every rightward arrow is multiplication by $t$,
Every leftward arrow by $(1-t)$
Bézier Curve – Algebraic Formula

Example: quadratic Bézier curve from three points

\[
\begin{align*}
\mathbf{b}_0^1(t) &= (1 - t)\mathbf{b}_0 + t\mathbf{b}_1 \\
\mathbf{b}_1^1(t) &= (1 - t)\mathbf{b}_1 + t\mathbf{b}_2 \\
\mathbf{b}_0^2(t) &= (1 - t)^2\mathbf{b}_0 + 2t(1 - t)\mathbf{b}_1 + t^2\mathbf{b}_2
\end{align*}
\]
Bézier Curve – General Algebraic Formula

Bernstein form of a Bézier curve of order $n$:

$$b^n(t) = b^n_0(t) = \sum_{j=0}^{n} b_j B^*_j(t)$$

Bézier curve order $n$
(vector polynomial of degree $n$)

Bernstein polynomial
(scalar polynomial of degree $n$)

Bézier control points
(vector in $\mathbb{R}^N$)

Bernstein polynomials:

$$B^n_i(t) = \binom{n}{i} t^i (1 - t)^{n-i}$$
Bézier Curve – Algebraic Formula: Example

Bernstein form of a Bézier curve of order \( n \):

\[
b^n(t) = \sum_{j=0}^{n} b_j B^n_j(t)
\]

Example: assume \( n = 3 \) and we are in \( \mathbb{R}^3 \)
i.e. we could have control points in 3D such as:

\[
\begin{align*}
b_0 &= (0, 2, 3), & b_1 &= (2, 3, 5), & b_2 &= (6, 7, 9), & b_3 &= (3, 4, 5) \\
\end{align*}
\]

These points define a Bezier curve in 3D that is a cubic polynomial in \( t \):

\[
b^n(t) = b_0 (1 - t)^3 + b_1 3t(1 - t)^2 + b_2 3t^2(1 - t) + b_3 t^3
\]
Cubic Bézier Basis Functions

Bernstein polynomials:

\[ B^n_i(t) = \binom{n}{i} t^i (1 - t)^{n-i} \]
Piecewise Bézier Curves
(Bézier Spline)
Higher-Order Bézier Curves?

High-degree Bernstein polynomials don’t interpolate well

Very hard to control!
Uncommon

\[ n = 10 \]
Piecewise Bézier Curves

Instead, chain many low-order Bézier curve
Piecewise cubic Bézier the most common technique

Widely used (fonts, paths, Illustrator, Keynote, …)
Piecewise Bézier Curve – Continuity

Two Bézier curves

\[ a : [k, k + 1] \rightarrow \mathbb{R}^N \]

\[ b : [k + 1, k + 2] \rightarrow \mathbb{R}^N \]

Assuming integer partitions here, can generalize
Piecewise Bézier Curve – Continuity

$C^0$ continuity: \[ a_n = b_0 \]
**Piecewise Bézier Curve – Continuity**

\[ C^1 \text{ continuity: } \quad a_n = b_0 = \frac{1}{2} (a_{n-1} + b_1) \]
Piecewise Bézier Curve – Continuity

$C^2$ continuity: “A-frame” construction

$$d_{i+1}$$
Properties of Bézier Curves

Interpolates endpoints
  • For cubic Bézier: \( b(0) = b_0; \quad b(1) = b_3 \)

Tangent to end segments
  • Cubic case: \( b'(0) = 3(b_1 - b_0); \quad b'(1) = 3(b_3 - b_2) \)

Affine transformation property
  • Transform curve by transforming control points

Convex hull property
  • Curve is within convex hull of control points
Bézier Surfaces
Bézier Surfaces

Extend Bézier curves to surfaces

Ed Catmull’s “Gumbo” model

Utah Teapot
Bicubic Bézier Surface Patch

Bezier surface and 4 x 4 array of control points
Visualizing Bicubic Bézier Surface Patch

Visualizing Bicubic Bézier Surface Patch

4x4 control points

• Each 4x1 control points in u define a Bezier curve
  • (4 Bezier curves in u)

• Corresponding points on these 4 Bezier curves define 4 control points for a “moving curve” in v
  • This “moving” curve sweeps out the 2D surface
Evaluating Bézier Surfaces
Evaluating Surface Position For Parameters \((u,v)\)

For bi-cubic Bezier surface patch,
Input: 4x4 control points
Output is 2D surface parameterized by \((u,v)\) in \([0,1]^2\)
Method 1: Separable 1D de Casteljau Algorithm

Goal: Evaluate surface position corresponding to (u,v)

(u,v)-separable application of de Casteljau algorithm

- Use de Casteljau to evaluate point u on each of the 4 Bezier curves in u. This gives 4 control points for the “moving” Bezier curve
- Use 1D de Casteljau to evaluate point v on the “moving” curve
Method 1: Separable 1D de Casteljau Algorithm
Method 2: Algebraic Evaluation

Let the moving curve be a degree $m$ Bézier curve

$$b^m(u) = \sum_{i=0}^{m} b_i B_i^m(u)$$

(remember, Bernstein polynomials)

$$B_i^n(t) = \binom{n}{i} t^i (1 - t)^{n-i}$$

Let each control point $b_i$ be moving along a Bézier curve of degree $n$

$$b_i = b_i(v) = \sum_{j=0}^{n} b_{i,j} B_j^n(v)$$

Tensor product Bézier patch

$$b^{m,n}(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{i,j} B_i^m(u) B_j^n(v)$$
Bézier Surface Continuity
Piecewise Bézier Surfaces

$C^0$ continuity: Boundary curves
Piecewise Bézier Surfaces

$C^1$ continuity: Collinearity
Piecewise Bézier Surfaces

$C^2$ continuity: A-frames
Things to Remember

Splines

- Cubic Hermite and Catmull-Rom interpolation
- Matrix representation of cubic polynomials

Bézier curves

- Easy-to-control spline
- Recursive linear interpolation – de Casteljau algorithm
- Properties of Bézier curves
- Piecewise Bézier curve – continuity types and how to achieve

Bézier surfaces

- Bicubic Bézier patches – tensor product surface
- 2D de Casteljau algorithm
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